Heegaard splittings of the form $H + nK$

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Suppose that a three-manifold $M$ contains infinitely many distinct strongly irreducible Heegaard splittings $H + nK$, obtained by Haken summing the surface $H$ with $n$ copies of the surface $K$. We show that $K$ is incompressible. All known examples, of manifolds containing infinitely many irreducible Heegaard splittings, are of this form. We also give new examples of such manifolds.

1. Introduction.

F. Waldhausen, in his 1978 paper [19], asked if every closed orientable three-manifold contains only finitely many unstabilized Heegaard splittings. A. Casson and C. Gordon (see [1] or [11]), using a result of R. Parris [13], obtain a definitive “no” answer; they obtain examples of closed hyperbolic three-manifolds each of which contains strongly irreducible splittings of arbitrarily large genus. These examples have been studied and generalized by T. Kobayashi [5], [6], M. Lustig and Y. Moriah [10], E. Sedgwick [17], and K. Hartshorn [3].

The goal of this paper is three-fold. We first show, in Section 3, that all of the examples studied so far are of the form $H + nK$: There is a pair of surfaces $H$ and $K$ in the manifold so that the strongly irreducible splittings are obtained via a cut-and-paste construction, Haken sum, of $H$ with $n$ copies of $K$. See Section 2 for a precise definition of Haken sum.

Next, and of more interest, we show when such a sequence exists the surface $K$ must be incompressible (in Sections 5 through 6). We claim:

**Theorem 1.1.** Suppose $M$ is a closed, orientable three-manifold and $H$ and $K$ are closed orientable transverse surfaces in $M$. Suppose that a Haken sum $H + K$ is given so that, for arbitrarily large values of $n$, the surfaces $H + nK$
are pairwise non-isotopic strongly irreducible Heegaard splittings. Then the surface \( K \) is incompressible.

**Theorem 1.2.** shows that all of the counter-examples to Waldhausen’s question found thus far are Haken manifolds. This was already known but required somewhat subtle techniques (see Lemmas 3.2 and 3.3 and Theorem 4.9 of Y.-Q. Wu’s paper [20]).

**Theorem 1.3.** was originally conjectured by Sedgwick along with the much stronger:

**Conjecture 1.4.** Let \( M \) be a closed, orientable 3-manifold which contains infinitely many irreducible Heegaard splittings that are pairwise non-isotopic. Then \( M \) is Haken\(^4\).

We also produce new counter-examples, which are quite different from those previously studied. These examples are discussed in Section 7. The paper concludes in Section 8 by listing several conjectures.

### 2. Preliminaries.

Fix \( M \), a closed, orientable three-manifold. If \( X \) is a submanifold of \( M \) we denote an open regular neighborhood of \( X \) by \( \eta(X) \).

A surface \( K \) is incompressible in \( M \) if \( K \) is embedded, orientable, closed, not a two-sphere, and a simple closed curve \( \gamma \subset K \) bounds an embedded disk in \( M \) if and only if \( \gamma \) bounds a disk in \( K \). The three-manifold \( M \) is irreducible if every embedded two-sphere bounds a three-ball in \( M \). If \( M \) is irreducible and contains an incompressible surface, then \( M \) is a Haken manifold.

A surface \( H \) is a Heegaard splitting for \( M \) if \( H \) is embedded, connected, and separates \( M \) into a pair of handlebodies, say \( V \) and \( W \). A disk \( D \) properly embedded in a handlebody \( V \) is essential if \( \partial D \subset \partial V \) is not null-homotopic in \( \partial V \).

**Definition 2.1.** A Heegaard splitting \( H \subset M \) is reducible if there is a pair of essential disks \( D \subset V \) and \( E \subset W \) with \( \partial D = \partial E \). If \( H \) is not reducible, it is irreducible.

\(^4\)After our paper was submitted this conjecture, and the other conjectures in Section 8, were claimed by T. Li. See [8] and [9].
Definition 2.2. A Heegaard splitting $H \subset M$ is \textit{weakly reducible} if there is a pair of essential disks $D \subset V$ and $E \subset W$ with $\partial D \cap \partial E = \emptyset$. (See [2].) If $H$ is not weakly reducible it is \textit{strongly irreducible}.

One reason to study strongly irreducible Heegaard splittings is that these surfaces have many of the properties of incompressible surfaces. An important example of this is:

Lemma 2.3 (Scharlemann’s No Nesting Lemma [15]). Suppose that $H \subset M$ is a strongly irreducible Heegaard splitting. Suppose $\gamma \subset H$ bounds a disk $D$ that is embedded in $M$ and transverse to $H$. Then, $\gamma$ bounds a disk in either $V$ or $W$. \hfill $\Box$

We now turn from Heegaard splittings to the concept of the \textit{Haken sum} of a pair of surfaces. See Figure 1 for an illustration.

Suppose $F, G \subset M$ are a pair of closed, orientable, embedded, transverse surfaces. Assume that $\Gamma = F \cap G$ is non-empty. Note that, for every $\gamma \in \Gamma$, the open regular neighborhood $T(\gamma) = \eta(\gamma)$ is an open solid torus in $M$. Note that $\partial T(\gamma) \setminus (F \cup G)$ is a union of four open annuli $A_1(\gamma) \cup A_2(\gamma) \cup A_3(\gamma) \cup A_4(\gamma)$, ordered cyclically. We collect these into two opposite pairs; $A_+(\gamma) = A_1 \cup A_3$ and $A_-(\gamma) = A_2 \cup A_4$. For every $\gamma \in \Gamma$, choose an $\epsilon(\gamma) \in \{+, -\}$ and form the \textit{Haken sum}:

$$F + G = (F \cup G) \setminus \left( \bigcup_\gamma T(\gamma) \right) \cup \left( \bigcup_\gamma A_{\epsilon(\gamma)}(\gamma) \right)$$
Note that the Haken sum depends heavily on our choices of $\epsilon(\gamma)$. As a bit of notation, we call the core curves of the annuli $A_\epsilon$ the *seams* of the Haken sum. Also there is an obvious generalization of Haken sum to properly embedded surfaces.

**Remark 2.4.** If $F$ and $G$ are compatible normal surfaces, carried by a single branched surface, or transversely oriented, there is a natural choice for the function $\epsilon(\gamma)$.

