Outer Automorphisms of Free Coxeter Groups

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A hyperelliptic involution of a surface of genus $g$ is an involution $\iota$ defining a degree-$2$ branched cover

$$S_g \rightarrow S^2$$

with $2g + 2$ branch points.

It turns out $\iota_* : H_1(S_g; \mathbb{Z}) \rightarrow H_1(S_g; \mathbb{Z})$ is $-1$ (finger proof!)
A famous theorem of Birman-Hilden implies the existence of a short exact sequence

\[ 1 \rightarrow \langle l \rangle \rightarrow \mathbb{C}_{\text{Mod}(S_g)} \langle l \rangle \rightarrow \text{Mod}(S_{0,2g+2}) \rightarrow 1. \]

\[ \mathbb{C}_2 \quad \text{HMod}(S_g) \]

**NB:** An index-\((2g+2)\) subgroup of \(\text{Mod}(S_{0,2g+2})\) is isomorphic to \(\mathbb{B}_{2g+1} / \mathbb{Z}(\mathbb{B}_{2g+1})\).

**NB:** \(\text{HMod}(S_2) = \text{Mod}(S_2)\), so this gave the first finite presentation of \(\text{Mod}(S_2)\).

Braid groups have some nice properties that remain unknown for \(\text{Mod}(S_g)\).
What about for $\text{Out}(F_n)$?

*A hyperelliptic involution* of a graph is an involution $\iota$ defining a degree-2 branched cover

$$
\Gamma \longrightarrow \mathcal{T}
$$

with $n+1$ branch points ($b_1(\Gamma) = n$).

It turns out there is a generating set $\chi_1, \ldots, \chi_n$ for $\pi_1(\Gamma) \cong F_n$ s.t.

$$
\iota_\# : \pi_1(\Gamma) \longrightarrow \pi(\Gamma) \quad \iota_\#(\chi_i) = \chi_i^{-1}
$$
A hyperelliptic involution of a graph may have \( n+1 \) components of fixed points:

![Diagram showing a hyperelliptic involution with fixed points and separating edges.]

**Lemma:** having fixed components that are not a single point happens \( \iff \) \( \Gamma \) has separating edges.
Work of Krstić implies a group-theoretic Birman-Hilden result:

\[ 1 \rightarrow \langle \ell \rangle \rightarrow C_{\text{Out}(F_n)}(\ell) \rightarrow \text{Out}(W_{n+1}) \rightarrow 1 \]

\[ \overset{\text{H}}{\cong} \]

Here \( W_n = \underbrace{C_2 \ast \cdots \ast C_2}_{\text{n copies}} \) is the free (universal) Coxeter group of rank \( n \).

**NB:** \( \text{HOut}(F_2) = \text{Out}(F_2) \cong \text{GL}_2(\mathbb{Z}) \), so \( \text{Out}(W_3) \cong \text{PGL}_2(\mathbb{Z}) \).

**NB:** \( \text{Aut}(F_2) \cong \text{Aut}(W_3) \cong \text{Aut}(B_4) \)

In the analogy between \( \text{Out}(F_{2g}) \) and \( \text{Mod}(S_g) \), \( \text{Out}(W_{2g+1}) \) plays a similar role to \( \text{Mod}(S_{0,2g+2}) \) or \( B_{2g+1}/\mathbb{Z}(B_{2g+1}) \).
Just as every $X \in \text{Out}(F_n)$ can be represented as a homotopy equivalence of a graph, so too can every $X \in \text{Out}(W_n)$ be represented by a homotopy equivalence of a graph of groups as below:

\[ \mathcal{W}_3 = \langle a, b, c : a^2 = b^2 = c^2 = 1 \rangle = \pi_1(\Gamma, \ast) \]

**Ex.**

\[ \Phi = \begin{cases} 
    a &\mapsto a \\
    b &\mapsto b \\
    c &\mapsto a^{-1}ca \\
\end{cases} \]

\[ f : \pi_1(\Gamma, \ast) \rightarrow \pi_1(\Gamma, \ast) \]
\[ f_{\#} = \Phi \]

