Chapter 1

Results

Volumes of knot complements

A hyperbolic structure on a three-manifold is a complete metric of constant negative sectional curvature. On knot and link complements such a structure always has a finite volume. Not all knot and link complements admit a hyperbolic structure, but for those that do Mostow's Rigidity Theorem states that the hyperbolic structure is unique. This implies that the volume of the hyperbolic structure is a topological invariant of the knot or link.

A hyperbolic manifold is one which admits a hyperbolic structure, and a hyperbolic knot or link is one whose complement is hyperbolic.

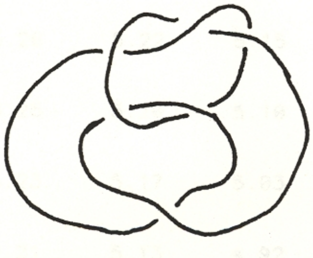
Bill Thurston [TH1] has proved that every knot which is neither a torus knot, a connected sum, nor a satellite is hyperbolic. Since connected sums are excluded from the knot tables, and satellites tend to have large numbers of crossings, one would expect most knots in the knot tables to be hyperbolic, and the remaining few to be torus knots. This is in fact the case. With the exception of 3_1, 5_1, 7_1, 8_{19} and 9_1, which are all torus knots, all the knots up through nine crossings are hyperbolic. The surprise is that, for a given number of crossings, the knots are more or less arranged in order of increasing volume (see Table 1). (The exceptions are that the nonalternating knots—8_{19} through 8_{21} and 9_{42} through 9_{49}—have been pushed to the end of each group, and for some reason a new series of 9-crossing knots starts at 9_{35}.) This suggests that the measure of complexity originally used to order these knots within their groups is compatible with using volume as a measure of complexity. This is remarkable in that the knots were originally ordered in 1927 (or possibly earlier), 50 years before any discussion of hyperbolic volumes. On the other hand, the whole idea of grouping knots by crossing number seems to have very little to do with hyperbolic volume.

The volumes listed in Table 1 are all distinct, so volume is a complete knot invariant for

Table 1
VOLUMES OF KNOT COMPLEMENTS

knot31	torus knot
knot41	2.029883
knot51	torus knot
knot52	2.828122
knot61	3.163963
knot62	4.400833
knot63	5.693021
knot71	torus knot
knot72	3.331744
knot73	4.592126
knot74	5.137941
knot75	6.443537
knot76	7.084926
knot77	7.643375
knot81	3.427205
knot82	4.935243
knot83	5.238684
knot84	5.500486
knot85	6.997189
knot86	7.475237
knot87	7.022197
knot88	7.801341
knot89	7.588180
knot810	8.651149
knot811	8.286317
knot812	8.935857
knot813	8.531232
knot814	9.217800
knot815	9.930648
knot816	10.579022
knot817	10.985908
knot818	12.350906
knot819	torus knot
knot820	4.124903
knot821	6.783714
knot91	torus knot
knot92	3.486660
knot93	4.994856
knot94	5.556519
knot95	5.698442
knot96	7.203601
knot97	8.014861
knot98	8.192348
knot99	8.016816
knot910	8.773457
knot911	8.288589
knot912	8.836642
knot913	9.135094
knot914	8.954989
knot915	9.885499
knot916	9.883007
knot917	9.474580
knot918	10.057730
knot919	10.032547
knot920	9.644304
knot921	10.183266
knot922	10.620727
knot923	10.611348
knot924	10.833729
knot925	11.390305
knot926	10.595841
knot927	10.999981
knot928	11.563177
knot929	12.205856
knot930	11.954527
knot931	11.686312
knot932	13.099900
knot933	13.280456
knot934	14.344581
knot935	7.940579
knot936	9.884579
knot937	10.989450
knot938	12.932859
knot939	12.810310
knot940	15.018343
knot941	12.098936
knot942	4.056860
knot943	5.904086
knot944	7.406768
knot945	8.602031
knot946	4.751702
knot947	10.049958
knot948	9.531880
knot949	9.427074

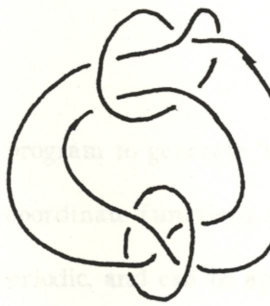
hyperbolic knots of at most 9 crossings. However, this is no longer true once knots of 10 crossings are permitted: 9_{42} and 10_{132} both have volume 4.056860..., but their complements are not homeomorphic. You can tell the complements are not homeomorphic because the set of volumes you get by doing Dehn surgery on 9_{42} is different than the set you get by doing Dehn surgery on 10_{132}. Similarly, knot 5_2 and the 12-crossing knot shown below have the same volume 2.828122... even though their complements are not homeomorphic.