We now define the Haken sum $F + nG$: Take $n$ parallel copies of $G$ in $\eta(G)$ and number these $\{G_i\}^n$. For every curve $\gamma \in \Gamma$, we now have $n$ curves $\{\gamma_i \subset F \cap G_i\}^n$. A Haken sum $F + G$ is determined by labelings $A_\pm(\gamma)$ and choices $\epsilon(\gamma) \in \{+, -\}$. Using the parallelism of the $G_i$, we take identical labelings for $A_\pm(\gamma_i)$ and make identical choices for $\epsilon(\gamma_i)$. See Figure 6 for a cross-sectional view at $\gamma$.

The surface $F + nG$ is now the usual Haken sum of $F$ and $nG$ with these induced choices, $A_\pm(\gamma_i)$ and $\epsilon(\gamma_i)$.

### 3. Existing examples.

This section shows that the Casson–Gordon examples are of the form $H + nK$. At the end of the section, we briefly discuss the examples of Kobayashi [6], and Lustig and Moriah [10].

Let $k = k(n_1, \ldots, n_m) \subset S^3$ be a *pretzel knot* [4] with *twist boxes* of order $n_i$. Here, we choose $m$ and the $n_i$ to be odd, positive, and greater than 4. See Figure 2 for an example.

![Figure 2: The $k(5, 5, 5, 5, 5)$-pretzel knot.](image)

A pretzel knot has an associated Seifert surface, $F$. This is the compact checkerboard surface for the standard diagram. Again, see Figure 2. Let
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$B$ be the three-ball containing the pair of consecutive twist boxes of order $n_i$ and $n_{i+1}$. Let $S = \partial B$. Note that $|k \cap S| = 4$; see Figure 3. There is a well-known twisting procedure which, twists $k = k(n_1, \ldots, n_m)$ along $S$ giving

$$k_1 = k(n_1, \ldots, n_{i-1}, -1, n_i, n_{i+1}, 1, n_{i+2}, \ldots, n_m).$$

Again, see Figure 3.

Figure 3: After twisting the $k(5, 5, 5, 5, 5)$-pretzel knot, we obtain the $k(5, 5, -1, 5, 5, 1, 5)$-pretzel knot.

So, given the pretzel knot $k$ and the sphere $S$, we can produce the sequence $\{k_n\}$ of $n$-times twisted pretzels:

$$k_n = k(n_1, \ldots, n_{i-1}, \underbrace{-1, \ldots, -1}_{n}, n_i, n_{i+1}, \underbrace{1, \ldots, 1}_{n}, n_{i+2}, \ldots, n_m).$$

Denote the associated Seifert surface for $k_n$ by $F_n$. Note that $k_n$ is isotopic to $k = k_0$ and that $F_0 = F$.

In his thesis, Parris proves:

**Theorem 3.1 (Parris [13]).** The surfaces $F_n$ are free incompressible Seifert surfaces for $k$. \hfill \Box

Let $X = S^3 \setminus \eta(k_n)$. Let $\widehat{V}_n$ be a closed regular neighborhood of $F_n \cup \eta(k_n)$. So $k_n \subset \widehat{V}_n$. Let $W_n = S^3 \setminus \widehat{V}_n$. Now, as $k_n$ is isotopic into $H_n = \partial \widehat{V}_n$, doing $1/l$ Dehn surgery along $k$ makes $\widehat{V}_n$ into a handlebody, which we denote by $V_n$. Here, $l$ is any positive integer greater than 4. Let $M = X(1/l)$ be the $1/l$ Dehn surgery of $S^3$ along $k$. Let $H_n = \partial V_n = \partial W_n \subset M$. Note that the genus of $H_n$ is $2n + 4$. We have:

**Theorem 3.2 (Casson and Gordon [1], [11]).** The Heegaard splittings $H_n \subset M$ are strongly irreducible. \hfill \Box
Now, let $G$ be the surface $\partial (B \setminus \eta(k)) = (S \setminus \eta(k)) \cup (\partial \eta(k) \cap B)$. We now state the main theorem of this section:

**Theorem 3.3.** The Heegaard surfaces $H_n$ are isotopic to a Haken sum $H_0 + 2nG$.

We require several lemmas for the proof of Theorem 3.3.

**Lemma 3.4.** The surface $F_n$ is isotopic to $F_0 + nS$.

**Proof.** Let $\alpha$ and $\beta$ be the arcs of intersection between $S$ and $F = F_0$. Let $B_\alpha$ be a closed regular neighborhood of $\alpha$. Let $S_\alpha$ be the boundary of $B_\alpha$. See the left side of Figure 4 for a picture of $S \cup F$ inside $B_\alpha$.

![Figure 4: The knot $k$ has been thickened a bit. On the left, $F$ is vertical while $S$ is horizontal. The middle is their Haken sum. The right shows the isotopy of $\alpha' \cup k' \cup \alpha'' \cup k''$ to be horizontal.](image)

We choose the Haken sum which glues the top sheet of $(F \cap B_\alpha) \setminus \alpha$ to the back sheet of $(S \cap B_\alpha) \setminus \alpha$. Glue the bottom sheet of $(F \cap B_\alpha) \setminus \alpha$ to the front sheet of $(S \cap B_\alpha) \setminus \alpha$. See the center of Figure 4 for a picture of the Haken sum.

Let $\alpha'$ and $\alpha''$ be the seams along which the sheets of $F$ and $S$ are glued. Let $k'$ and $k''$ be the arcs of $k \setminus (\partial \alpha' \cup \partial \alpha'')$ inside of $B_\alpha$. Do a small isotopy of the loop $\gamma = \alpha' \cup k' \cup \alpha'' \cup k''$ as shown in Figure 4. After this isotopy, the image of $\gamma$ lies in a regular neighborhood of the curve $S_\alpha \cap S$.

We perform the same sequence of steps near $\beta$. Recall that $S_\alpha \cap S$ and $S_\beta \cap S$ cobound an annulus, $A \subset S$. Isotope the surface $F + S$ to move $k$ close to the core curve of $A$ – this isotopy is illustrated in a sequence of steps in Figure 5.
Now flatten out the right-hand side of Figure 5 by rotating the two twist boxes inside of $S$ by $180^\circ$. Also flatten the annulus into the plane containing the standard diagram of $k$. See Figure 6.

Note that the result is the Seifert surface associated to the pretzel knot $k_1 = k(5, 5, -1, 5, 5, 1, 5)$. Thus, by induction, the proof of Lemma 3.4 is complete.

\[ \square \]

Figure 5: Isotoping $F + S$, moving $k$ near the equator of $S$.

