Chapter 4

An Interesting Example

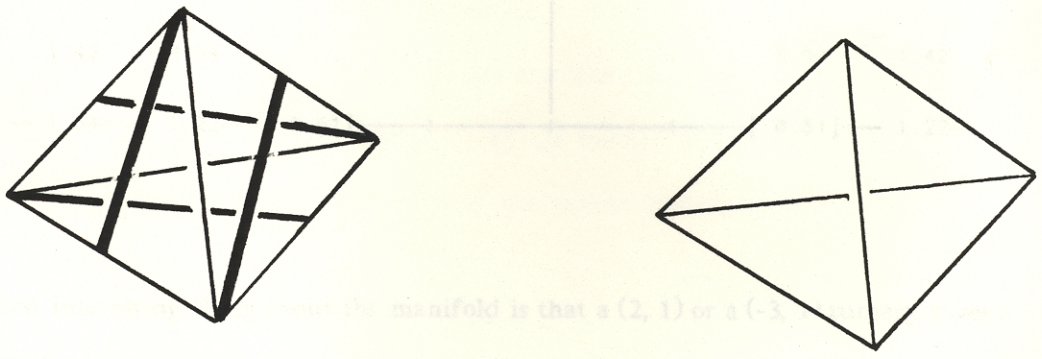
and

A Census of Simple Manifolds

Overview

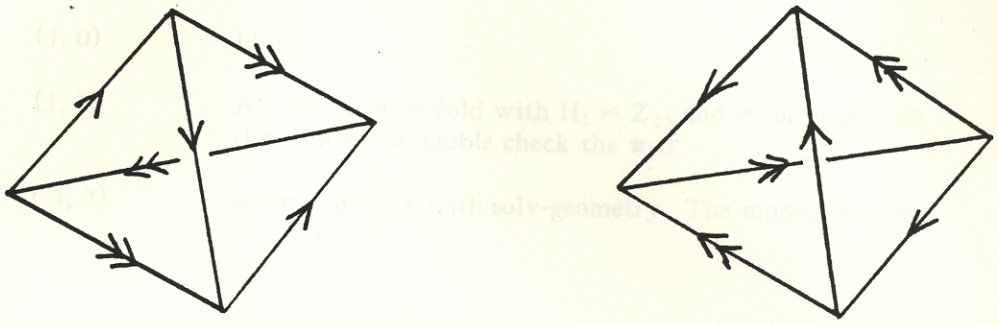
The first section of this chapter describes an interesting manifold which turned up in a census of orientable manifolds made from two ideal tetrahedra. The manifold is, in a certain sense, a cousin of the figure-eight knot complement, although it is not a knot complement itself. A surgery on this manifold gives a closed manifold with volume .942707...; this is the smallest known volume for a hyperbolic three-manifold.

The second section of this chapter reports on some initial steps towards a census of three-manifolds with cusps, and discusses prospects for a more extensive computer search.



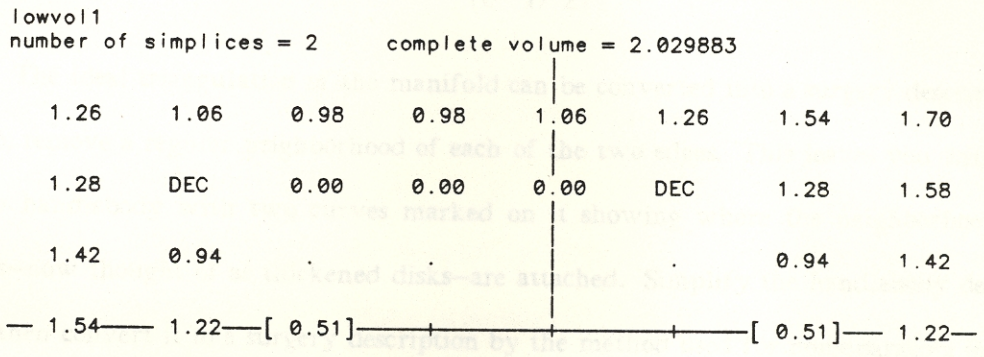
An interesting manifold

The manifold is made from two ideal tetrahedra. (See above figure.) Each face of the left hand tetrahedron is glued to the corresponding face of the right hand tetrahedron with a flip about the indicated axis. From the above figure it's easy to deduce that the edges of the tetrahedra are grouped into two edge classes of six edges each.



Thus the gluing conditions are satisfied when both tetrahedra are regular. Like the figure-eight knot complement, this manifold has volume 2.029883... . However, this manifold is not a knot complement: its first homology is $Z + Z_5$.

Here's its Dehn surgery space:



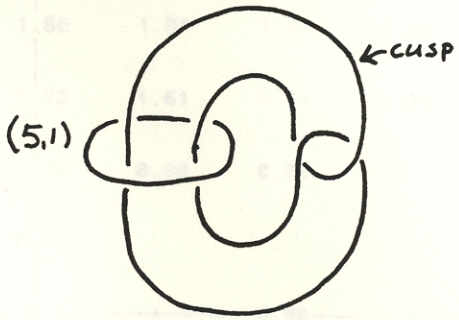
The first interesting thing about the manifold is that a (2, 1) or a (-3, 1) surgery gives a manifold of volume .942707..., the smallest volume yet found for a three-manifold. The next interesting thing is the large number of nonhyperbolic integer points. The symmetry which seems to exist is real: the manifold obtained by (a, b) surgery is homeomorphic to the one obtained by (-a-b, b) surgery. Thus there are not 10 but only 6 nonhomeomorphic, nonhyperbolic spaces obtainable by generalized Dehn surgery. Of these six, five are actually manifolds.

surgery	*manifold*
(0, 1)	L(5,1)
(1, 0)	L(10, 3)
(1, 1)	An elliptic manifold with $H_1 = Z_{15}$ and π_1 of order 120. (I still need to double check the π_1.)
(-1, 2)	A torus bundle with solv-geometry. The monodromy is $\begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix}$
(1, 2)	Decomposes into an orientable interval bundle over a Klein bottle and a trefoil complement.