5_2 (nonstandard projection)

A 12-crossing knot
with the same volume

This last pair of knots are part of an infinite family of such examples which can be obtained by surgery on the following two-component link. The Dehn surgery space (for surgeries on the unknotted component with the knotted component left complete) is shown below.



surgery on cusp #2
cusp #1 complete
number of simplices = 6 complete volume = 5.333490

5.28	5.26	5.23	5.18	5.07	4.85	4.46	4.58	4.89
5.28	5.25	5.21	5.14	4.98	4.61	4.06	4.55	4.93
5.27	5.24	5.19	5.08	4.83	4.12	3.76	4.58	4.98
5.26	5.22	5.15	4.99	4.56	2.83	3.66	4.67	5.03
5.25	5.20	5.10	4.85	4.02	0.00	3.76	4.79	5.08
5.23	5.17	5.03	4.61	[2.67]	0.00	4.06	4.91	5.13
5.21	5.13	4.92	4.12		0.00	4.46	5.01	5.16
5.19	5.08	4.75	2.83		2.83	4.75	5.08	5.19

Dan Ruberman recently found an example of two nonequivalent knots which have identical volumes at corresponding points in their Dehn surgery spaces. (Such knots are so-called 'mutants' of one another.)

Random knots

Bruce Ramsay has written a program to generate 'random' knots. The program thinks of a knot in E^3 in terms of its three coordinate functions, x(t), y(t) and z(t). Because the knot is a closed loop, these functions are periodic, and can be approximated by Fourier series. Reversing this chain of reasoning, Bruce's program generates random Fourier coefficients (from some predetermined distribution) and uses them to construct a knot. As you might expect, a projection of a random knot has a lot of unnecessary crossings. Bruce has written another program which removes unnecessary crossings, and does some additional simplification. I ran Bruce's programs in tandem, recording the number of crossings in each simplified projection. Here are the results:

number of crossings	number of occurrences
0	1448
1	0
2	0
3	433
4	154
5	161
6	135
7	116
8	72
9	50
10	35
11	25
12	21
13	10
14	11
15	8
16	4
17	5
18	2
19	1
20	1
>20	0

(My program was not prepared to interpret some supplementary messages produced by Bruce's program, and as a result a few unknots were erroneously counted as knots of higher complexity.)

I intentionally set the complexity fairly low, and the program produced lots of unknots and knots of few crossings. I saved the knots whose projections had exactly 9 crossings, and later identified them by computing their volumes. Here are the results:

number of occurrences	knot	volume
8	9-42	4.056860
6	9-44	7.406768
5	9-43	5.904086
3	9-45	8.602031
3	9-19	10.032547
3	9-18	10.057730
2	9-9	8.016816
2	9-46	4.751702
1	9-32	13.099900
1	9-23	10.611348
1	9-17	9.474580
1	9-12	8.836642
1	9-11	8.288589
1	9-5	5.698442
1	8-21	6.783714
1	6-2	4.400833

In addition, there was one knot which didn't admit a hyperbolic structure; it turned out to be the connected sum of a trefoil and knot 6_2.

These results are interesting for two reasons:

(1) These 'random' 9-crossing knots, which turned up in a batch of low complexity knots, have an average volume of 7.13..., whereas the set of all 9-crossings knots has an average volume of 9.56... . This suggests that volume is a better measure of complexity than is the number of crossings.

(2) These 'random' knots tend to be nonalternating. Mathematicians often restrict themselves to alternating knots because they are easier to work with, but it seems that if we are looking for generic behavior we should look at nonalternating knots instead.

Dehn surgery on knots

Here is a picture of the Dehn surgery space for the knot 9_{44}:

```
knot944
number of simplices = 9      complete volume = 7.406768
                                          |
  7.20    7.19    7.18    7.17    7.17    7.16    7.16    7.16    7.16
                                          |
  7.00    6.96    6.93    6.89    6.87    6.86    6.85    6.87    6.89
                                          |
  6.35    6.09    5.80    5.52    5.33    5.27    5.35    5.56    5.89
                                          |
-[ 4.84]—[ 3.10]———+————————+———————+———————+————————+———[ 3.10]—[ 4.84]-
```

(See the appendix for an explanation of the square brackets.)

This knot has the following two properties:

(1) Only the trivial surgery gives a nonhyperbolic manifold.

(2) The only other nonhyperbolic integer point is (2,0). This 'surgery' corresponds to an orbifold with an order 2 singularity along the knot.