Figure 6: Flatten the resulting figure into the plane of the diagram.

Recall that $k$ is the given pretzel knot, $F = F_0$ is the associated Seifert surface, and $S$ is the two-sphere bounding the three-ball $B$, as above.

Lemma 3.5. The surface $F_n$ is isotopic to $F_0 + nG$.

Proof. Consider a single component of $\eta(k) \cap \eta(B)$. This component $B'$ is a ball. Let $k' = k \cap B'$. The disk $F' = F \cap B'$ is a boundary compression for $k'$ in $B'$. The two disks $S' \cup S'' = S \cap B'$ each intersect $k'$ in a single
point. See the left-hand side of Figure 7 for a picture. (The knot $k$ has been thickened a bit.)

The arcs $S' \cap F'$ and $S'' \cap F'$ are both part of $\alpha \subset S \cap F$. Thus, Haken summing along $S' \cap F'$ agrees with Haken summing along $S'' \cap F'$. See the right-hand side of Figure 7.

Turn now to $F + G$. Recall that $G = \partial (B \setminus \eta(k))$. Note that $G \cap \eta(k)$ is a pair of annuli. Isotope these annuli, rel boundary, slightly into $\eta(k)$ so that $G \setminus \eta(k)$ is identical to $S \setminus \eta(k)$. Thus, obtain the picture of $F \cap B'$ and $G \cap B'$, shown on the left in Figure 8.

Finally, take the Haken sum of $F'$ with $G' = G \cap B'$ as forced by our previous choices. See the right of Figure 8. Note that $F' + G'$ is isotopic to $F' + (S' \cup S'')$, rel boundary. The same holds inside the other component of $\eta(k) \cap B$. Finally, $F + S$ is identical to $F + G$ outside of $\eta(k)$. The lemma is proved. $\square$

![Figure 7: Forming the Haken sum of $F$ (longitudinal) and $S$ (meridional).](image)

![Figure 8: Forming the Haken sum of $F$ and $G$.](image)

We are now equipped to prove Theorem 3.3:
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Proof. Notice now that $H_n$ is isotopic to the boundary of a regular neighborhood of $F_n$. As $\partial F_n = k_n$, the splitting $H_n$ is obtained by gluing two parallel copies of $F_n$ with an annulus $A_n \subset \partial \eta(k_n)$, where the core curve of $A_n$ has longitudinal slope $\partial \eta(k_n) \cap F_n$. Note that $A_0$ is taken to $A_n$ by the twisting isotopy taking $k = k_0$ to $k_n$. We thus have the following:

\[
H_n = 2F_n \cup A_n 
\]

(3.1)

\[
\approx 2(F_0 + nG) \cup A_0 
\]

(3.2)

\[
= (2F_0 \cup A_0) + 2nG 
\]

(3.3)

\[
= H_0 + 2nG. 
\]

(3.4)

The second line follows from Lemma 3.5. The third line holds because $G$ has no boundary. This concludes the proof of Theorem 3.3. \qed

Remark 3.6. The examples of [6] and [10] are very similar – they begin with a knot admitting a Conway sphere $S$ and a natural Seifert surface $F$. They then isotope the knot by twisting inside $S$. Thus their examples of high genus Heegaard splittings may also be obtained via Haken sum.

4. Removing trivial curves.

Here we discuss a method for “cleaning” Haken sums. To be precise, we have:

Lemma 4.1. Suppose $H + nK$ is a sequence of Haken sums. Let $m$ be the number of curves of $H \cap K$ which are inessential on $K$. Then there is an isotopy of $H' = H + mK$ and a Haken sum $H' + K$ so that

- all curves of $H' \cap K$ are essential on $K$ and
- for all $n > m$ the surface $H + nK$ is isotopic to $H' + (n - m)K$.

Definition 4.2. We call such sequences essential in $K$.

Proof of Lemma 4.1. If $m = 0$ there is nothing to prove. If not, we claim there is a surface $\tilde{H}$ such that: $\tilde{H}$ is isotopic to $H + K$, $\tilde{H} \cap K$ has fewer inessential (on $K$) curves than $H \cap K$ does, and $\tilde{H} + (n - 1)K$ is isotopic to $H + nK$ for all $n > 0$. Applying this $m$ times will prove the lemma.
So suppose $\alpha \subset H \cap K$ is inessential on $K$. Assume that the disk $D \subset K$ bounded by $\alpha$ is *innermost*. That is, $D \cap H = \alpha$.

Let $N = \eta(K) \cong K \times [0, 1]$. We identify $K$ with $K \times \{1/2\}$. Let $D'$ be the component of $(H + K) \setminus \partial N$ containing $D$. Suppose that $\overline{D'}$ has boundary in $K \times \{1\}$. (The case $\overline{D'} \subset K \times \{0\}$ is similar.)

Isotope $D'$ up, relative to $(H + K) \cap \partial N$, to lie in $\eta(K \times \{1\})$, while isotoping all other components of $K \setminus H$ down into $\eta(K \times \{0\})$. See Figure 9.

![Figure 9](image)

Figure 9: On the left, we see $H + K$ intersecting $\eta(K)$. On the right, $H + K$ has been isotoped to $\hat{H}$.

Let $\hat{H}$ be this new position of $H + K$ and note that $\hat{H} \cap (K \times \{1/2\})$ has at least one fewer trivial curve of intersection with $K$.

We now must prove that $\hat{H} + (n - 1)K$ is isotopic to $H + nK$, for all $n > 0$. Recall that $\alpha$ was the chosen innermost curve of $H \cap K$, bounding $D \subset K$. Form $H + nK$ and isotope all subdisks parallel to $D$ up. Isotope the lowest copy of $K \setminus D$ down. This yields $\hat{H} + (n - 1)K$. (See Figure 10.) This completes the claim and thus the lemma.

![Figure 10](image)

Figure 10: $H + 3K$ is isotopic to $\hat{H} + 2K$. 

□
5. Adding surfaces of genus greater than two.

Theorem 1.3 divides into two statements. The first addresses the case \( \text{genus}(K) > 1 \) while the second deals with the case \( K \) a torus. We begin with:

**Theorem 5.1.** Suppose \( M \) is a closed, orientable three-manifold and \( H \) and \( K \) are closed orientable transverse surfaces in \( M \), with \( \text{genus}(K) \geq 2 \). Suppose that a Haken sum \( H + K \) is given so that the surface \( H + nK \) is a strongly irreducible Heegaard splitting for arbitrarily large values of \( n \). Then the surface \( K \) is incompressible.