Bob Meyerhoff has pointed out that this manifold has the same Dehn surgery space as the figure-eight knot complement, up to a linear transformation. The following matrix takes this manifold's Dehn surgery space to that of the figure-eight knot.

$$\begin{pmatrix} 2 & 1 \\ 0 & 1/2 \end{pmatrix}$$

The ideal triangulation of the manifold can be converted into a surgery description. (To do so, remove a regular neighborhood of each of the two edges. This leaves you with a genus three handlebody with two curves marked on it showing where the neighborhoods of the edges—now thought of as thickened disks—are attached. Simplify the handlebody description, and then convert it to a surgery description by the method used for nonseparating incompressible surfaces in Chapter 3.) The surgery description reveals that this manifold can be obtained from the Whitehead link:



This prompts us to take a closer look at the Whitehead link. Here is a portion of its Dehn surgery space:

```
wh_link
surgery on cusp #1
        cusp #2 complete
number of simplices = 4      complete volume = 3.663862
                             |
   3.46      3.43      3.38  |  3.33      3.28      3.22      3.18      3.15      3.14
                             |
   3.28      3.16      3.01  |  2.83      2.67      2.57      2.54      2.57      2.67
                             |
   2.67      2.03      DEC   |  0.00      0.00      0.00      DEC       2.03      2.67
                             |
— 0.00———————+———————————————+——— 0.00——— 1.78——— 2.54——— 2.92——— 3.14—
```

There is a symmetry to the volumes in the Whitehead link's Dehn surgery space, only now matching manifolds are not homeomorphic (proof: compare homology). The manifold we've been working with matches with the figure-eight knot complement. The symmetry was found independently by Bob Meyerhoff, who also discovered additional symmetries involving the half-integer points.

By the way, here's what our manifold's Dehn surgery space looks like when we use coordinates which are natural relative to its description as as (5, 1) surgery on the Whitehead link.

```
lowvol
number of simplices = 2      complete volume = 2.029883
                             |
   1.92      1.89      1.86  |  1.81      1.74      1.63      1.48      1.26      1.28
                             |
   1.86      1.81      1.73  |  1.61      1.41      1.06      DEC       0.94      1.22
                             |
   1.74      1.61      1.39  |  0.98      0.00      .       [ 0.51]     0.94      1.28
                             |
   1.41      0.98      0.00  |  .         .         .         DEC       1.26      1.59
                             |
— 0.00———————+———————————————+——— 0.00——— 1.06——— 1.48——— 1.68——— 1.79—
```

A census of simple three-manifolds with cusps

By hand I have studied all three-manifolds obtained by identifying pairs of faces of one tetrahedron, and all orientable three-manifolds obtained from two tetrahedra. I studied the closed examples and the examples with cusps (for which the vertices of the tetrahedra were, of course, deleted), but I ignored the examples having 'cusps' with cross-sections of genus greater than 1. The only closed 3-manifolds obtainable with one tetrahedron are S^3, L(4,1) and L(5,2), and the only such manifold with a cusp is the Gieseking manifold (whose hyperbolic structure apparently cannot be deformed). The closed manifolds you can get with two tetrahedra are S^3, RP^3, L(3,1), L(5,?), L(8,?) and another elliptic manifold which is an 8-fold quotient of the 3-sphere. The manifolds with cusps are a punctured-torus bundle, a thrice-punctured-sphere bundle, a solid torus, a trefoil complement, a manifold which decomposes into a (2,5) torus knot and a (2,4) torus link, a figure-eight knot complement, and the interesting manifold described in the first part of this chapter. In particular, the latter two manifolds are the only orientable hyperbolic manifolds of finite volume obtainable from two tetrahedra. However there may well be nonorientable manifolds of finite volume which can be assembled from two ideal tetrahedra, and I wouldn't be at all surprised if some of them had volumes less than the volume of the figure-eight knot complement. If their cusps were tori then perhaps their hyperbolic structures could be deformed to give nonorientable closed manifolds with very low volumes. (This last point is still not at all clear, however.)

One of my next projects is to do a computer census of manifolds obtainable from small numbers of ideal tetrahedra. As well as providing some answers to questions like the ones just raised, such a study would reveal the typical behavior of simple manifolds with cusps. This would complement the study of random manifolds with cusps (described in Chapter 1), which could be expected to reveal typical complicated behavior.

I believe it's important to study manifolds with cusps in general, rather than just knot and link complements. It's an artificial restriction to insist on studying only manifolds which embed in the three-sphere, and if we want to really understand what's going on we must

avoid artificial restrictions as much as possible. In particular, we'll never see the general structure of Dehn surgery spaces by looking at knot and link complements alone, because knot and link complements--by definition--have a Dehn surgery which gives the three-sphere.