TIGHTENING ALMOST NORMAL SURFACES

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ABSTRACT. We present a specialized version of Haken's normalization procedure. Our main theorem states that there is a compression body canonically associated to a given transversely oriented almost normal surface. Several applications are given.

1. Introduction

Haken's normalization procedure [5] takes an arbitrary surface S inside a triangulated three-manifold and, after a sequence of isotopies and compressions, produces a *normal surface* — a surface which is essentially rectilinear with respect to the triangulation. This process is key in many of the algorithmic results in three-manifold topology such as; deciding if a surface is incompressible [4] and deciding if a manifold is irreducible [16], atoroidal [12], or Haken [9].

However, a certain amount of information is lost in this process. For example, we cannot expect S and its normalization to cobound a product region or even a compression body or, indeed, to be disjoint. This is not a problem while considering incompressible surfaces, but is a serious obstacle to dealing with Heegaard splittings.

If we restrict our attention to *almost normal surfaces* and keep careful track of the normalization isotopies then these complexities are removed. Theorem 3.3 shows that our procedure yields a compression body. The compression body has S as one boundary component while the other boundaries form the normal surface which is "as close as possible" to S. This is made precise in Section 3.

As a first application we obtain a new proof of Lemma 1 from [15] which is restated here as Lemma 3.4. Stocking's lemma states that any almost normal surface which is incompressible to one side is in fact isotopic to a normal surface on that side. This lemma gives the induction step in Stocking's proof that all strongly irreducible Heegaard splittings are isotopic to a normal or almost normal surface.

Our second application, Theorem 4.1, is a description of the one-skeleton of efficiently triangulated three-spheres. In *Thin position and*

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bridge number for knots in the 3-sphere [17], Thompson proved that a knot which does not contain a planar meridional incompressible surface in its complement has the property that any thin position realizes bridge position for the knot. Bachman [1] and Heath and Kobayashi [6] have made further progress along these lines.

Theorem 4.1 states that, in any zero-efficient, one-vertex triangulation of S^3, thin position of the one-skeleton is also bridge position. This gives an algorithm to find the bridge number of such one-skeleta. Note that Thompson's proof in [17] does not apply directly in this context. This is due to the presence of normal spheres near the vertices of the triangulation.

Theorem 3.3 has further uses. For example, it gives an algorithm to decide if a closed three-manifold is homeomorphic to a surface bundle over the circle. See the author's thesis [14] as well as [**?**]. In fact, the methods of this paper may be used to show that the recognition problem for S^3, as well as the recognition of surface bundles, lies in the complexity class **NP**. We plan on developing these ideas in a future paper. Also, suitably generalized, our technology has applications to the study of *distances* of Heegaard splittings, as defined by Hempel [7]. (Again, see [14].)

Section 2 provides a brief synopsis of normal surface theory. Section 3 gives the definitions necessary to precisely state our main theorem and quickly proves Stocking's lemma as a corollary. Section 4 is devoted to defining the concepts of thin and bridge position for one-skeleta and stating our second application. The tightening map is defined in Section 5. A collection of lemmata in Section 6 carefully analyzes the intersection of this isotopy with the two-skeleton. Finally, Section 7 proves the main theorem and Theorem 4.1.

I would like to thank Andrew Casson for his invaluable advice and Danny Calegari for reading an early version of this paper.

2. Surfaces, in pieces

Throughout this paper M^3 will denote a compact three-manifold. This section develops a few rudiments of normal surface theory. For a more complete treatment see [16] or [13].

2.1. **Normal surfaces.** Fix T, a triangulation of M. This triangulation is not assumed to be simplical. Let t be a tetrahedron of T and let f be a triangular face of t.

A *normal arc* of f is an arc, properly embedded in f, with its endpoints in distinct edges of f. A *normal curve* in ∂t is a simple closed

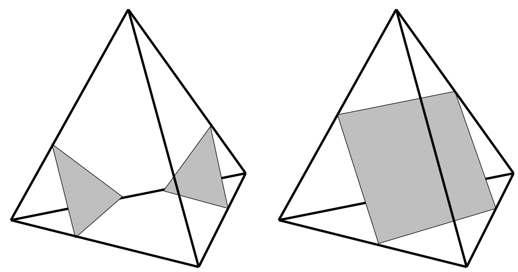

Figure 1. Normal disks

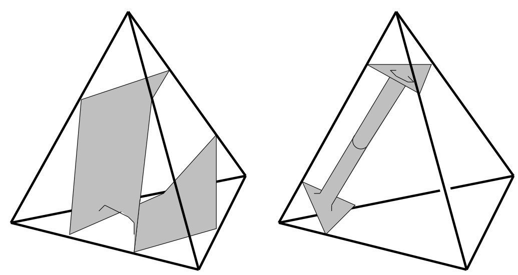

Figure 2. Almost normal pieces

curve, embedded in ∂t, which is transverse to the edges of t and whose intersection with each face of t is a collection of normal arcs.

