Chapter 3

Nonhyperbolic Manifolds

Overview

This chapter presents some elementary techniques for working with the nonhyperbolic manifolds which arise from doing Dehn surgery on a knot or link. The chapter is divided into three sections. Surgered manifolds containing a nonseparating torus are treated in the first section, those containing a separating torus are treated in the second, and Seifert fibered spaces are treated in the third. These techniques all work well in the very simplest examples (through 8 crossings), but I must admit that when I recently tried them on more complicated--and presumably more generic--examples I had limited success. Nevertheless I am presenting them here anyhow because they are fun and they do at least shed some light on the simpler examples.

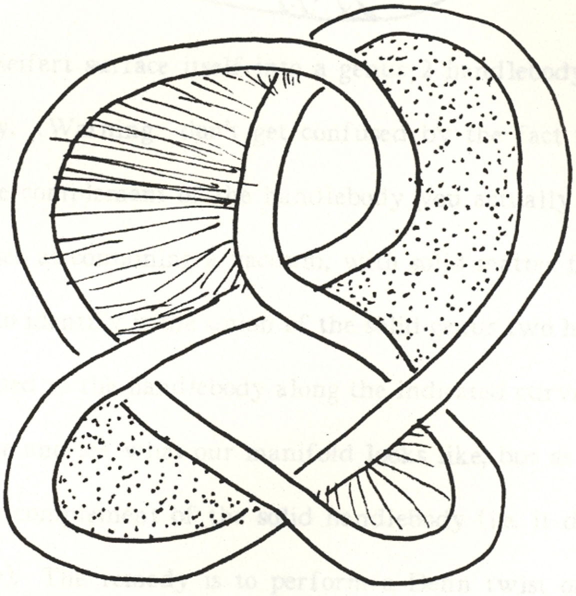
I have developed no analogous techniques for connected sums because so far none have shown up! I'm not sure whether connected sums can't arise from surgery on a hyperbolic knot, or whether I just haven't looked at enough examples.

By the way, when using the program SNAPPEA you suspect an incompressible surface whenever some or all the simplices are approaching 0, 1 or infinity. It's good to check the volume as you approach such points, because the volume approaches a limit equal to the sum of the volumes of the pieces your manifold is decomposing into. Seifert fibered spaces are recognizable by the fact that the simplices approach real values not equal to 0, 1 or infinity.

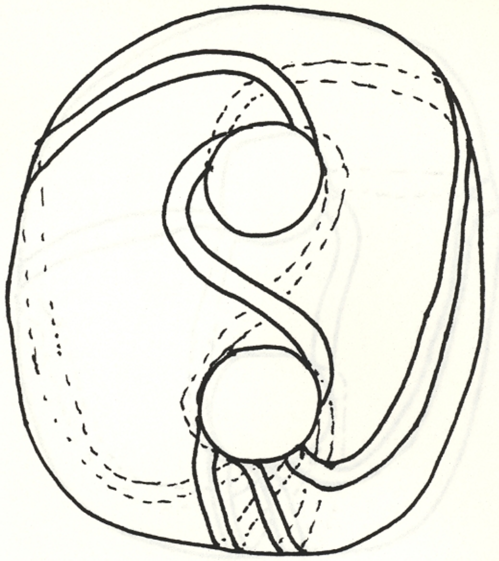
Readers wanting to know more about the seven homogeneous nonhyperbolic geometries should read Peter Scott's survey article [S].

Nonseparating incompressible surfaces

Nonseparating incompressible surfaces are much easier to work with than separating ones. At least in the examples I've seen so far they all occur as the Seifert surface for a genus one knot, so finding the surface is no problem. The only question is what do you get when you cut the (0,1)-surgered manifold open along the surface, and even that question is pretty easy. When you cut open a closed manifold along a nonseparating torus you will of course get a manifold with two cusps. To see how to identify this manifold we'll consider two examples. The first, and simpler, example is the knot 5_2. First draw the knot and sketch in the Seifert surface.

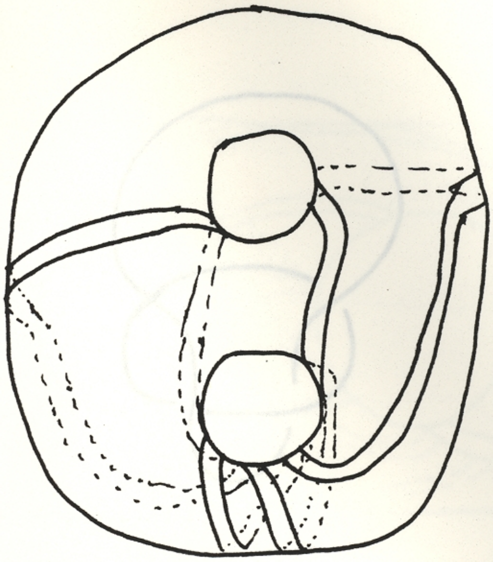


The incompressible surface we're cutting along is the Seifert surface you actually see plus a disk which is a cross-section of the surgered-in solid torus. Imagine thickening up the incompressible surface—this is OK since we are interested only in its complement.