Other knots with these two properties are 6_3, 7_5–7_7, 8_6–8_{17}, 8_{21}, 9_6–9_{33}, 9_{36}–9_{38}, 9_{43}–9_{45} and 9_{48}.

Some knots are even more restrictive. On 9_{47}, for example, the trivial surgery is the only nonhyperbolic integer point:

```
knot947
number of simplices = 11        complete volume = 10.049958
                                       |
  9.97     9.97     9.96     9.95     9.95     9.94     9.94     9.93     9.92
                                       |
  9.89     9.87     9.86     9.84     9.82     9.80     9.78     9.75     9.73
                                       |
  9.58     9.50     9.40     9.27     9.12     8.95     8.78     8.61     8.47
                                       |
-[ 6.91]—[ 4.86]—[ 0.79]———+————————+————————+————[ 0.79]—[ 4.86]—[ 6.91]-
```

The knots 8_{18}, 9_{34}, 9_{39}—9_{41} and 9_{49} also have this property. It may seem that having two nonhyperbolic points, like 9_{44}, is much more common than having only one nonhyperbolic point like the knots just mentioned. But this is only because the majority of knots of 9 or fewer crossings are algebraic in the sense of Conway (i.e. they are obtained as boundaries of unknotted twisted bands plumbed together in a tree-like pattern, see Dave Gabai's thesis [G] for a discussion). Most knots of higher complexity are not algebraic, and have hyperbolic structures at all integer point except (1,0). To demonstrate this I had Bruce's program generate ten random knots of high complexity. Their volumes are as shown below:

knot #1	70.156482
knot #2	54.470278
knot #3	54.097557
knot #4	38.183820
knot #5	83.514835
knot #6	63.659180
knot #7	66.381787
knot #8	88.412151
knot #9	74.326097
knot #10	65.231157

One of these knots (#7) was hyperbolic at all integer points except (1, 0) and (2, 0); all the rest were hyperbolic at (2, 0) as well.

The figure-eight knot is the only knot which isn't hyperbolic at (3,0) (it admits a Euclidean structure there) and all knots are hyperbolic at (n,0) when $n>3$. This follows from Bill Dunbar's work [D], which shows that the figure-eight knot is the only knot with a nonhyperbolic geometric structure at (3,0), together with Thurston's theorem that orbifolds with 1-dimensional singular sets admit geometric structures. (Knots whose complements aren't hyperbolic to begin with are, of course, excluded from the discussion.)

The very simplest knots have more nonhyperbolic integer points. For knots of 9 or fewer crossings these 'interesting' Dehn surgery spaces fall into two categories. The first category is typified by the structure of knot 6_1:

```
knot61
number of simplices = 4        complete volume = 3.163963
                           |
                           |
   2.94       2.91       2.88       2.84       2.81       2.78       2.76       2.75       2.74
                           |
                           |
   2.72       2.63       2.52       2.41       2.31       2.25       2.24       2.27       2.35
                           |
                           |
   1.94       1.40       DEC        0.00       0.00       0.00       DEC     [ 1.61]      2.13
                           |
                           |
-----+----------+----------+----------+----------+----[ 0.65]--[ 1.65]--[ 2.18]--- 2.47---
```

There are two integer points, namely (0,1) and (4,1), corresponding to manifolds which decompose along incompressible tori. These two points are the endpoints of a line of manifolds with degenerate hyperbolic structures. These manifolds are Seifert fibered spaces. This line of Seifert fibered spaces is horizontal in all examples up through 9 crossings. The Dehn surgery spaces for the other knots in this category appear in an appendix. In the case of (0,1) surgery on the figure-eight knot the degenerate hyperbolic structure turns out to be a solv-geometry structure, but this is the only such example among all surgeries on knots (this is a consequence of Dave Gabai's recent result that if a surgery on a knot gives a manifold which fibers over a circle, then the knot complement itself also fibers over a circle).

knot820
number of simplices = 5 complete volume = 4.124903

3.90	3.87	3.82	3.78	3.73	3.68	3.64	3.62	3.62
3.75	3.66	3.54	3.40	3.24	3.08	2.99	3.00	3.12
3.42	[3.12]	[2.65]	[1.91]	DEC	0.00	0.00	[1.44]	2.57
2.77	1.71	+	+	+	+	+	1.71	2.77

In some Dehn surgery spaces, such as the one for knot 8_{20} shown above, one of the endpoints of the horizontal line occurs at a noninteger point. In the case of 8_{20} it occurs at (2 2/3, 1). These endpoints seem to occur only at points with rational coordinates.