The length of a normal curve is the number of normal arcs it contains. A normal curve is *short* if its length is four or less. Otherwise it is *long*.

Each normal curve bounds a disk in t. Such disks with boundary of length three are *normal triangles* and those with boundary of length four are *normal quadrilaterals* or "normal quads." These normal disks are illustrated in Figure 1. A surface S properly embedded in M is *normal* if it intersects each tetrahedron in a collection of normal disks.

2.2. **Almost normal surfaces.** In order to capture certain behaviors it is necessary to expand the allowable intersections of S with a single tetrahedron beyond normal disks. The *almost normal pieces* shown in Figure 2 are one of three *almost normal octagons* and one of twenty-five *almost normal annuli*. The tubes of the almost normal annuli are required to be unknotted. (See [13].)

A surface properly embedded in M is *almost normal* if it intersects all tetrahedra but one in normal disks, and it meets the exceptional tetrahedron in exactly one almost normal piece and possibly some normal disks.

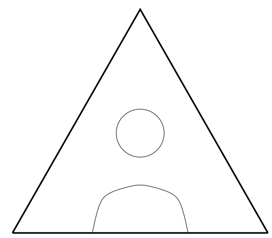

FIGURE 3. Non-normal curves

2.3. **Non-normal surfaces.** Let S be a surface properly embedded in a triangulated three-manifold M. Assuming that S is transverse to the skeleta of M we will characterize some of the ways S can fail to be normal. Denote the i-skeleton of (M, T) by T^i.

Definition. The *weight*, $w(S)$, of a surface $S \subset M$ is the number of points in $S \cap T^1$.

A *simple curve* of S in a triangle $f \in T^2$ is a properly embedded closed curve in interior(f), the interior of f. Also, a *bent arc* is a properly embedded arc with both of its endpoints contained in a single edge of f. Both of these are drawn in Figure 3.

Definition. An embedded disk D is a *surgery disk* for S if $D \cap S = \partial D$, $D \subset T^2$ or $D \cap T^2 = \emptyset$, and $D \cap T^1 = \emptyset$.

We may *surger* S along D: Remove a small neighborhood of ∂D from S and cap off the boundaries thus created with disjoint, parallel copies of D. Note that we do not require ∂D to be essential in S. A simple curve of S is *innermost* if it is the boundary of a surgery disk embedded in a triangle of T^2.

Definition. An embedded disk D is a *tightening disk* for S if $\partial D = \alpha \cup \beta$ where α and β are closed intervals, $D \cap S = \alpha$, $D \subset T^2$ or $D \cap T^2 = \beta$, $D \cap T^1 = \beta$, and β does not meet T^0.

There is a *tightening isotopy* of S across D: Push α along the disk D, via ambient isotopy of S supported in a small neighborhood of D, until we have moved α past β. This procedure reduces $w(S)$ by exactly two. A bent arc of S is *outermost* if it lies on the boundary of a tightening disk embedded in a triangle of T^2.

Suppose S contains an almost normal octagon. There are two tightening disks on opposite sides of the octagon both giving tightening isotopies of S to a possibly non-normal surface of lesser weight. (See [16].) These are the *exceptional tightening disks*. If S contains an almost normal annulus then the tube is parallel to at least one edge of the containing tetrahedron. For every such edge there is an exceptional tightening disk. Also, the disk which surgers the almost normal annulus will be called the *exceptional surgery disk*.

2.4. **Common notions.** Given a triangulated manifold (M, T) there is a standard notion of equivalence:

Definition. An isotopy $H : M \times I \to M$ is a *normal isotopy* if, for all $s \in I$ and for every simplex σ in T, $H_s(\sigma) = \sigma$.

Two submanifolds of M are *normally isotopic*, or simply *equivalent*, if there is a normal isotopy taking one to the other.

Let $S \subset M$ be a transversely oriented surface. In Figures 5 and 6 the transverse orientation is indicated by arrows pointing in the appropriate direction. The side of S pointed towards is *below* S while the opposite side is *above*. When S is *thickened* to obtain a product neighborhood $S \times I \to M$ then $S \times 1$ is above $S \cong S \times 0$.