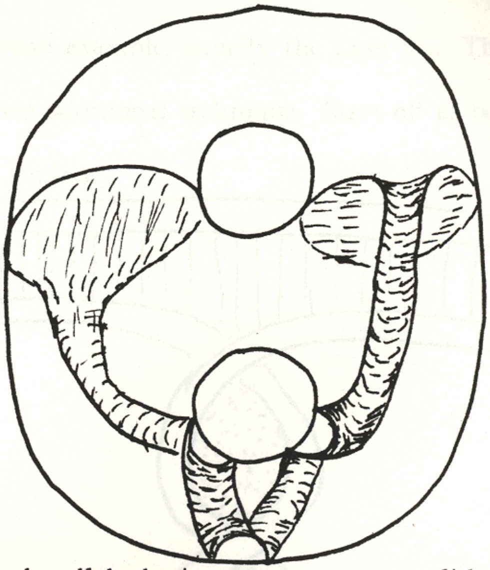


opaque version:

We've turned the Seifert surface itself into a genus 2 handlebody. Its complement is also a genus 2 handlebody. (Warning: don't get confused by the fact that the handlebody we're working with is the complement of the handlebody you actually see. Think of the handlebody you actually see as containing a vacuum, with solid matter filling its complement.) The manifold we want to identify is the union of the solid genus two handlebody and a thickened disk. The disk is glued to the handlebody along the indicated curve. We would like to attach the disk to the curve and see what our manifold looks like, but as it stands the curve doesn't bound a disk in the complement of the solid handlebody (i.e. it doesn't bound a disk in the vacuous handlebody). The remedy is to perform a Dehn twist on one of the handlebody's handles so that the curve will bound a disk in the vacuous handlebody (look ahead a few drawings and you'll see where we're headed). A left-handed Dehn twist on the handle which passes through the upper hole of the vacuous handlebody gives the following result:



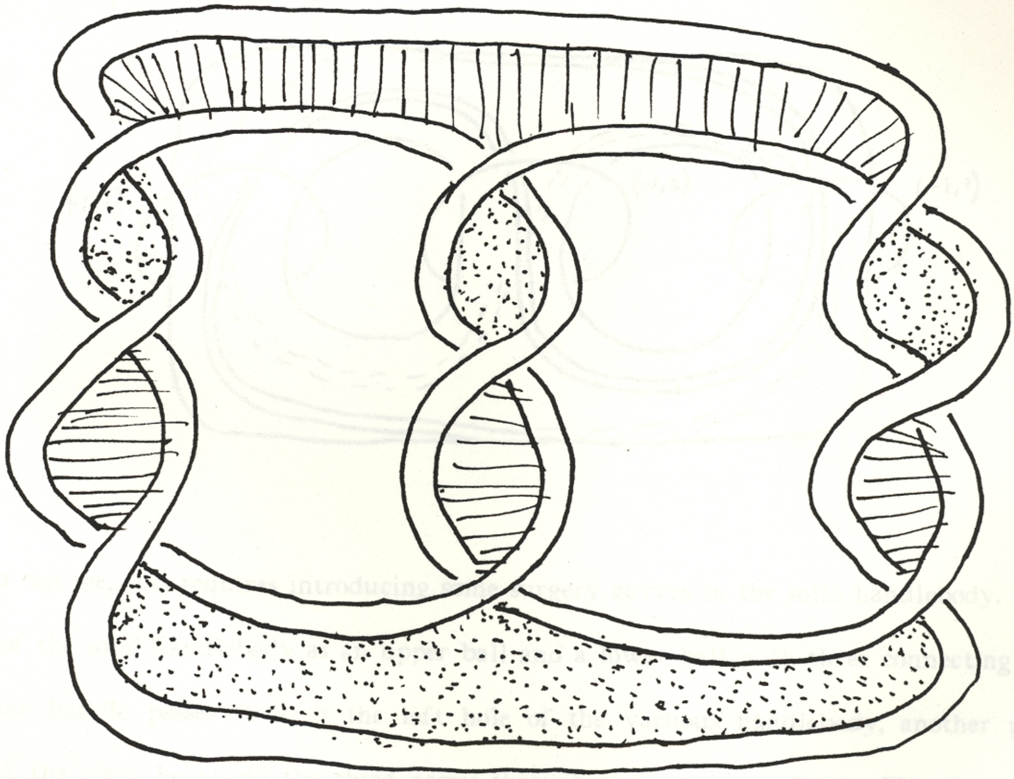
The curve now bounds a disk in the vacuous handlebody, as shown below.



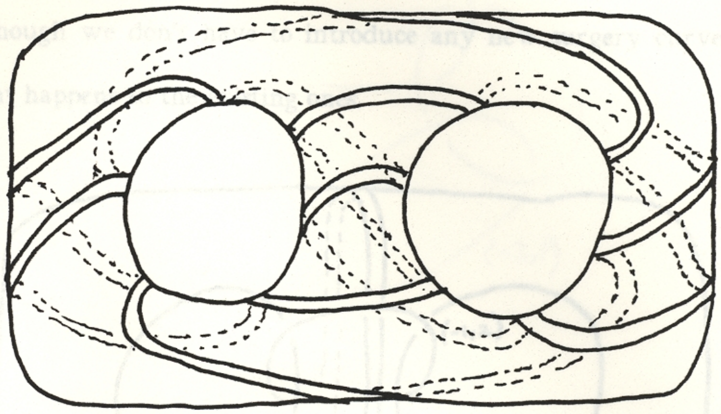
The disk cuts the vacuous handlebody into two vacuous solid tori, and we see that our manifold--the one obtained by doing (0, 1) surgery on the knot 5_2 and then cutting along the incompressible torus--is the complement of a (2,4) torus link:

(Only in topology can you talk about objects which are vacuous and solid at the same time.)

Now let's look at a second example, namely the knot 9_{35}. The basic idea is the same as for knot 5_2, but we'll use one additional technique. Start off as before by drawing the knot and sketching in the Seifert surface.