* or an equivariant map of Bass–Serre trees
Ex. lifting homotopy equivalences

\[ \tilde{f} \]

\[ \begin{array}{c}
\text{\( L_1 \) } \\
\text{\( \uparrow \)} \\
\text{\( L_2 \) }
\end{array} \quad \begin{array}{c}
\text{\( \downarrow \)} \\
\text{\( \uparrow \)} \\
\text{\( \text{\( \ell \) } \)}
\end{array} \]

\[ \begin{array}{c}
\text{\( \langle a \rangle \) } \\
\text{\( \text{\( \ell \) } \)} \\
\text{\( \langle c \rangle \) }
\end{array} \quad \begin{array}{c}
\text{\( \text{\( \ell \) } \)} \\
\text{\( \langle a \rangle \) } \\
\text{\( \langle b \rangle \) } \\
\text{\( \langle c \rangle \) }
\end{array} \]

\[ \begin{array}{c}
\text{\( \langle a \rangle \) } \\
\text{\( \text{\( \ell \) } \)} \\
\text{\( \langle c \rangle \) }
\end{array} \quad \begin{array}{c}
\text{\( \text{\( \ell \) } \)} \\
\text{\( \langle a \rangle \) } \\
\text{\( \langle b \rangle \) } \\
\text{\( \langle c \rangle \) }
\end{array} \]

NB: Every (core) graph in rank 2 admits a hyperelliptic involution
A few similarities between $B_n/E(B_n)$ and $\text{Out}(W_n)$:

write \[ \text{POut}(W_n) = \ker (\text{Out}(W_n) \rightarrow S_n) \]

**Thm** (essentially Varghese ’19) \"forgetful map\"

\[ \text{POut}(W_n) \rightarrow \text{POut}(W_{n-1}). \]

\[ \Rightarrow \text{POut}(W_n) \] does not have property FA

\[ \Rightarrow \text{Out}(W_n) \] does not have property (T)

**Ex.** \"forget a cone point\"
The first two statements are true of $B_n / Z(B_n)$.

**Thm** (Guerch '20)

\[
\begin{align*}
\text{Out}(\text{Out}(W_n)) &= 1 & n &\geq 5 \\
\text{Out}(\text{Out}(W_4)) &\cong C_2 & n &= 4
\end{align*}
\]

**Thm** (Krstić–Vogtmann '93, McCullough–Miller '90)

\[
\text{vd}(\text{Out}(W_n)) = n - 2
\]

**Thm** (Birman–Hilden '73) if $W_n = \langle a_1, \ldots, a_n \rangle$, the subgroup of $\text{Out}(W_n)$ preserving the conjugacy class $[a_1 a_2 \cdots a_n]$ is isomorphic to $B_n / Z(B_n)$.

**NB:** This subgroup has infinite index when $n \geq 4$. 
Theorem (L ’20) There is a natural map from Guirardel-Levitt’s Outer Space for $W_n$ to Culler-Vogtmann’s Outer Space for $F_{n-1}$.

The map is an isometric embedding in the (asymmetric) Lipschitz metric and its image is the intersection of the fixed-point set of a hyperelliptic involution with reduced Outer Space.

The map sends a marked, metric $W_n$ graph of groups to its characteristic double cover.
Culler-Vogtmann's Outer Space $CV_n$ plays the role of Teichmüller space for $\text{Out}(F_n)$. Points of $CV_n$ are equiv. classes of pairs $(\Gamma, \varepsilon)$

- $\Gamma$ is a metric graph w/
  vertices of valence $\geq 3$ ("core" graph)

- $\varepsilon: R_n \rightarrow \Gamma$ is a homotopy
  rose w/ equivalence
  $n$ petals "marking"

- $(\Gamma, \varepsilon) \sim (\Gamma', \varepsilon')$ when there exists $h$ a homothety s.t.
  the diagram commutes up to homotopy

Guirardel-Levitt's Outer Space for a free product is the same idea.
Reduced Outer Space only includes graphs without separating edges.

The lemma \( \Rightarrow \) the double cover of a pt in Guirardel–Levitt Outer Space lives in Reduced Outer Space.

markings!
The Lipschitz distance between graphs \((\Delta, \sigma), (\Gamma, \tau)\) in Outer Space is

\[ d(\Delta, \Gamma) = \log \inf \left\{ L : \frac{\text{vol}(\Delta)}{\text{vol}(\Gamma)} \leq L \right\} \]

where there exists an \(L\)-Lipschitz homotopy equivalence \(f: \Delta \longrightarrow \Gamma\) s.t.

\[ f\sigma = \tau : \mathbb{R}_n \longrightarrow \Gamma \]

**NB:** Typically \(d(\Delta, \Gamma) \neq d(\Gamma, \Delta)\)!