We begin by giving a brief sketch of the proof. Aiming for a contradiction, we assume that \( K \) is compressible. Using Lemma 5.2 below, we find a compressing disk \( D \) for \( K \) with \( \partial D \) separating in \( K \).

For large \( n \), the disk \( D \) intersects \( H + nK \) in a fairly controlled way – in particular, there is a large family of parallel curves \( \{ \gamma_i \} \) in the intersection \((H + nK) \cap D\). We will show that many of the \( \{ \gamma_i \} \) are essential curves on \( H + nK \). By Scharlemann’s “No Nesting” Lemma 2.3, all of these \( \gamma_i \)'s bound disks \( D_i \) in one of the two handlebodies \( V_n \) or \( W_n \). (Here \( \partial V_n = \partial W_n \) equals \( H + nK \).) Finally, the two curves \( \gamma_i \) and \( \gamma_{i+1} \) cobound a subannulus \( A_i \subset D \). Compressing or boundary compressing \( A_i \), will give an essential disk \( E_i \) disjoint from \( D_i \). This demonstrates that \( H + nK \) is weakly reducible, a contradiction.

5.1. Finding a separating compressing disk.

We will need a simple lemma:

**Lemma 5.2.** If \( G \subset M \) is a compressible surface, which is not a torus, then there is a compressing disk \( D \subset M \) so that \( \partial D \) is a separating curve, on \( G \).

**Proof.** Let \( E \) be any compressing disk for \( G \). If \( \partial E \) is a separating curve then take \( D = E \) and we are done. So suppose instead that \( \partial E \) is non-separating in \( G \). Choose \( \gamma \subset G \) to be any simple closed curve which meets \( \partial E \) exactly once. Let \( N \) be a closed regular neighborhood of \( \gamma \cup E \), taken in \( M \). Let \( D \) be the closure of the disk component of \( \partial N \setminus G \). This is the desired disk. \( \square \)

5.2. The intersection with the compressing disk.

We now begin the proof of Theorem 5.1.
Recall that $H$ and $K$ are a pair of surfaces so that $H + nK$ is a strongly irreducible Heegaard splitting for arbitrarily large $n$. Applying Lemma 4.1, we may assume that every curve of intersection between $H$ and $K$ is essential in $K$.

In order to obtain a contradiction assume that $K$ is compressible. Use Lemma 5.2 to obtain a compressing disk $D$ for $K$, transverse to $H$, where $\partial D$ is separating in $K$. We may choose $D$ to minimize the size of the intersection $|(H \cap K) \cap D|$. Denote the two components of $K \setminus \partial D$ by $K'$ and $K''$.

For any $n > 0$ such that $H + nK$ is a strongly irreducible Heegaard splitting, proceed as follows: Label the components of $nK$ as $K_1, \ldots, K_n$. Isotope $nK$ so that all of the $K_i$ lie inside of $\eta(K)$, are disjoint from $K$, and meet interior($D$) in a single curve. Choose subscripts for the $K_i$ consecutively so that $K_1 \cap D$ is innermost among the curves of intersection $(\cup K_i) \cap D$. See Figure 11 for a picture of how the $K_i$ and $H$ intersect $D$.

Figure 11: A picture of $D$. The concentric circles are the curves of $K_i \cap D$. The arcs and small circles make up $H \cap D$.

Note that $H \cap D$ is a collection of arcs and simple closed curves. The arcs’ intersection with $K_i \cap D$ will give a cross-sectional view of the Haken sum of $H$ with $nK$.

Fix attention on a stack of intersections: a collection of $n$ consecutive points of intersection between an arc of $H \cap D$ and $nK$, all of which are close to a point of $H \cap \partial D$. Again, see Figure 11. Choose a transverse orientation on $D$. Assign a parity to the stack as follows: A stack is positive if, after the Haken sum, the segment of $(K_i \cap D) \setminus \eta(K_i \cap H)$ on the left is attached to the segment of $(K_{i+1} \cap D) \setminus \eta(K_{i+1} \cap H)$ on the right. Otherwise, the stack is negative. See Figure 12.
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Figure 12: In both cases, we are looking at $D$ in the direction of the transverse orientation.

**Claim 5.3.** The number of positive stacks equals the number of negative stacks.

**Proof.** Recall $\partial D$ separates $K$ into two pieces, $K'$ and $K''$. So every component of $H \cap K'$ is either a simple closed curve, disjoint from $\partial D$, or is a properly embedded arc. Pick one of these arcs, say $\alpha \subset H \cap K'$. Note the endpoints of $\alpha$ lie in $\partial D$ and give rise to stacks of opposite parity. \hfill \Box

Figure 13.

Next, analyze how the intersection $(H + nK) \cap D$ lies in $D$: As in Figure 13, fix any point $x \in (\partial D \setminus H)$. Let $x_i$ be the corresponding point of $K_i \cap D$.

An arc of $(K_i \cap D) \setminus \eta(H \cap nK)$ is a horizontal arc at level $i$. In particular, the arc containing $x_i$ is at level $i$. Orient these arcs in a clockwise fashion. Note that horizontal arcs are also subarcs of $(H + nK) \cap D$. When a horizontal arc at level $i$ enters a positive stack, it ascends and when it enters a negative stack it descends a single level.
Consider now an arc of \((H \cap D) \setminus \eta(H \cap nK)\). These are the vertical arcs. If a vertical arc meets \(\partial D\), call it an external arc. If a vertical arc is contained in the subdisk of \(D\) bounded by \(K \cap D\) call it an internal arc. See Figure 11.

Suppose the component of \((H + nK) \cap D\) which contains \(x_i\) does not contain any internal or external vertical arcs. Then, call that component \(\gamma_i\). For each value of \(i\) where the property above does not hold, \(\gamma_i\) is left undefined.

Set
\[c_1 = |H \cap \partial D|.\] (5.1)

Note that \(c_1\) is even.

Claim 5.4. The collection \((H + nK) \cap D\) consists of

- exactly \(c_1/2\) arcs,
- the curves \(\{\gamma_i\}\), and
- at most another \(|H \cap D|\) simple closed curves.

Furthermore, each \(\gamma_i\) is a simple closed curve. Also \(|\{\gamma_i\}| \geq n - c_1\). Finally, \(\gamma_i\) and \(\gamma_{i+1}\) cobound an annulus component \(A_i\) of \(D \setminus (H + nK)\).