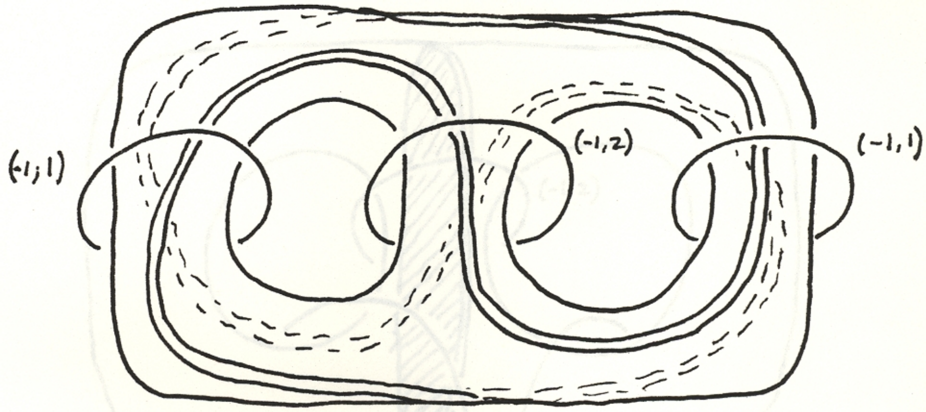


Thicken up the incompressible surface.

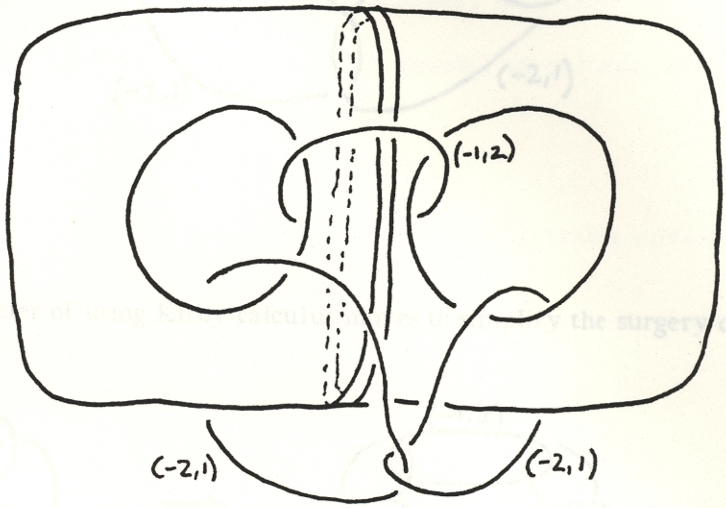


opaque version:

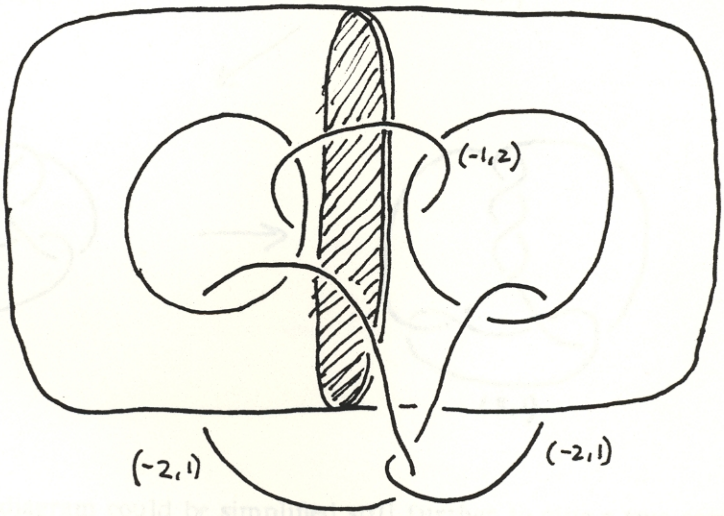
This example differs from 5_2 in that we now have to perform Dehn twists on the vacuous handlebody as well as the solid one. Start by doing Dehn twists on the vacuous handlebody as shown.



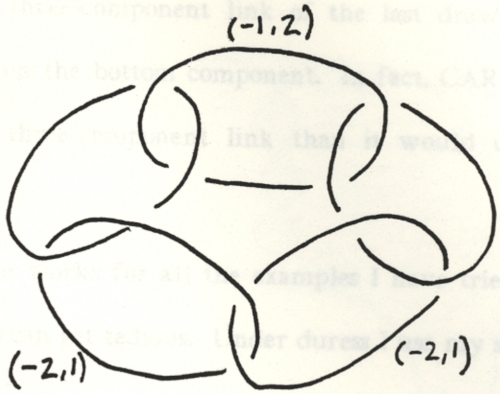
As you can see, this requires introducing some surgery curves in the solid handlebody. Now think of the solid handlebody as an upper ball and a lower ball with three connecting handles: one handle passes through the left hole of the vacuous handlebody, another passes through the right hole, and the third passes through the point at infinity. The situation is symmetrical in S^3, even through it doesn't look very symmetrical in our sketches. We can perform a Dehn twist on any of the three handles without introducing new surgery curves (remember—the handles are surrounded by vacuum!). Just to keep things interesting (and symmetrical) we'll do the Dehn twist on the handle passing through the point at infinity. Note that even though we don't have to introduce any new surgery curves, we do have to keep track of what happens to the existing ones.



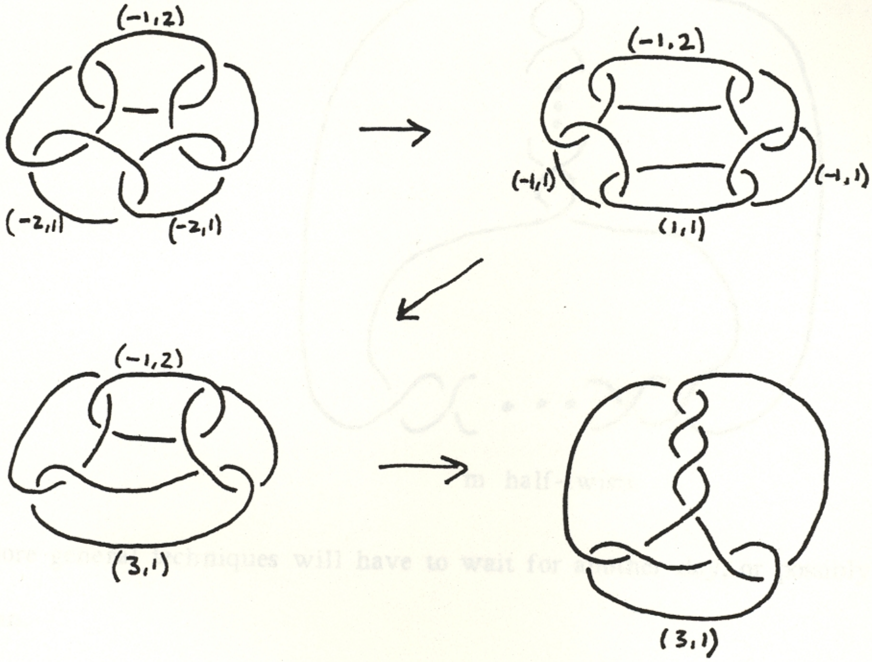
We now have a place to glue in the thickened disk.