3. Canonical Compression Bodies

This section defines the notions required to state Theorem 3.3 and its first corollary.

3.1. **Triangulations.** Fix (M, T), a closed triangulated three-manifold.

Definition. T is *0-efficient* or simply, *efficient* if every normal two-sphere is a *vertex link*; that is, every normal sphere contains only normal triangles. See [10].

Remark 3.1. Note that the only orientable prime three-manifolds which do not admit such triangulations are $\mathbb{RP}^3$ and $S^2 \times S^1$. This is due to Jaco and Rubinstein [10] and may also be found in unpublished work of Casson's. Jaco and Sedgwick [11] have further shown that all other lens spaces (including S^3) admit infinitely many efficient triangulations.

Remark 3.2. As shown in Lemma 3.6 if M is a closed, efficiently triangulated three-manifold with more than one vertex then M is homeomorphic to the three-sphere. Jaco and Segdwick [11] have given an argument to prove that an efficient triangulation of S^3 has at most two vertices; Ben Berton has given an infinite collection of such triangulations of S^3 [8].

3.2. **Compression bodies.** Let $S \subset M$ be a closed transversely oriented surface. A *compression body based above* S is a three-dimensional submanifold of M obtained by thickening S and then attaching thickened disks to $S \times 1$. If the disks are attached to $S \times 0$ then the compression body is based *under* S. Note that our compression bodies may have two-sphere boundary components. See [2].

Let $\mathcal{V}_S$ be the set of all compression bodies V based over S such that $\partial_- V = \partial V - (S \times 0)$ is normal. (Given our conventions, $\partial_+ V = S \times 0$.) If we only consider elements of $\mathcal{V}_S$ up to normal isotopy then there is a natural partial order on the set $\mathcal{V}_S$. Namely, $V \leq V'$ if $V \subset V'$, perhaps after a normal isotopy. Note that the product neighborhood $S \times I$ is only an element of $\mathcal{V}_S$ when S itself is normal.

Theorem 3.3. *Let $S \subset M$ be a transversely oriented almost normal surface. Then the partially ordered set $\mathcal{V}_S$ has a unique minimal element.*

This is proved in Section 7.

3.3. **Stocking's Lemma.** We can now prove:

Lemma 3.4. *Let $S \subset M$ be a two-sided, almost normal surface which is incompressible on one side.*

($*$) *Assume that M is efficiently triangulated and S is not a sphere.*

Then there is an embedding $\mathcal{F} : S \times I \to M$ with the following properties:

1. $\mathcal{F}(S \times 0) = S$.
2. $\mathcal{F}(S \times 1)$ *is normal.*
3. $\mathcal{F}(S \times I)$ *is on the incompressible side of S.*

Proof. Choose the transverse orientation on S so that it points towards the incompressible side. Let V be the unique minimal element supplied by Theorem 3.3. We have $\partial_- V = \{S'$ together with a collection of normal spheres$\}$, where S' is homeomorphic to S and, by efficiency, all of the normal spheres are vertex links. It follows that there is at most one copy of each vertex link.

We may cap off these spheres (if they occur) with regular neighborhoods of the relevant vertices. Note that these neighborhoods do not intersect S' because S' is a not a sphere and thus contains normal quads. Call the capped off compression body V'. V' admits the desired product structure by the incompressibility of S and we are done. □

Remark 3.5. The technical assumption ($*$) in the lemma may be replaced by the following: M is irreducible and S is not contained in a three-ball which is embedded in M.

At this point we can also show that most efficient triangulations have exactly one vertex.

Lemma 3.6. *Suppose that (M, T) is an efficiently triangulated closed three-manifold. If $|T^0| > 1$ then M is homeomorphic to S^3.*

Proof. As M is connected, so is T^1. Suppose that x and y are distinct vertices of T^0 which are connected by an edge, $e \in T^1$. Let S_x and

S_y be vertex links about x and y, respectively. Let t be a tetrahedron adjacent to e. Connect S_x to S_y by an unknotted tube which is parallel to e inside of t. Call the almost normal two-sphere obtained S.

By Theorem 3.3, there is a pair of canonical compression bodies, V and W, based below and above S. Each of these is homeomorphic to a three-ball, perhaps with a collection of smaller three-balls removed. The boundary of $V \cup W$ is a collection of vertex links. It follows that $M \cong S^3$. □

Remark 3.7. This gives a version of Lemma 2 in [16].