The claim follows from Figure 13. For completeness, a proof is included.

Proof of Claim 5.4.. The first statement in the claim is trivial: \(H \cap \partial D\) and \((H + nK) \cap \partial D\) are the same set of points. Next, count the \(\gamma_i\)'s: Choose any \(i\) with \(c_1/2 < i < n - c_1/2\) and let \(\alpha\) be the component of \((H + nK) \cap D\) containing \(x_i\). Starting at \(x_i\), and moving along \(\alpha\) in a clockwise fashion, we ascend whenever we go through a positive stack and descend through the negative stacks. As there are \(c_1/2\) positive stacks and the same number of negative stacks \(\alpha\) contains no internal or external vertical arcs. Also \(\alpha\) goes through none of the other \(x_j\)'s. So \(\alpha\) is a simple closed curve and is labeled \(\gamma_i\).

It follows that there are at least \(n - c_1\) of the \(\gamma_i\)'s in \((H + nK) \cap D\). These are all parallel in \(D\), yielding the annuli \(\{A_i\}\). Again, see Figure 5.2.

To finish the claim, note that any simple closed curve of \((H + nK) \cap D\), which is not a \(\gamma_i\), is either a simple closed curve component of \(H \cap D\) or contains an internal vertical arc. Thus, there are at most \(|H \cap D|\) such simple closed curves.
In short, if \( n \) is sufficiently large then \((H + nK) \cap D\) cuts \( D \) into pieces and most of these pieces are the parallel annuli, \( A_i \).

### 5.3. Finding a “cover” of \( K \).

Recall that \( K \cap \partial D = K' \amalg K'' \). Let \( \{ \alpha'_j \} = H \cap K' \). Similarly, let \( \{ \alpha''_j \} = H \cap K'' \). Due to the minimality assumptions (see the beginning of Section 5.2) every loop of \( H \cap K \) is essential in \( K \) and every arc \( \alpha'_j \subset K' \) and \( \alpha''_j \subset K'' \) is also essential.

Choose a collection of oriented arcs \( \{ \beta'_j \} \) with the following properties:

- Every arc \( \beta'_j \) is simple and is embedded in \( K' \).
- Both endpoints of \( \beta'_j \) are at the point \( x \).
- The interiors of the \( \beta'_j \) are disjoint.
- The union of the \( \beta'_j \), together with \( \partial D \), forms a one-vertex triangulation of \( K' \).
- The chosen arcs \( \{ \beta'_j \} \) minimize the quantity \(|(\bigcup_j \alpha'_j) \cap (\bigcup_j \beta'_j)|\).

Similarly, choose a collection of arcs \( \{ \beta''_j \} \) for \( K'' \).

Now, lift everything to a subsurface of \( H + nK \) which is “almost” a cyclic cover of \( K \): Let \( \tilde{K} = (H + nK) \cap \eta(K) \). Let \( \pi: \tilde{K} \to K \) be the natural projection map. So \( \pi \) is the composition of the homeomorphism of \( \eta(K) \cong K \times (0,1) \) with projection onto the first factor, restricted to \( \tilde{K} \subset \eta(K) \). (It is necessary to slightly tilt the vertical annuli coming from \( H \cap nK \).

This makes \( \pi \), a local homeomorphism.)

Thus \( \{ x_i \} = \pi^{-1}(x) \). As discussed above, for most values of \( i \) the curve \( \gamma_i \) is the component of \( \pi^{-1}(\partial D) \) which contains \( x_i \).

Now lift the set of curves \( \alpha', \alpha'', \beta', \beta'' \): To be precise, let \( \alpha'_{j,i} \) be the component of \( \pi^{-1}(\alpha'_j) \) which is contained in the annulus connecting \( K_i \) and \( K_{i+1} \). Define \( \alpha''_{j,i} \) similarly. See Figure 14.

Let \( \beta'_{j,i} \) be the component of \( \pi^{-1}(\beta'_j) \) which, given the orientation of \( \beta'_j \), starts at the point \( x_i \). Define \( \beta''_{j,i} \) similarly. Not every \( \beta'_{j,i} \) is useful. However, letting

\[
c_2 = \max_k \left\{ \left| \left( \bigcup_j \alpha'_j \right) \cap \beta'_k \right|, \left| \left( \bigcup_j \alpha''_j \right) \cap \beta''_k \right| \right\}
\]

(5.2)
we have:

**Claim 5.5.** For all \( j \) and for all \( i \) with \( c_2 < i < n - c_2 \), we have \( \pi(\beta'_{j,i}) = \beta'_j \). The same holds for \( \pi|\beta''_{j,i} \).

**Proof.** Every time \( \beta'_{j,i} \) crosses one of the \( \alpha'_{j,i} \)'s it goes up (or down) exactly one level. Thus, any \( \beta'_{j,i} \), with \( i \) as in the hypothesis, has both endpoints on some lift of \( x \) and the claim holds. \( \square \)

**Definition 5.6.** Suppose that \( c_2 < i < n - c_2 \). Suppose that the final point of \( \beta'_{j,i} \) is \( x_k \). By definition of \( \beta'_{j,i} \) the starting point is \( x_i \). Define the *shift* of \( \beta'_{j,i} \) to be \( \sigma(\beta'_{j,i}) = k - i \).

An important observation is:

**Claim 5.7.** The shift \( \sigma(\beta'_{j,i}) \) does not depend on the value of \( i \). \( \square \)

**Remark 5.8.** Note that \( c_2 \) is an upper bound on the absolute value of any shift \( \sigma(\beta'_{j,i}) \) or \( \sigma(\beta''_{j,i}) \).

Henceforth, we will use \( \sigma(\beta'_j) \) to denote the shift of \( \beta'_{j,i} \), for any \( i \). The same notation will be used for arcs of \( K'' \).

### 5.4. Finding essential curves and annuli.

Now to gain some control over the parallel curves \( \gamma_i \subset D \). Set

\[
c_3 = \max \{ c_1, 2c_2 \}.
\] (5.3)
**Claim 5.9.** If, for all \( j \), the shifts \( \sigma(\beta'_j) \) are zero then, for all \( i \) with \( c_2 < i < n - c_2 \), the curve \( \gamma_i \) separates \( H + nK \) into two surfaces. One of these is homeomorphic to \( K' \) (and in fact is isotopic, relative to \( \gamma_i \), to \( K'_i \)). The similar statement holds on the \( K'' \) side. \( \square \)

**Remark 5.10.** It follows immediately that there is at least one non-zero shift on at least one side. Otherwise, \( H + nK \) would be disconnected for large \( n \).