The disk splits the vacuum into two solid tori. We can therefore redraw the above picture as a link, where the link components without numbers represents drilled out circles and the link components with numbers represent Dehn surgeries.



It's now just a matter of using Kirby calculus moves to simplify the surgery diagram.

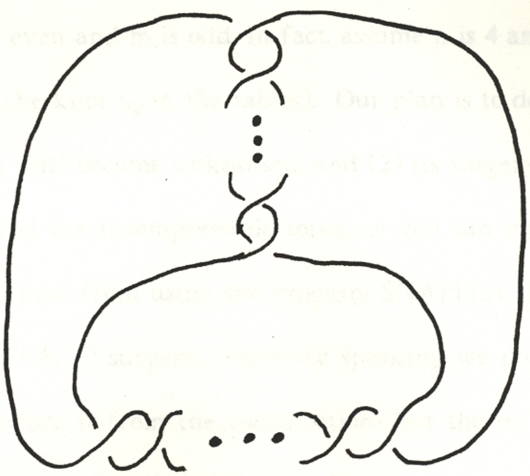


The last surgery diagram could be simplified still further to give a two component link. (Do two left-hand twists on the right unsurgered component. This will leave the surgered component unknotted but will change its surgery coefficient to (1, 1). You then do a left-hand twist on the surgered component, which eliminates it and leaves you with a link complement.) Even though it would be interesting to see the final link, it would be tedious to carry out the simplification. Fortunately one can use SNAPPEA to study the link complement simply by entering the three-component link of the last drawing above and immediately doing the (3, 1) surgery on the bottom component. In fact, CARROT might produce a better triangulation using the three-component link than it would using the complicated two-component link directly.

The above technique works for all the examples I have tried. As we just saw, though, the Kirby calculus moves can get tedious. Under duress I use my shoelaces.

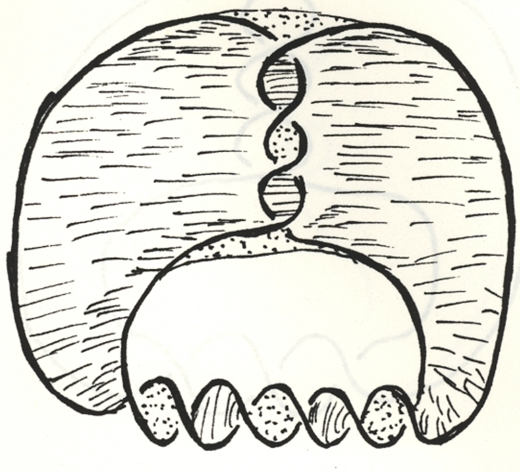
Separating incompressible surfaces

I will describe a technique for quickly finding incompressible surfaces for knots of the following form:

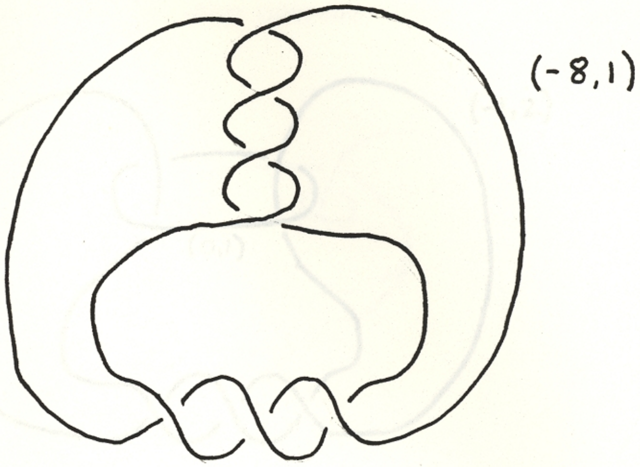


More general techniques will have to wait for another day, or possibly another mathematician.

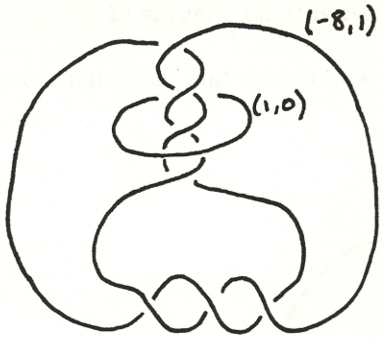
Consider a knot as shown above, with n half-twists in one part and m half-twists in the other. If m and n are both even, then the knot has genus one and contains a nonseparating incompressible torus as described above. Here's a picture of it in the case where n = 4 and m = 6:

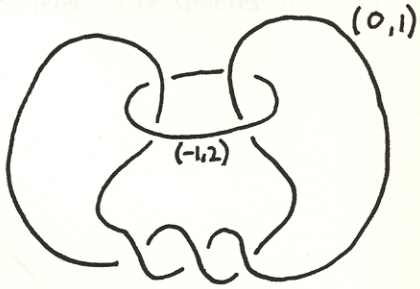


If n and m are both odd the knot won't be a knot but a link, and we won't worry about that here. So assume that n is even and m is odd. In fact, assume n is 4 and m is 3 (it's not immediately obvious, but this is the knot 6_2 in the tables). Our plan is to do Kirby calculus moves so that (1) the original knot will become unknotted, and (2) its surgery coefficient will be (0, 1). It will then be easy to find the incompressible torus, as you can see by looking ahead at the next few drawings. We know from using the program SNAPPEA that an incompressible surface occurs when you do (-8, 1) surgery. (Strictly speaking we don't need this last piece of information--we could deduce it from the construction--but the information is readily available and provides motivation, so I will include it.) Thus we are starting with a 6_2 knot with (-8, 1) surgery.