4. Products and Positions

In this section we give definitions for thin and bridge position of one-skeleta and state Theorem 4.1. Let C be a disjoint union of three-dimensional submanifolds of M, called the *cores* of M. Let $S \subset M$ be a properly embedded surface.

Definition. A *product structure* $\mathcal{F}$ on (M, C) is a homeomorphism $\mathcal{F} : S \times I \to \overline{M - C}$.

We will refer to $F_r = \mathcal{F}(S \times r)$ as a *level* of the product structure.

For the rest of this section only we assume that $M = S^3$ and that M is given with a one-vertex triangulation. Let B_V be a regular neighborhood of this vertex. Let B_W be a small ball in the interior of some tetrahedron. Let $\mathcal{F}$ be a product structure on $(M, B_V \cup B_W)$ with $\mathcal{F}(S \times 0) = \partial B_V$.

Following [16] we will adapt Gabai's notion of *thin position* (see [3]) to our situation.

Definition. $\mathcal{F}$ is *transverse* to T^1 if the following conditions hold:

(1) All but finitely many levels of $\mathcal{F}$ are transverse to T^1.
(2) Each nontransverse level F_{c_i}, $i \in \{1, \ldots, n\}$, has exactly one singular intersection with T^1.

All nontransverse intersections look like local maxima or minima. The open submanifold obtained by taking the union of all levels between a maximum immediately above and a minimum immediately below is called a *thick region*. Similarly a *thin region* is contained between a minimum above and a maximum below. See Figure 4.

The set $\{c_i \in [0, 1] \mid F_{c_i}$ is not transverse to $T^1\}$ are the *critical points*. Chose now a collection of points in the interval, $R' \subset I$, such that for each adjacent pair of critical points there is exactly one point of R' between them. Let $R = R' \cup \{0\}$.

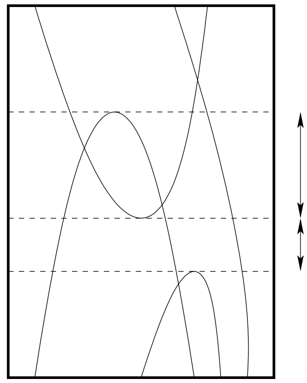

FIGURE 4. Thick and thin regions

Definition. The *width* of a transverse $\mathcal{F}$ is

$$w(\mathcal{F}) = \sum_{r \in R} w(F_r).$$

Definition. $\mathcal{F}$ realizes *thin position for* T^1 if $\mathcal{F}$ is transverse to T^1 and $w(\mathcal{F}) \leq w(\mathcal{F}')$ for all product structures $\mathcal{F}'$ on $(S^3, B_V \cup B_W)$ which are transverse to T^1.

Definition. $\mathcal{F}$ realizes *bridge position* for T^1 if $\mathcal{F}$ is transverse to T^1 and contains no thin region.

Our second corollary of Theorem 3.3 is:

Theorem 4.1. *Suppose that T is an efficient triangulation of S^3. Then any thin $\mathcal{F}$ realizes bridge position.*

This consequence is somewhat delicate. A proof is given at the end of Section 7.

5. HOW TO TIGHTEN A SURFACE

This section presents our main tool, the *tightening map.*

Suppose that S is a transversely orientable almost normal surface with respect to some triangulation of M. We wish to isotope S off of itself while reducing the weight of S as efficiently as possible. The next section closely studies the track of this isotopy.

As S has at most one exceptional surgery disk choose a transverse orientation for S which points towards an exceptional tightening disk, D.

Construct an isotopy $\mathcal{F} : S \times I \to M$ as follows:

(1) Thicken S to obtain $\mathcal{F}_0 : S \times I \to M$. Note that $\mathcal{F}_0(S \times 0) = S$. Abuse notation and set $F_0 = \mathcal{F}_0(S \times 1)$. F_0 is almost normal,

transversely oriented, and has an exceptional tightening disk, $D_0 = D - \text{image}(\mathcal{F}_0)$, which does not intersect the image of $\mathcal{F}_0$. Here image$(\mathcal{F}_0)$ is the image of $\mathcal{F}_0$.

(2) Do a small normal isotopy of F_0 in the transverse direction while tightening F_0 along D_0. This extends $\mathcal{F}_0$ to $\mathcal{F}_1$, with $F_1 = \mathcal{F}_1(S \times 1)$. F_1 inherits a transverse orientation from F_0.