**Claim 5.11.** For all \( i \) with \( c_3 < i < n - c_3 \), the curve \( \gamma_i \) is essential in \( H + nK \).

*Proof.* Consider some curve \( \gamma_i \) with \( i \) in the indicated range.

First, suppose that all shifts on one side, say \( K' \), are zero. Take \( n > 7 \cdot c_3 \) (this lower bound is used here and in Claim 5.12 below). Recall that \( \chi(K) < 0 \) and that Euler characteristic is additive under Haken sum. Thus \( \chi(K') + 1 > \chi(H) + n\chi(K) = \chi(H + nK) \). Now, if \( \gamma_i \) is inessential then, by Claim 5.9, \( \gamma_i \) bounds a surface homeomorphic to \( K' \) on one side and bounds a disk on the other side. It would follow that \( \chi(H + nK) = \chi(K') + 1 \), a contradiction. So if all shifts on one side are zero, then \( \gamma_i \) is essential.

Now suppose that there are non-zero shifts on both sides. Reversing the orientation of some \( \beta'_j \) or \( \beta''_k \), we may assume that the shifts \( \sigma(\beta'_j) = r \) and \( \sigma(\beta''_k) = s \) are both positive. We may further assume that \( r \leq s \). If \( r = s \), take \( \delta = \beta'_j,i \cup \beta''_k,i \). If \( r < s \), take \( \delta = \beta'_j,i \cup \beta''_k,i+r-s \cup \beta'_j,i-s \cup \beta''_k,i-s \).

So \( \delta \) is a simple closed curve embedded in \( \tilde{K} \): the important point, that \( \delta \) is closed, follows from Claim 5.5 and the fact that \( c_3 \) is at least twice as large as \( c_2 \). Note that \( \delta \) meets \( \gamma_i \) exactly once at the point \( x_i \). So \( \gamma_i \) is essential. \( \square \)

Similar ideas will give some control over the annuli \( A_i \subset D \). Recall that \( \partial A_i = \gamma_i \cup \gamma_{i+1} \).

**Claim 5.12.** For all \( i \) with \( 3c_3 < i < n - 3c_3 - 1 \), the annuli \( A_i \) and \( A_{i+1} \) are not boundary parallel into \( H + nK \).

*Proof.* Suppose that \( A_i \) is boundary parallel into \( H + nK \). (The situation for \( A_{i+1} \) is similar.) Let \( B \subset H + nK \) be the annulus with which \( A_i \) cobounds a solid torus. So \( \partial B = \partial A_i = \gamma_i \cup \gamma_{i+1} \). As the other case is similar, suppose that \( B \) is adjacent to the curve \( \gamma_i \) from the \( K' \)-side.
As $B$ is not homeomorphic to $K'$, it follows from Claim 5.9 that there is a non-zero shift on the $K'$ side. Let $r = \sigma(\beta'_j)$ be the smallest non-zero shift (in absolute value) on the $K'$ side. Now the arc $\beta'_{j,i}$, by Claim 5.5, runs from $x_i$ to $x_{i+r}$. Also the interior of $\beta'_{j,i}$ does not meet any $\gamma_k$. Since $\partial B = \gamma_i \cup \gamma_{i+1}$ it follows that $\gamma_{i+r} \subset B$. Given the assumed bounds on $i$ it follows from Claim 5.11 that $\gamma_{i+r}$ is essential in $H+nK$ and thus in $B$. So $\gamma_{i+r}$ is parallel in $B$ to $\gamma_i$.

Let $B' \subset B$ be the annulus cobounded by $\gamma_i$ and $\gamma_{i+r}$. Now, $B'$ is adjacent to both $\gamma_i$ and $\gamma_{i+r}$ on the $K'$ side. Note that $r = \sigma(\beta'_{j,i}) = \sigma(\beta'_{j,i+r})$, by Claim 5.7. As above deduce from Claim 5.5 that the arc $\beta'_{j,i+r}$ runs from $x_i+r$ to $x_{i+2r}$. Also the interior of $\beta'_{j,i+r}$ does not meet any $\gamma_k$. Since $\partial B' = \gamma_i \cup \gamma_{i+r}$, it follows that $\gamma_{i+2r} \subset B'$. Given the assumed bounds on $i$, it follows from Claim 5.11 that $\gamma_{i+2r}$ is essential in $H+nK$ and thus in $B'$. So $\gamma_{i+2r}$ separates $\gamma_i$ from $\gamma_{i+r}$ in $B'$. See Figure 15. This is a contradiction, as $\beta'_{j,i}$ connects $x_i \in \gamma_i$ to $x_{i+r} \in \gamma_{i+r}$ and does not meet $\gamma_{i+2r}$.

\[\begin{align*}
  \gamma_i & \quad \text{at} \quad x_i \\
  \gamma_{i+r} & \quad \text{at} \quad x_{i+r}
\end{align*}\]

Figure 15: The curve $\gamma_{i+2r}$ cannot be a core curve for $B'$ without crossing $\beta'_{j,i}$.

5.5. Finishing the proof of the theorem.

Recall that all of the curves $\gamma_i$ bound embedded disks in the manifold because they bound disks in $D$. Thus, by Scharlemann’s “No Nesting” Lemma 2.3, all of the $\gamma_i$’s bound disks in one of the two handlebodies bounded by $H+nK$, $V_n$ or $W_n$. From strong irreducibility of $H+nK$ and Claim 5.11, it follows that all the $\gamma_i$’s bound essential disks on the same side. As the other case is identical, suppose that $\gamma_i$ bounds $D_i \subset V_n$ for all $i$. 

Now either $A_i$ or $A_{i+1}$ lies in the opposite handlebody $W_n$. As the two possibilities are symmetric, suppose $A_i \subset W_n$. There are two final cases. If $A_i$ is compressible in $W_n$, then compress to obtain two disks, say $E_i, E_{i+1} \subset W_n$. Here, $\partial E_i = \gamma_i = \partial D_i$. It follows that $H + nK$ is reducible, a contradiction.