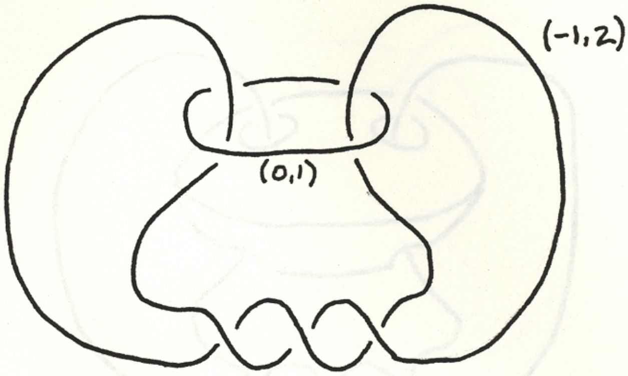


Introduce a trivially surgered loop as shown, and perform two clockwise twists.

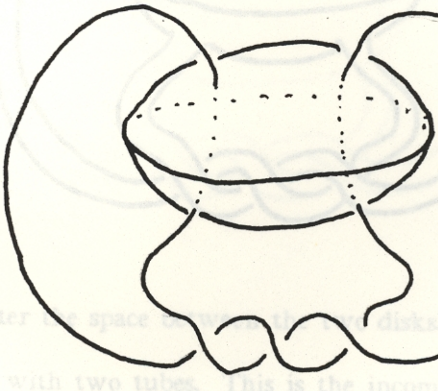




This accomplishes two things: it unknots the original knot, and it changes the surgery coefficient to (0, 1). For clarity, interchange the two components of the resulting link. (In all the examples we do the resulting link will be a two-link chain with some number of twists, so we will always be able to interchange the components at this stage.)



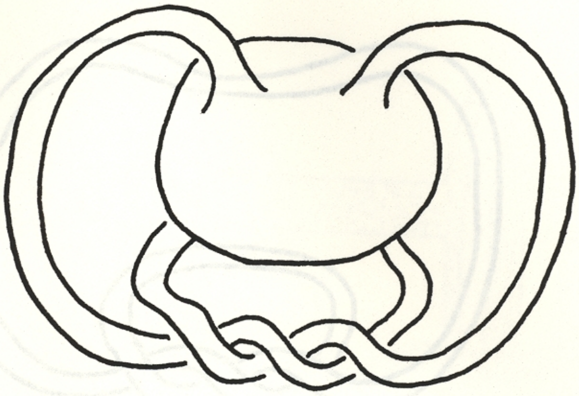
It's now easy to draw in the incompressible torus. Start with two disks spanning the smaller component of the link (one bulges upward and the other bulges downward). Because we're doing (0, 1) surgery on this component, the disks can be completed to be spheres in the surgered manifold, and that is how we will think of them.



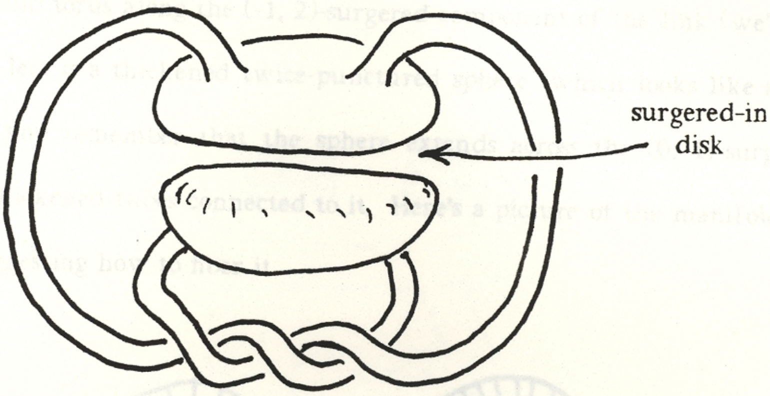
Cut two holes in each disk to allow the other component of the link to pass through.

Now make two tubes, each of which runs from a hole in the upper disk to a hole in the lower disk, surrounding part of the other link component.

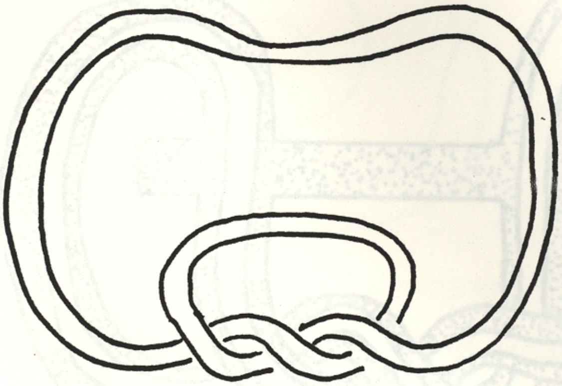
(The tubes do not enter the space between the two disks.) What we have is two spheres connected to each other with two tubes. This is the incompressible torus we're looking for. It separates the manifold into two pieces. Let's see what the pieces are. First look at the exterior piece. It's the complement of the following solid,