(3) Let $i \in \{1, 2, 3, \ldots\}$. If F_i has no outermost bent arc with transverse orientation pointing towards a tightening disk then the construction is complete. Otherwise extend $\mathcal{F}_i$ to $\mathcal{F}_{i+1}$ by doing a small normal isotopy of F_i in the transverse direction while tightening F_i across D_i, the i^{th} tightening disk. This produces F_{i+1} together with its induced transverse orientation.

Remark 5.1. As $w(F_{i+1}) = w(F_i) - 2$ this process terminates.

Let F_n be the last surface produced.

Definition. The map $\mathcal{F}_n : S \times I \to M$ is called a *tightening map.*

By construction $\mathcal{F}_n(S \times 0) = S$. We may assume that each F_i in the construction is represented by some $t_i \in I$, i.e. $\mathcal{F}_n(S \times t_i) = F_i$ and $t_n = 1$. Thus $\mathcal{F}_i = \mathcal{F}_n|[0, t_i]$. To simplify notation, let $\mathcal{F} = \mathcal{F}_n$.

6. Tracking an isotopy

In this section we analyze how $\mathcal{F}$ intersects the skeleta of the triangulation. Let $S \subset M$, $\mathcal{F}$, $\mathcal{F}_i$, and F_i be as defined in Section 5.

Figures 5 and 6 display a few of the possible types of intersection, image$(\mathcal{F}_i) \cap T^2$, were $\mathcal{F}_i$ an embedding. Lemma 6.1 below shows that this collection is complete up to symmetry. Note that the arcs bounding the types receive a transverse orientation from the surface they lie in. Arcs of S are always pointed towards while arcs of F_i are pointed away from by the transverse orientation. The two types with a normal arc of F_i are called *critical.* Those with a bent arc are called *temporary* while the rest are called *terminal.*

During the tightening procedure, the critical types are combined in various ways while a temporary type always results in a terminal type which is stable. Note also that there is a second critical rectangle which "points upward." The non-critical types may be foliated by the levels of $\mathcal{F}_i$ in multiple ways, depending on the ordering of the tightening isotopies.

Lemma 6.1. *The maps $\mathcal{F}_i : S \times I \to M$ are embeddings. Furthermore, for every $f \in T^2$ the connected components of* image$(\mathcal{F}_i) \cap f$ *are given, up to symmetry, by Figures 5 and 6.*

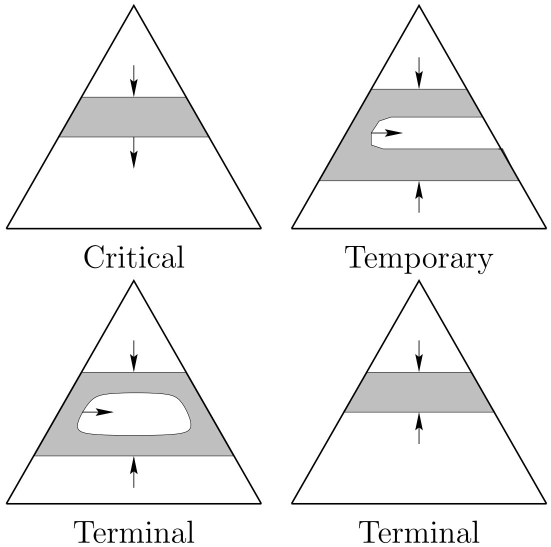

FIGURE 5. The Rectangles

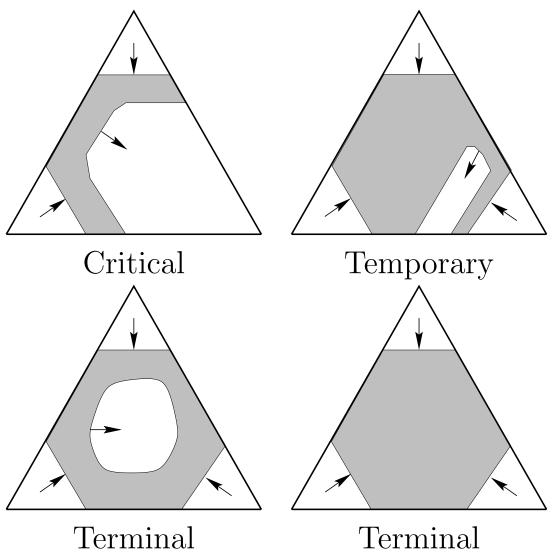

FIGURE 6. The Hexagons

Proof. Proceed by induction: Both claims hold trivially for $i = 0$, as all components of image$(\mathcal{F}_0) \cap f$ are critical rectangles.