The other category of Dehn surgery space found among knots of at most 9 crossings is represented by the Dehn surgery space of knot 9_3:

knot93
number of simplices = 6 complete volume = 4.994856

4.97	4.96	4.96	4.96	4.96	4.96	4.95	4.95	4.95
4.91	4.90	4.89	4.88	4.87	4.85	4.83	4.80	4.77
4.14	3.83	[3.34]	[2.52]	DEC	[1.90]	[2.50]	2.98	3.43
4.55	4.64	4.71	4.76	4.80	4.82	4.85	4.87	4.88
8	9	10	11	12	13	14	15	16

There is only one nonhyperbolic integer point off the horizontal axis, and it represents a manifold containing an incompressible torus which decomposes the manifold into two pieces. The Dehn surgery spaces for the other knots in this category also appear in the appendix.

I invented the above two categories purely for expository purposes. For knots of more than 9 crossings they are inadequate. For example, the Dehn surgery space of the 12-crossing knot referred to earlier has nonhyperbolic points at (-16, 1) through (-20, 1) and also at (-37, 2). There seems to be some regularity to the structure of all these Dehn surgery spaces, but I'm not yet ready to conjecture what it is. The same patterns seem to appear in the Dehn surgery spaces of knots of high complexity, only there the corners of the nonhyperbolic region are all at noninteger points (presumably as the volume of the knot goes up the nonhyperbolic region gets smaller).

Table 2 shows the number of nonhyperbolic integer surgeries for each knot of 9 or fewer crossings. As it stands the table is not particularly orderly. The number of nonhyperbolic surgeries doesn't correlate well with the number of crossings, and even within a group, e.g. the 9-crossing knots, some of the knots with lots of nonhyperbolic surgeries occur near the beginning of the group while others occur near the end. In contrast, look at Table 3 and see what happens when the knots are ordered by volume. The number of nonhyperbolic surgeries correlates well with volume.

Bill Thurston has often said that crossing number is a silly knot invariant. We now have two more pieces of evidence to support this opinion:

(1) The character of the Dehn surgery space correlates well with volume, but poorly with crossing number.

(2) Bruce Ramsay's random knot program, when set for low complexity, generates knots of low volume but not necessarily low crossing number.

It would be interesting to compile knot tables ordered by volume without regard to crossing number. As well as reflecting a more natural property of the knots, this table would be more convenient to use than the standard tables: you wouldn't have to find any special projection of the knot; you could start with any projection, compute the volume, and quickly look up the knot. The major drawback of this scheme is that the set of volumes of knot complements has accumulation points. For example, the twist knots—which can be all be obtained by surgery

Table 2
Number of Nonhyperbolic Dehn Surgeries

knot31	all
knot41	12
knot51	all
knot52	7
knot61	7
knot62	3
knot63	2
knot71	all
knot72	7
knot73	3
knot74	3
knot75	2
knot76	2
knot77	2
knot81	7
knot82	3
knot83	3
knot84	3
knot85	3
knot86	2
knot87	2
knot88	2
knot89	2
knot810	2
knot811	2
knot812	2
knot813	2
knot814	2
knot815	2
knot816	2
knot817	2
knot818	1
knot819	all
knot820	5
knot821	2
knot91	all
knot92	7
knot93	3
knot94	3
knot95	3
knot96	2
knot97	2
knot98	2
knot99	2
knot910	2
knot911	2
knot912	2
knot913	2
knot914	2
knot915	2
knot916	2
knot917	2
knot918	2
knot919	2
knot920	2
knot921	2
knot922	2
knot923	2
knot924	2
knot925	2
knot926	2
knot927	2
knot928	2
knot929	2
knot930	2
knot931	2
knot932	2
knot933	2
knot934	2
knot935	3
knot936	2
knot937	2
knot938	2
knot939	1
knot940	1
knot941	1
knot942	6
knot943	2
knot944	2
knot945	2
knot946	5
knot947	1
knot948	2
knot949	1