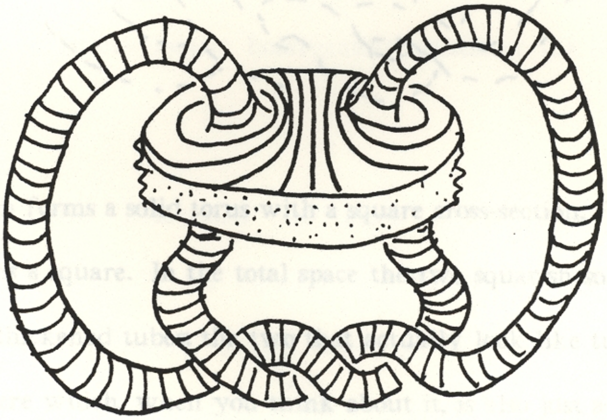
plus a thickened disk which gets added when you do the (0, 1) surgery. Surgering in the thickened disk is very, very easy. It can cut across the central ball like this:



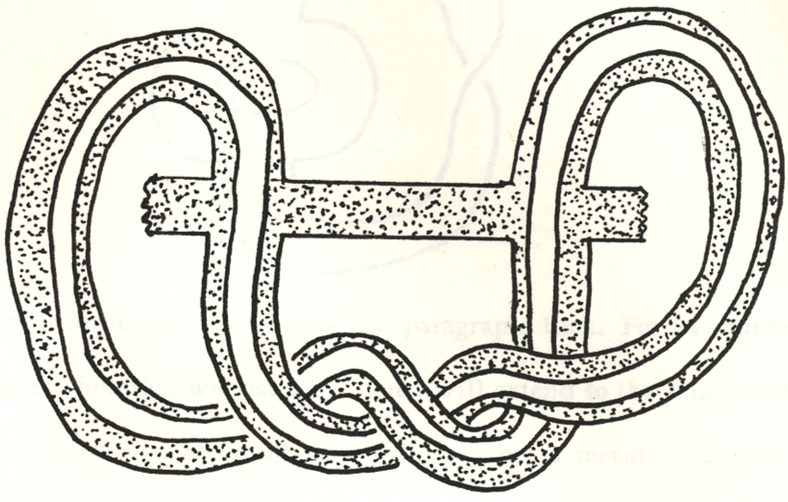
This is obviously a trefoil complement. Here is the final, cleaned up picture.



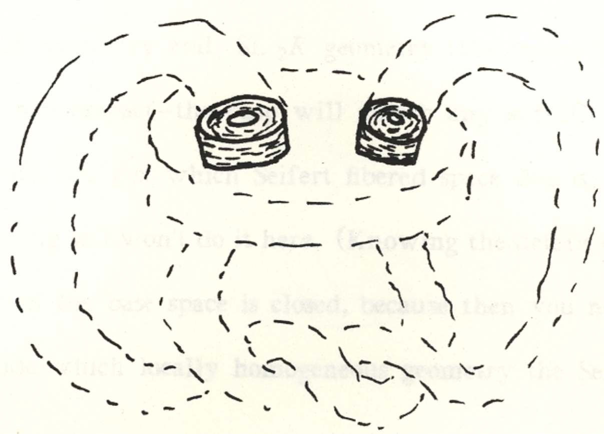
We just saw that the incompressible surface cuts the manifold into two pieces, one of which is a trefoil complement. The other piece is a Seifert fibered space. Here's how to see it. First drill out a solid torus along the (-1, 2)-surgered component of the link (we'll add it back in later). What's left is a thickened twice-punctured sphere (which looks like a twice punctured disk until you remember that the sphere extends across the (0, 1)-surgered-in solid torus) with two thickened tubes connected to it. Here's a picture of the manifold, along with some markings suggesting how to fiber it.



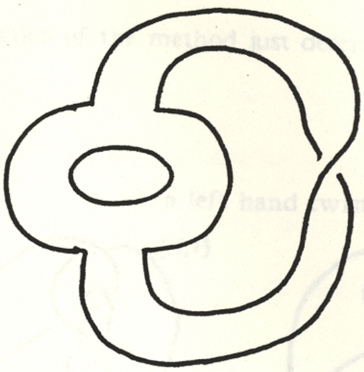
Here's a cross-sectional picture of the same thing. The dots are cross-sections of the fibers.



We can find out what the base of the Seifert fibration is by compressing each fiber to a point. First consider the fibers in the shaded region shown below:



In the total space each forms a solid torus with a square cross-section. When compressed down to the base each forms a square. In the total space the two squarish solid tori are connected to each other by three thickened tubes: the two that actually look like tubes, plus the thickened twice-punctured sphere which, when you think about it, is also just a tube. In the base space the two squares are therefore connected to each other with three strips, as shown:

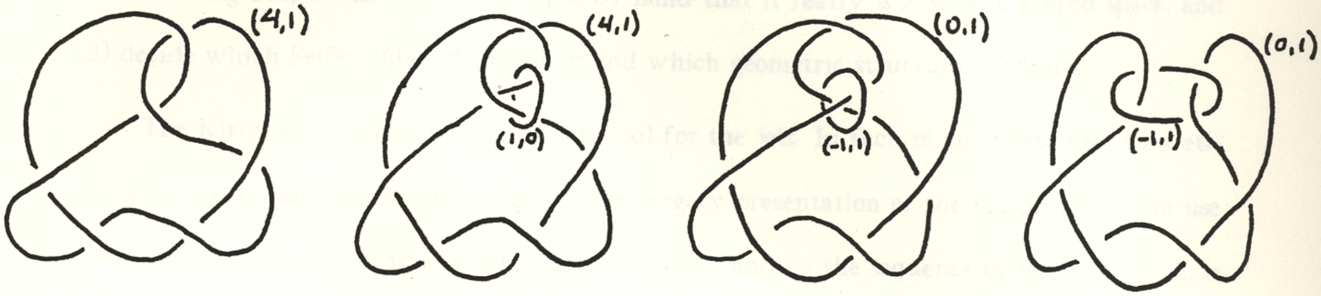


Now replace the solid torus we drilled out a few paragraphs back. For any surgery except the trivial one the Seifert fibration we just constructed will extend to the solid torus (because the fiber will map to a curve on the solid torus which is not a meridian, and that's all that's required to extend the fibration). Adding the solid torus adds a disk to the base space, and typically the disk will have a cone point of some order, depending on the surgery. Therefore the base space of the fibration is a Mobius strip with one cone point. When the order of the cone point is nontrivial the base space will be hyperbolic and the Seifert fibered space itself will admit both H^2xE geometry and SL_2R geometry (the reason you have your choice is that the manifold is noncompact—the cusp will absorb any surplus shear). It's completely straightforward to work out just which Seifert fibered space this is, but it is rather tedious and not very enlightening so I won't do it here. (Knowing the details of the exceptional fibers is more important when the base space is closed, because then you need to know about the shear in order to decide which locally homogeneous geometry the Seifert fibered space will admit.)