Suppose both claims hold at $i = k$. We now verify the first claim for $i = k + 1$: Suppose that $\alpha \subset F_k$ is the bent arc on the boundary of $D_k \subset f \in T^2$, the next tightening disk in the sequence. Suppose that interior(D_k) meets image($\mathcal{F}_k$). By the second induction hypothesis there is a component, s, of $f \cap \text{image}(\mathcal{F}_k)$ which meets interior(D_k) and appears among the types listed in Figures 5 and 6. Observe that each type meets at least two edges of f while α meets only one. It follows that the interior of s must meet α. Thus $\mathcal{F}_k$ was not an embedding, a contradiction.

It follows that $D_k \cap \text{image}(\mathcal{F}_k) = \alpha$. Since the $k+1^{\text{th}}$ stage of the isotopy is supported in a small neighborhood of $F_k \cup D_k$ it follows that $\mathcal{F}_{k+1}$ is an embedding.

Now, the transverse orientation on F_k gives rise to a transverse orientation on F_{k+1}. To verify the second claim list the possible cases:

(1) Two critical rectangles may be combined to produce a temporary rectangle, a terminal rectangle with a hole, or a critical hexagon.
(2) Three critical rectangles may be combined to produce a temporary hexagon or a terminal hexagon with a hole.
(3) A critical rectangle and critical hexagon may be combined to produce a temporary hexagon or a terminal hexagon with a hole.
(4) A temporary type can lead to either terminal type.

This completes the induction step. □

Remark 6.2. By maximality of $\mathcal{F}$, the surface $F_n = \mathcal{F}(S \times 1)$ has no outermost bent arcs with outward orientation. A bent arc with inward orientation would violate the second induction hypothesis. So F_n contains no bent arcs. F_n may contain simple curves, but the second induction hypotheses shows that all of these are innermost with transverse orientation pointing toward the bounded surgery disk.

Given that $\mathcal{F}$ is an embedding, in the sequel image($\mathcal{F}_i$) is denoted by $\mathcal{F}_i$. Replacing S in Lemma 6.1 by a disjoint union of S with a collection of normal surfaces gives:

Corollary 6.3. *If S' is any normal surface in M which does not intersect S then $\mathcal{F} \cap S' = \emptyset$, perhaps after a normal isotopy.*

Note that $\mathcal{F}$ naturally imposes a product structure on $(M, \overline{M - \mathcal{F}})$. This allows us to examine the intersection of $\mathcal{F}$ with the one-skeleton.

Lemma 6.4. *$T^1 \cap \mathcal{F}$ meets the nontransverse levels of $\mathcal{F}$ only in maxima.*

Proof. Lemma 6.1 shows that $\mathcal{F}$ gives a foliation of image($\mathcal{F}$) $\subset M$. Every bent arc of every F_i is outermost and has a transverse orientation pointing toward its tightening disk. It follows that all nontransverse levels occur at maxima of the one-skeleton with respect to $\mathcal{F}$. □

Let t be any tetrahedron in the given triangulation of M.

Lemma 6.5. *For all i, $t - \mathcal{F}_i$ is a disjoint collection of balls.*

Proof. Again we use induction. Our induction hypothesis is as follows: $t - \mathcal{F}_i$ is a disjoint collection of balls, unless $i = 0$ and t contains the almost normal annulus of S.

Let B be a component of $t - \mathcal{F}_k$. There are two cases to consider. Either B is cut by an exceptional tightening disk or it is not. Assume the latter. After the $k + 1^{\text{th}}$ stage of the isotopy $B \cap \mathcal{F}_{k+1}$ is a regular neighborhood (in B) of a collection of disjoint arcs and disks in ∂B. Hence $B - \mathcal{F}_{k+1}$ is still a ball.

If B is adjacent to the almost normal piece of F_0 then let D_0 be the exceptional tightening disk. Set $B_\epsilon = B - D_0$. Each component of B_ϵ is a ball, and the argument of the above paragraph shows that they persist in the complement of $\mathcal{F}_1$. □

A similar induction argument proves:

Lemma 6.6. *For all i, $t \cap \mathcal{F}_i$ is a disjoint collection of handlebodies.*

This lemma is not used in what follows and its proof is accordingly left to the interested reader. Recall that $\partial\mathcal{F}_i = S \cup F_i$. A trivial corollary of Lemma 6.5 is:

Corollary 6.7. *For all i, the connected components of $t \cap F_i$ are planar.*

The connected components of $t \cap F_n$ warrant closer attention:

Lemma 6.8. *Each component of $t \cap F_n$ has at most one normal curve boundary component. This normal curve must be short.*

Proof. Let $t \in T^3$ be a tetrahedron. Let P be a connected component of $t \cap F_n$. By Lemma 6.1 ∂P is a collection of simple curves and normal curves. Let α be any normal curve in ∂P. Let $\{\alpha_j\}$ be the normal arcs of α.

