High crossing knot complements with few tetrahedra

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May 12, 2020

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Big Question(s)

Diagram to triangulation: Given a diagram $D$ of a knot $K$ how many tetrahedra are needed to make up a complement?

Triangulation to Diagram: Given a triangulation $\mathcal{T}$ of a knot complement $S^3 \setminus K$, how many crossings could $K$ have?
Restatement:
c(K) minimum crossing number over all diagrams of K.
t(K) minimum number of tetrahedra needed to triangulate a complement of K.

Diagram to triangulation: Coarsely bound t(K) by a function in c(K).

Triangulation to Diagram: Coarsely bound c(K) by a function in t(K).
Octahedral Decomposition (attributed to D. Thurston)

\[ t(K) \leq 4c(K) \] using octahedra.
Triangulation to Diagram: Is $c(K)$ bounded by a polynomial function in $t(K)$?

No!

**Theorem (Haraway-H)**

There is a constant $C$ such that the complement of the torus knot $T_{F_{n+3}, F_{n+2}}$ in $S^3$ can be triangulated with at most $(2n - 1) + C$ tetrahedra and $c(T_{F_{n+3}, F_{n+2}}) \geq \varphi^{2n}$, where $\varphi = \frac{1 + \sqrt{5}}{2}$.
**Triangulation to Diagram:** If $S^3 \setminus K$ hyperbolic, is $c(K)$ bounded by a **polynomial** function in $t(K)$?

Still no!

**Theorem (Haraway-H)**

*The complement of twisted torus knot $T(F_{n+5}, F_{n+4}, 2, 4)$ in $S^3$ can be triangulated with at most $2n - 1 + D_1 + D_2$ tetrahedra and $c(T(F_{n+5}, F_{n+4}, 2, 4)) \geq \varphi^{2n}$, where $\varphi = \frac{1+\sqrt{5}}{2}$.***

Our construction here can be adapted to Satellite knot complements as well.
Theorem (Murasugi)

A $p/q$ torus knot $K_{p,q}$ with $p \geq q \geq 2$ has at least $p(q - 1)$ crossings. More generally, if $K$ is any knot presented as a homogeneous $n$-braid with braid index $n$, $c(K)$ can be read from that diagram.

Two Gadgets

1.

2.
Jaco and Rubinstein’s Layered Solid Tori
t(K) bounding c(K)

Proposition (H-Haraway)

If $K$ is a torus knot, there exists globally defined exponential function in $t(K)$ that bounds $c(K)$.

Theorem (Greene, Howie)

It is decidable if $T$ is the triangulation of an alternating knot complement.

Corollary (Juhász–Lackenby)

If $K$ is alternating, $c(K)$ is bounded by an function of $7t(K)^3 \cdot 2^{14t(K)} + 4$. 
Thank you for your attention!