Suppose instead that $A_i$ is incompressible. Since $A_i$ is not boundary parallel (Claim 5.12), there is a boundary compression of $A_i$ yielding an essential disk $E_i$ with $\partial E_i$ disjoint from $\partial A_i = \gamma_i \cup \gamma_{i+1}$. So $H + nK$ is weakly reducible, another contradiction. This final contradiction completes the proof of Theorem 5.1.

\[\square\]

6. Adding copies of a torus.

For the remaining part of Theorem 1.3, the surface added is a torus, $T$. Hence, we deal with sequences of strongly irreducible Heegaard splittings of the form $H + nT$.

**Theorem 6.1.** Suppose $M$ is a closed, orientable three-manifold and $H$ and $T$ are closed orientable transverse surfaces in $M$, with $T$ a two-torus. Suppose that a Haken sum $H + T$ is given so that the surface $H + nT$ is a strongly irreducible Heegaard splitting for arbitrarily large values of $n$. Assume also that no pair of these splittings are isotopic in $M$. Then, the surface $T$ is incompressible.

Assume that $T$ is compressible to obtain a contradiction. As $M$ is irreducible there are two cases: Either $T$ bounds a solid torus or $T$ bounds a cube with a knotted hole. Denote this submanifold which $T$ bounds by $X \subset M$.

Before considering these cases in detail, apply Lemma 4.1 so that $H \cap T$ consists of curves essential on $T$. These all have the same slope. Further, assign a parity to the curves of $H \cap T$ as follows: Choose any oriented curve $\alpha$ in $T$ which meets each of the components of $H \cap T$ exactly once. Then, traveling along $\alpha$ in the chosen direction, we cross the curves of $H \cap T$ and, according to the Haken sum, $H + nT$ either descends into the submanifold $X$ or ascends out of $X$. Assign the former a negative parity and the latter a positive. As the other case is similar, we assume that there are more curves of $H \cap T$ of positive parity than negative. (There cannot be equal numbers of both as then, for large values of $n$, the surface $H + nT$ fails to be connected.)

Recalling Definition 4.2 of an essential sequence we now have:
**Lemma 6.2.** Suppose the sequence \( H + nT \) is essential in \( T \). Let \( m \) be the number of positive curves of \( H \cap T \) minus the number of negative. Let \( m' = (|H \cap T| - m)/2 \). Then, there is an isotopy of \( H' = H + m'T \) so that

- all curves of \( H' \cap T \) are essential in \( T \),
- all curves of \( H' \cap T \) are positive, and
- for all \( n > m' \), the surface \( H + nT \) is isotopic to \( H' + (n - m')T \).

As the proof of Lemma 6.2 is essentially identical to that of Lemma 4.1, we omit it. An essential sequence \( H + nT \) is **reduced** if all of the curves of \( H \cap T \) have the same parity.

## 6.1. Bounding a solid torus.

Suppose now that \( T \) bounds a solid torus \( X \). We have:

**Claim 6.3.** If \( H + nT \) is reduced and \( m = |H \cap T| \) then, for any positive \( n \), the surface \( H + nT \) is isotopic in \( M \) to \( H + (n + m)T \).

**Proof.** Choose a homeomorphism \( X \cong \mathbb{D}^2 \times S^1 \), where \( \eta(T) \cap X \cong A \times S^1 \) with \( A \cong \{ z \in \mathbb{C} \mid 1/2 \leq |z| \leq 1 \} \). Set \( D_0 = \mathbb{D}_2 \setminus A \).

If the slope of \( H \cap T \) is meridional (isotopic to \( \partial \mathbb{D}^2 \times \{\text{pt}\} \)), then the desired isotopy is \( \varphi: M \times I \to M \) with \( \varphi_t(M \setminus X) = \text{Id}, \varphi_t(z, \theta) = (z, \theta \pm 2t\pi) \) for all \( z \in D_0 \), and \( \varphi_t(z, \theta) = (z, \theta \pm 2t\pi \cdot (2 - 2|z|)) \) for all \( z \in A \). Here, the sign \( \pm \) is determined by the parity of the curves \( H \cap T \). Note also that we only need to do this isotopy once, not \( m \) times.

For any other slope the desired isotopy is \( \varphi: M \times I \to M \) with \( \varphi_t(M \setminus X) = \text{Id}, \varphi_t(z, \theta) = (z \cdot \exp(\pm 2t\pi i), \theta) \) for all \( z \in D_0 \), and \( \varphi_t(z, \theta) = (z \cdot \exp(\pm 2t\pi i(2 - 2|z|)), \theta) \) for all \( z \in A \). Again, the sign \( \pm \) is determined by the parity of the curves \( H \cap T \).

Thus, when \( T \) bounds \( X \) a solid torus, the sequence \( H + nT \) contains only finitely many isotopy classes of Heegaard splittings. This is a contradiction.

## 6.2. Bounding a cube with a knotted hole.

Suppose now that the two-torus \( T \) bounds a **cube with a knotted hole**. That is, \( X \subset M \) is a submanifold contained in a three-ball \( Y \subset M \), and \( T = \partial X \)
compresses in Y but not in X. The unique slope of this compressing disk is called the meridian.

We require one more definition: A pair of transverse surfaces H and K in a three-manifold M are compression-free if all curves of $H \cap K$ are essential on both surfaces.

The main theorem of [7] is:

**Theorem 6.4.** Suppose $H \subset M$ is strongly irreducible and the two-torus $T$ bounds $X \subset M$, a cube with a knotted hole. Suppose also that $H$ and $T$ are compression-free with non-trivial intersection. Then:

- the components of $H \cap X$ are all annuli and
- there is at least one component of $H \setminus T$ which is a meridional annulus, boundary parallel into $T$.

So, choose $H$ and $T$ as provided by the hypotheses of Theorem 6.1. Suppose also, as provided by Lemmas 4.1 and 6.2, that $H + nT$ is reduced – all curves of $H \cap T$ are essential and of the same parity.

**Claim 6.5.** All curves of $H \cap T$ are meridional on $T$.

*Proof.* If $H$ and $T$ are compression-free, then apply Theorem 6.4 and we are done. If not, then there is a curve of intersection which bounds an innermost disk in $H$ and which is essential on $T$. As $T$ is not compressible into $X$, we are done. \qed

The proof of Theorem 6.1, with $X$ a cube with knotted hole, now splits into two subcases. Either $H \cap T$ is compression-free or not.

**6.2.1. The compression-free case.** Suppose that $H \cap T$ is compression-free and that $H + nT$ is a reduced sequence. We again wish to prove that infinitely many of the $H + nT$ are pairwise isotopic.