Claim 6.9. α has length 3 or 4.

Each α_j lies on a critical rectangle or hexagon lying in ∂t. If no α_j is on a hexagon, then α is normally isotopic to a normal curve $\beta \subset S$. The first step of the tightening procedure prevents β from being a boundary of the almost normal piece of S. It follows that α must be short.

Otherwise α_1, say, is on the boundary of a critical hexagon $h \subset f$. Let β be a normal curve of S incident on h and let $\beta_1 \subset \beta$ be one of the normal arcs in ∂h. Let e be the edge of f which α_1 does not meet. This edge is partitioned into three pieces; $e_h \subset h$, e', and e''. We may assume that β_1 separates e_h from e'. Note that a normal curve of length ≤ 8 has no parallel normal arcs in a single face. Thus β meets e'

exactly once, at a endpoint of e'. Since α and β do not cross it follows that β separates α from e' in ∂t.

Similarly, α is separated from e''. Thus α does not meet e at all. It follows that α is short. (Assign to each normal arc type in ∂t a variable, v_i. This counts the number of times the i^{th} arc type is present in α. Deduce an equation of the form $v_i + v_j = v_k + v_l$ for each of the six edges of the tetrahedron t. Since α misses e four of the v_i's must be zero. A bit of linear algebra gives the desired conclusion.)

Claim 6.10. *P, the component of $F_n \cap t$ in question, has at most one boundary component which is a normal curve.*

Suppose that P has two such: α and β. Let A be the annulus cobounded by α and β in ∂t. Suppose the transverse orientation F_n induces on α points away from A. There are several cases, depending on the length of α and the types of skeletal faces to which α is adjacent.

(1) Suppose α has length three:
 (a) If α meets only critical rectangles then a normal triangle of S separates α and β.
 (b) If α meets one critical hexagon then the almost normal octagon and the exceptional tightening disk together separate α and β.
 (c) If α meets two critical hexagons then either a normal triangle or normal quad of S separates α and β.
 (d) If α meets only critical hexagons then a normal triangle of S separates α and β.

(2) Suppose α has length four:
 (a) If α meets only critical rectangles then a normal quad of S separates α and β.
 (b) If α meets one critical hexagon then S could not have been an almost normal surface.
 (c) If α meets two critical hexagons then a normal triangle of S separates α and β.

In all cases except 1(b) and 2(b), observe that $S \cap P \neq \emptyset$ and thus $S \cap F_n \neq \emptyset$. This contradicts the fact that $\mathcal{F}$ is an embedding (Lemma 6.1.) In case 1(b), P must intersect either S or the exceptional tightening disk whereas in case 2(b) S could not have been almost normal. Both are impossible.

We deduce that the transverse orientation which F_n gives α must point toward A. Let γ be an arc which runs along P from α to β. Let α' be a push-off of α along A, towards β. This push-off bounds a disk in one of the components of $t - \mathcal{F}$, by Lemma 6.5. This disk does not intersect $P \subset F_n \subset \mathcal{F}$ and hence fails to intersect γ. This is a contradiction. □

Remark 6.11. By Lemma 6.1 all simple curves of F_i are innermost. It follows that the "tubes" analyzed in Lemma 6.8 do not run through each other. Furthermore, Lemma 6.6 implies that these tubes are unknotted, but this fact is not needed in the sequel.

We have a corollary which is easy to deduce from Lemma 6.5, Lemma 6.8, and Corollary 6.7:

Corollary 6.12. *The surface obtained by surgering all simple curves of F_n is a disjoint collection of two-spheres and normal surfaces. The former are disjoint from T^2*

7. Proof of the Main Theorem

This section gives proofs for the main theorem (Theorem 3.3) and Theorem 4.1.

Proof. (of Theorem 3.3) Suppose that the surgery disk of the almost normal annulus is above S. Form V by thickening S and attaching a thickened copy of the exceptional surgery disk. If V' is another compression body in $\mathcal{V}_S$ then $\partial_- V'$ is disjoint from S and hence disjoint from $\partial_+ V$. Thus V is the desired minimal element.