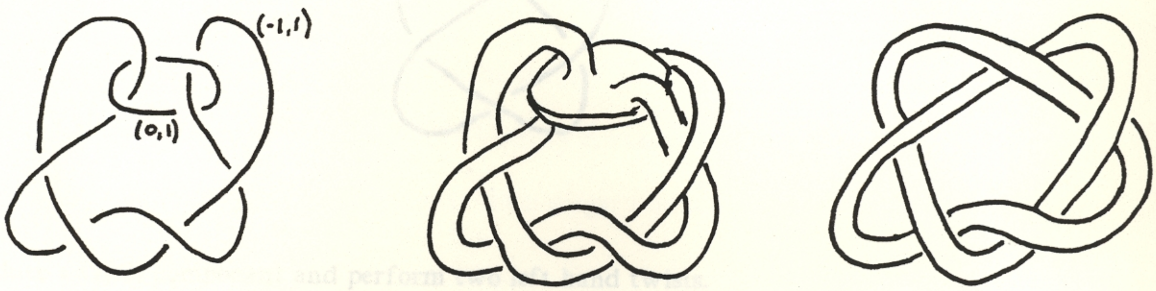
The techniques just illustrated for knot 6_2 apply equally well to knots 4_1, 5_2, 6_1, 7_2, 7_3, 8_1, 8_2, 8_4, 9_2, 9_3, 9_4, and, in a modified from, to knot 8_5. I'd like to make them work for other examples too (e.g. knot 9_{42}, which has an incompressible surface at (6, 1), and knot 9_{46}, which has one at (2, 1)), but so far I haven't had the time.

I should mention that when m is even and n is 2 (n and m are the number of half-twists as defined at the beginning of this section) you get not only the nonseparating incompressible torus at (0, 1), but also a separating one at (4, 1) or (-4, 1). The latter is constructed by a minor variation of the method just described. Here are the steps for the case of knot 6_1:

add a component   do a left hand twist   isotope



interchange components   draw the surface   glue in a disk

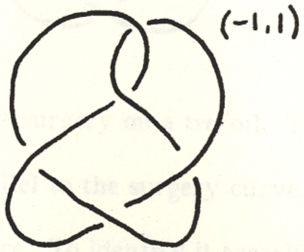


Seifert fibered spaces

Surgery on links of low complexity often yields Seifert fibered spaces. You can easily recognize these manifolds with SNAPPEA by seeing when the simplices flatten out, i.e. when the cross-ratios approach real values other than 0, 1 and infinity. (Doing (0, 1) surgery on the figure-eight knot gives flattened simplices which yield a torus bundle with a solv-geometry structure rather than a Seifert fibered space, but this is apparently the only such example among the knots in the tables.) Once you see that the simplices for a certain surgered manifold are flattening out, it's nice to (1) confirm by hand that it really is a Seifert fibered space, and (2) decide which Seifert fibered space it is and which geometric structure it admits.

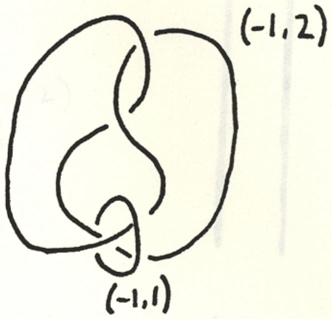
The Kirby calculus is the appropriate tool for the job. In fact, in the narrowest theoretical sense we are assured of success: given one surgery presentation of the manifold we can use the Kirby calculus to get to any other one. Actually finding the sequence of moves, though, is more an art than a science. Fortunately the simplest knots aren't too hard to work with.

As an example, consider the knot 5_2, for which the surgeries (-1, 1), (-2, 1) and (-3, 1) all give Seifert fibered spaces. First consider (-1, 1) surgery.

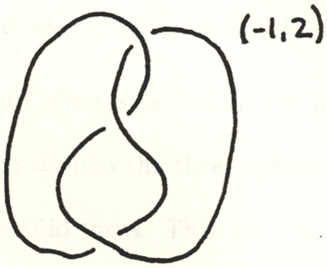


Introduce a (1, 0) component and perform two left hand twists.

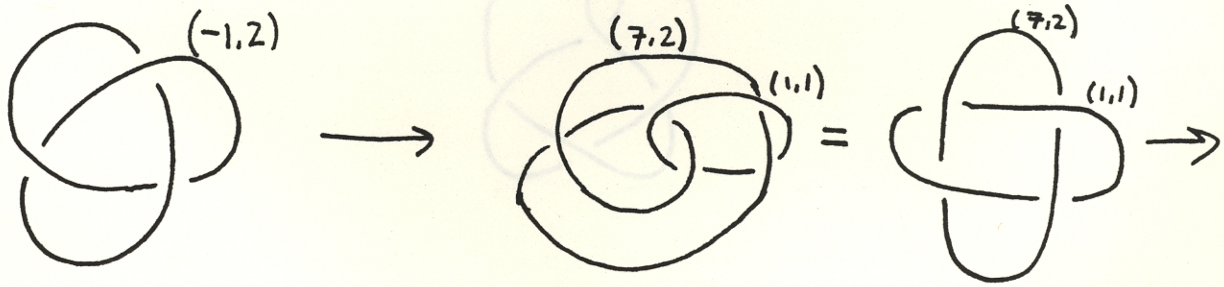
This is the Whitehead link, and we can interchange components.

