Hyperbolic Structures on Three-Manifolds

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Abstract

This thesis provides a practical means of finding hyperbolic structures for link complements and manifolds obtained by Dehn surgery on link complements. The algorithm is implemented as a pair of C programs, copies of which can be obtained free of charge from the author. Some initial applications of the programs suggest that the complexity of a knot or link is best measured by the volume of its complement, not by the number of crossings in its projection. The structure of Dehn surgery spaces is discussed, and some techniques are given for working with the nonhyperbolic manifolds which sometimes arise. One example, a manifold with one cusp which is in a certain sense a mate to figure-eight knot complement, is discussed in detail.

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Contents

Introduction

This thesis provides a practical way of finding hyperbolic structures for link complements and manifolds obtained by Dehn surgery on link complements. The algorithm is completely general: it applies to nonalternating links as well as alternating ones. So far it has been completely successful, too, finding hyperbolic structures in all cases where such a structure is possible. The algorithm is implemented as a pair of C programs (one does the triangulation and the other solves the resulting equations). Copies may be obtained free of charge from Jeff Weeks, Department of Math and Computer Science, Ithaca College, Ithaca, NY 14850.

The document you are now reading consists of four chapters. Chapter 1 presents results obtained using the programs. The central theme is that a knot's complexity is best measured by the volume of its complement, not by the number of crossings in its projection. Chapter 2 explains the triangulation algorithm, Chapter 3 analyzes some of the nonhyperbolic manifolds which the program finds, and Chapter 4 takes a look at a particularly interesting manifold. Chapter 1 is intended for a general mathematical audience, while Chapters 2, 3 and 4 will be of interest only to specialists.

Proper background material for this thesis could easily fill a book. I may write it some day, but in the meantime I will assume that readers venturing beyond Chapter 1 are already familiar with hyperbolic geometry, Dehn surgery, and the theory of geometric structures on three-manifolds. I can recommend some books for those wanting to learn more about these subjects. Rolfsen's *Knots and Links* is an outstanding introduction to low-dimensional topology in general, and gives a good account of Dehn surgery in particular. Rolfsen assumes no prerequisites beyond undergraduate math and the very basics of algebraic topology. Thurston's *Three-dimensional Geometry and Topology* will be the definitive source for hyperbolic geometry and the geometric theory of three-manifolds. For a very elementary account of the subject see Weeks' *The Shape of Space: How to Visualize Surfaces and Three-dimensional Manifolds.*

The notation used in this thesis is all very standard. The only exception is that what Rolfsen calls a 5/3 Dehn surgery is called a (5,3) Dehn surgery here (following Thurston). So in particular Rolfsen's 0 surgery is (0, 1) here, and his ∞ surgery is (1, 0) here. The 'standard knot tables' referred to throughout this thesis can be found in the back of Rolfsen.