Table 3

Number of Nonhyperbolic Dehn Surgeries

(knots ordered by volume)

knot31	torus knot	all
knot51	torus knot	all
knot71	torus knot	all
knot819	torus knot	all
knot91	torus knot	all
knot41	2.029883	12
knot52	2.828122	7
knot61	3.163963	7
knot72	3.331744	7
knot81	3.427205	7
knot92	3.486660	7
knot942	4.056860	6
knot820	4.124903	5
knot62	4.400833	3
knot73	4.592126	3
knot946	4.751702	5
knot82	4.935243	3
knot93	4.994856	3
knot74	5.137941	3
knot83	5.238684	3
knot84	5.500486	3
knot94	5.556519	3
knot63	5.693021	2
knot95	5.698442	3
knot943	5.904086	2
knot75	6.443537	2
knot821	6.783714	2
knot85	6.997189	3
knot87	7.022197	2
knot76	7.084926	2
knot96	7.203601	2
knot944	7.406768	2
knot86	7.475237	2
knot89	7.588180	2
knot77	7.643375	2
knot88	7.801341	2
knot935	7.940579	3
knot97	8.014861	2
knot99	8.016816	2
knot98	8.192348	2
knot811	8.286317	2
knot911	8.288589	2
knot813	8.531232	2
knot945	8.602031	2
knot810	8.651149	2
knot910	8.773457	2
knot912	8.836642	2
knot812	8.935857	2
knot914	8.954989	2
knot913	9.135094	2
knot814	9.217800	2
knot949	9.427074	1
knot917	9.474580	2
knot948	9.531880	2
knot920	9.644304	2
knot916	9.883007	2
knot936	9.884579	2
knot915	9.885499	2
knot815	9.930648	2
knot919	10.032547	2
knot947	10.049958	1
knot918	10.057730	2
knot921	10.183266	2
knot816	10.579022	2
knot926	10.595841	2
knot923	10.611348	2
knot922	10.620727	2
knot924	10.833729	2
knot817	10.985908	2
knot937	10.989450	2
knot927	10.999981	2
knot925	11.390305	2
knot928	11.563177	2
knot931	11.686312	2
knot930	11.954527	2
knot941	12.098936	1
knot929	12.205856	2
knot818	12.350906	1
knot939	12.810310	1
knot938	12.932859	2
knot932	13.099900	2
knot933	13.280456	2
knot934	14.344581	2
knot940	15.018343	1

on the Whitehead link–have volumes which approach the Whitehead link's volume of 3.66..., so you have infinitely many knots with volume less than 3.66... . Fortunately, this problem does not appear insurmountable: infinite classes of knots (e.g. the twists knots) can be grouped together with the two-component link which generates them (e.g. the Whitehead link). Similarly, infinite collections of two-component links could be grouped together under a three-component link, etc. A second problem is that there are different knots which have the same volume, and you'd need to find some other invariant–such as the Jones polynomial, or the volumes of various Dehn surgeries–to distinguish these. The advantage such tables would offer, though, is that they would encourage people to think about knots in a more natural–and hopefully more productive–way.

Other manifolds with cusps

It would be interesting to study all manifolds with cusps, rather than just knot and link complements. (By a 'cusp' I mean an end homeomorphic to $T^2 \times [0,1)$. One expects that in most–but not all–cases a manifold with cusps can be given a hyperbolic structure of finite volume.) Knot and link complements are atypical in that there is always a surgery which gives S^3. One method of getting at manifolds which are more typical would be to start with a link complement and do Dehn surgery on, say, all the components but one. All orientable manifolds with one cusp can be obtained in this way (the proof is an easy corollary of the theorem that all closed orientable 3-manifolds can be obtained by Dehn surgery on S^3).

This plan sounds nice in theory, but unfortunately certain computational difficulties arise when the plan is carried out. First note that if one is looking for generic behavior, one probably does not want to use the links in the standard tables. The reason is that the individual components of each link are usually unknotted. Instead one would prefer to look at links where the individual components are reasonably complex. As an example, I formed a link by interweaving a figure-eight knot with a 5_2 knot. The complement of this link has a volume of about 32.86, and its triangulation has 45 ideal simplices, so doing Dehn surgeries was computationally very slow. The next problem was that one would like to be able to, say, do a trivial surgery on the 5_2 component and be left with a figure-eight knot complement, but to do this you must reduce the volume of the manifold from 32.86 to 2.03. When you have 45 simplices, 2.03 is not a lot of volume to go around. The simplices get distorted, and many of them degenerate entirely.

A better approach to the study of typical manifolds with cusps may be to forget link complements entirely. After all, if we want to study typical manifolds, there is no need insist on starting with something which embeds in S^3. Instead, we could make manifolds by randomly assembling ideal tetrahedra and checking that the link of each ideal vertex is a torus. If we do this stupidly--completely assembling each manifold before checking the cusps--the computing time needed to generate a manifold will increase exponentially with the number of ideal tetrahedra used. But by monitoring the cusps as it goes along--and immediately backing off when the genus of a cusp gets too large--the process may run in polynomial time. (Another advantage of this plan is that the program to generate the manifolds would be noninteractive and could run at night. The interactive program SNAPPEA--the one with which the user explores the Dehn surgery space--would be expected to run as quickly as it does now.)