Suppose that there is an exceptional tightening disk above S. Form the tightening map $\mathcal{F}$. Surger F_n along all of its simple curves using thickened surgery disks. By Lemma 6.1 and Lemma 6.8, this cuts F_n into surfaces which are either normal or contained in a single tetrahedron. The latter are all spheres by Corollary 6.7 so we may cap them off with balls.

By Corollary 6.3 the compression body thus obtained is the desired minimal element. □

Let S be a separating almost normal surface containing an almost normal octagon. There are two exceptional tightening disks, one above and one below S. These allow us to construct a pair of product structures, $\mathcal{F}_+$ and $\mathcal{F}_-$. These intersect only at S by Lemma 6.1 and Corollary 6.3. Let $\mathcal{F}_S = \mathcal{F}_+ \cup \mathcal{F}_-$.

Lemma 7.1. *Let T be an efficient, one-vertex triangulation of M. Suppose that M contains S, an almost normal two-sphere with exceptional piece an octagon. Then $M = S^3$ and $\mathcal{F}_S$ is isotopic (rel T^1) to a product structure realizing bridge position for T^1.*

Proof. By Theorem 3.3, form the minimal compression bodies below and above S. Call these V and W respectively. By construction V and

W are homeomorphic to three-balls with a collection of disjoint open three-balls removed from their interior. The boundary components of V and W (not equal to S) are normal two-spheres. Because T is efficient and has one vertex deduce that $\partial V = S \cup \partial B_V$, where B_V is a regular neighborhood of the vertex, and $\partial W = S$. Thus $M = B_V \cup V \cup W$ and $M \cong S^3$.

Remove the last ball which was added to W during W's construction. Now both V and W are homeomorphic to $S^2 \times I$ and both are alterations of $\mathcal{F}_\pm$ only off of a regular neighborhood of the one-skeleton. It follows that $\mathcal{F}_S$ is isotopic to $V \cup W$ via an isotopy fixing T^1. By Lemma 6.4, T^1 must be in bridge position with respect to $\mathcal{F}_S$ and we are done. □

Proof. (of Theorem 4.1) Suppose that T is an efficient, one-vertex triangulation of S^3. Let $\mathcal{F}$ be a product structure on $(S^3, B_V \cup B_W)$ where B_V is a regular neighborhood of the unique vertex and B_W is a small ball inside some tetrahedron. Assume that $\mathcal{F}$ realizes thin position of $T^1 \cap (M - (B_V \cup B_W))$. Suppose there exists a minimum at level $F_b, b \in I$ immediately above a maximum at level $F_a, a \in I$. That is, $\mathcal{F}|[a,b]$ is a thin region.

By Claims 4.1 – 4.5 of Thompson's paper [16] there is a level F in the lowest thick region of $\mathcal{F}$ which contains (after surgering F along simple curves and possibly other surgery disks each contained in the interior of a tetrahedron) a connected component which is an almost normal sphere S. Since none of the surgery disks meet the one-skeleton we have $w(S) \leq w(F)$.

Now we must estimate the width of $\mathcal{F}$. Suppose that the number of edges in the one-skeleton is k. The weight of $F_0 = \partial B_V$ is $2k$. Recalling that there is a minimum above F, we have:

$$w(\mathcal{F}) \geq (w(F) + (w(F) - 2) + \ldots + 2k) + ((w(F) - 2) + \ldots + 2) + 4$$

The $+4$ in the above sum is contributed when the weight increases at the critical level F_b.

Now consider $\mathcal{F}_S$. By Lemma 7.1, $\mathcal{F}_S$ can be isotoped (rel T^1) until it realizes bridge position for the one-skeleton. The width of $\mathcal{F}_S$ is:

$$w(\mathcal{F}_S) = (w(S) + (w(S) - 2) + \ldots + 2k) + ((w(S) - 2) + \ldots + 2)$$

Recalling that $w(S) \leq w(F)$ this gives a contradiction to the assumed thinness of $\mathcal{F}$. □

Remark 7.2. It follows from this that there is an algorithm to compute the bridge number of the one-skeleton of a one-vertex, efficient triangulation T of S^3. By the Theorem 4.1, minimal bridge position is thin. Every such may be isotoped relative to T^1 so that the thick region contains an almost normal S^2. By Lemma 5 of Thompson's paper [16], all almost normal (with octagon) two-spheres of least weight are fundamental.

Thus to find the bridge number of the one-skeleton we need only list the fundamental almost normal two-spheres and pick one with smallest weight. One-half of this weight is the bridge number.

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