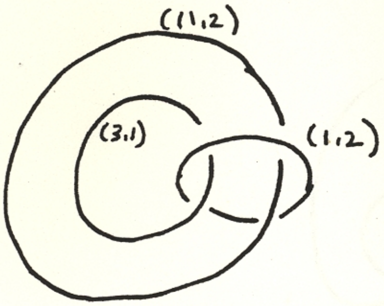


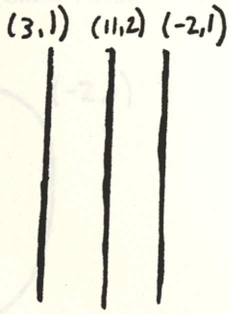
Now do a right hand twist to eliminate the (-1, 1) component.



We've represented the manifold as a surgery on a trefoil. The trefoil complement is a Seifert fibered space, and a fiber is not parallel to the surgery curve, so we know right away that our manifold is also a Seifert fibered space. To identify it exactly we do additional Kirby calculus moves to represent the manifold as $S^2 \times S^1$ with surgery on three fibers.





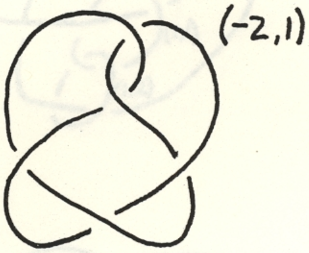


in $S^2 \times S^1$

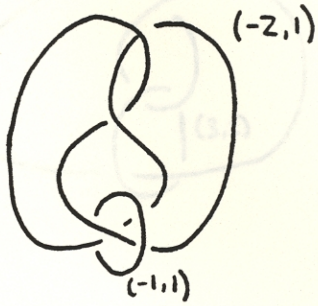
It's now easy to deduce that this is a Seifert fibered space over a sphere with cone points of orders 3, 11 and 2. The amount of shear concentrated at the three exceptional fibers does not add up to zero, so this Seifert fibered space admits an $SL_2 R$ structure.

(A surgery on a torus knot can always be put in the above form. Proof: Think of the torus knot as lying on a torus which divides the three-sphere into two solid tori. Add a (1, 0) component along the center of each solid torus. This lets you do Dehn twists on the torus the knot lies on. By working in a Euclidean-algorithm like fashion you can unknot the torus knot entirely.)

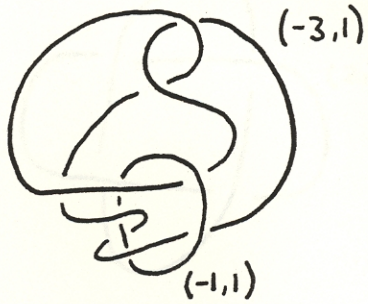
As a second example, consider (-2, 1) surgery on 5_2.



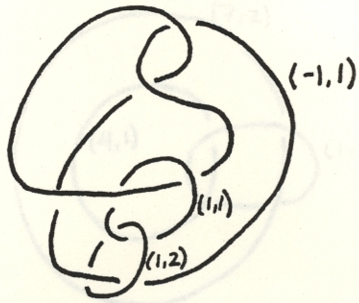
Introduce a trivial component and perform a left hand twist.



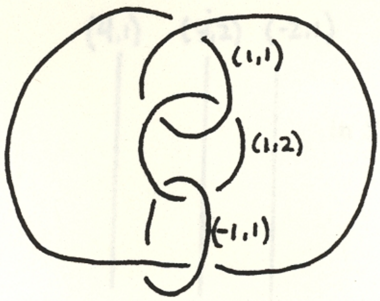
Slide the (-2, 1) component through the (-1, 1) component.



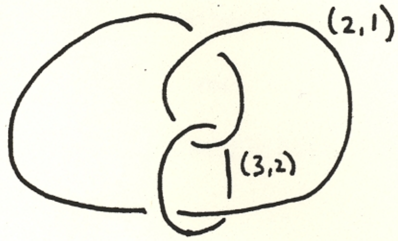
Introduce another trivial component and perform two right hand twists.



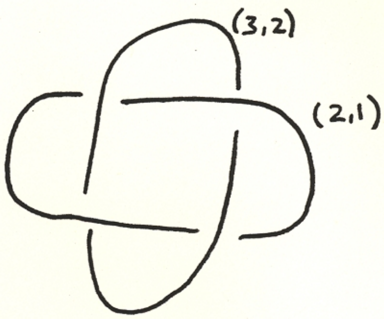
Permute the components.



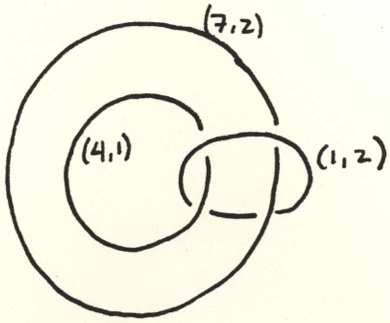
Do a right hand twist to eliminate the (-1, 1) component.



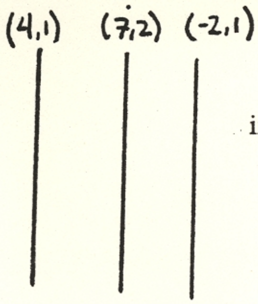
This is a (2,4) torus link.



Introduce a trivial component to take the twist out.



Redraw as three fibers in S^2 x S^1.



in S^2 x S^1

This Seifert fibered space fibers over a sphere with cone points of order 4, 7 and 2 and admits an SL_2R structure. The analysis of (-3, 1) surgery on 5_2 is similar, and it too admits an SL_2R structure.