Take $nT$ to be $n$ parallel copies of $T$, all inside of $X$. Note that $H \cap T = (H + nT) \cap T$ and $H \setminus X = (H + nT) \setminus X$. Hence, $H + nT$ and $T$ are compression-free.

We repeatedly isotope $H + nT$ via the following procedure: Apply Theorem 6.4 to $H + nT$ and $T$. Thus, there is a meridional annulus $A \subset (H + nT) \setminus T$ which is boundary parallel into $T$. Let $B \subset T$ be the annulus to which $A$ is parallel. Denote by $Z$ the solid torus which $A$ and $B$ cobound.
Now, if $A \subset M \setminus X$ then $Z \cap X = B$. In this case, isotope $A$ and all components of $(H + nT) \cap Z$ into $X$. Begin the procedure again applying to this new position of $H + nT$.

If $A \subset X$ then $Z \subset X$ as well. In this case all components of $(H + nT) \cap Z$ are meridional annuli which are parallel rel boundary into $T$. Isotope $A$ and all of the annuli of $(H + nT) \cap Z$ out of $X$, but keeping them parallel to $T$. See Figure 16.

At the end of the procedure, we have isotoped $H + nT$ out of $X$. The surface $H + nT$ is thus isotopic to the surface which is a union of components of $H \setminus X$ together with a union of annuli parallel to sub-annuli of $T$. There are only finitely many of the latter (as $H \cap T$ is bounded). This is a contradiction.

6.2.2. The meridional compression case. Suppose now that $H \setminus X$ contains a meridional disk $D \subset H$ for $T$. Let $Y$ be the three-ball $X \cup \eta(D)$. Note that all the curves $\{\gamma_j\} = H \cap \partial Y$ are parallel in $\partial Y$. This is because all of the curves $(H + nT) \cap T$ are meridional for $T$. We think of $Y$ as a copy of $\mathbb{D}^2 \times I$ – “a tall tuna can” – with all of the $\gamma_j$ of the form $\partial \mathbb{D}^2 \times \{\text{pt}\}$.

For each $n$, we carry out an inductive procedure: Fix $n$. Let $Y^0 = Y$ and let $H^0 = H_n = H + nT$. At stage $i$, there is a “stack of tuna cans” $Y^i \cong \mathbb{D}^2 \times I_i \subset Y^0$, where $I_i$ is a disjoint union of finitely many closed intervals in $I$. See either side of Figure 17.

Each component of $\partial Y^i$ contains at least one of the curves $\gamma_j$. Also, the surface $H^0$ has been isotoped to a surface $H^i$ so that $H^i \setminus Y^i \subset H^{i-1} \setminus Y^{i-1} \subset H \setminus Y$. It follows that $\partial Y^i \cap H^i$ is a subset of $\cup \gamma_j$. Note that all the components of $\partial Y^i \setminus H^i$ are “vertical” annuli or disks.

Suppose some annulus component of $\partial Y^i \setminus H^i$ is compressible in $M \setminus (Y^i \cup H^i)$. So do the “packing tuna” isotopy: There is a disk $D^i$ with interior in $M \setminus (Y^i \cup H^i)$ and with boundary $\partial D^i \subset \partial Y^i$ (see left side of Fig-
Figure 17: The packing step is illustrated on the left while the slicing step is on the right. The disk $D^i$ is depicted by the dotted line.

Let $Z$ be the component of $Y^i$ containing $\partial D^i$. Then $\partial D^i$ bounds two disks in $\partial Z$, say $E$ and $E'$. Then, either $D^i \cup E$ or $D^i \cup E'$ bound a three-ball in $M$ which has interior disjoint from $Z$. (This is because $M$ is irreducible.) So there is an isotopy of $H^i$ which moves some components of $H^i \setminus Z$ into $Z$. This reduces the number of curves of intersection $H^i \cap \partial Y^i$.

Let $H^{i+1}$ be the new position of $H + nT$. Let $Y^{i+1}$ be equal to the union of all the components of $Y^i$ which meet $H^{i+1}$. The induction hypotheses clearly hold.

Suppose instead some annulus component of $\partial Y^i \setminus H^i$ is compressible in $Y^i \setminus H^i$. Next, perform the “slice a can in half” move: Let $D^i \subset Y^i$ be such a compressing disk with $\partial D^i = \mathbb{D}^2 \times \{\text{pt}\}$ and $D^i \cap H^i = \emptyset$. See right side of Figure 17. Isotope $D^i \cup H^i$ until $D^i$ is level ($D^i = \mathbb{D}^2 \times \{\text{pt}\}$) while maintaining $D^i \cap H^i = \emptyset$. This isotopy is supported inside of $Y^i$. Let $H^{i+1}$ be the new position of $H + nT$ and let $Y^{i+1} = Y^i \setminus \eta(D^i)$. Again the induction hypotheses clearly hold.

The procedure terminates after at most $|\{\gamma_j\}| = |(H + nT) \cap Y|$ steps. To see this, note that we can never have $|Y^i|$ greater than the original number of curves $\{\gamma_j\}$. So, we cannot “slice” more than that number of times. Also, the number of components of $(H + nT) \setminus Y = H \setminus Y$ is bounded and $H^i \setminus Y^i$ is contained in $H \setminus Y$. So, we cannot “pack” more than that number of times.

Let $m$ be the largest value of $i$ reached in the above procedure. After the procedure terminates, we have every component of $\partial Y^m \setminus H^m$ being incompressible in both $M \setminus (Y^m \cup H^m)$ and inside $Y^m \setminus H^m$. An innermost disk argument shows that every component of $\partial Y^m \setminus H^m$ is incompressible in $M \setminus H^m$. 
Let $Z$ be a component of $Y^m$. Recall that the curves $\gamma_j \subset \partial Z$ are parallel. Now apply Scharlemann’s Local Detection Theorem [15] (for three-balls) to $\partial Z$. It follows that $H^m \cap Z$ is either a disk or an unknotted annulus.

At the end of the procedure, the surface $H + nT$ has been isotoped to a surface which is a union of components of $H \setminus Y$ together with a union of “vertical” annuli and disks of the form $D^2 \times \{pt\}$. There are only finitely many of the latter (as $H \cap \partial Y$ is bounded). So for all $n$, the splitting $H + nT$ is isotopic to one of these finitely many surfaces, a contradiction. This completes the proof of Theorem 6.1.

\[\square\]

7. New examples.

The goal of the next two sections is to give new examples of $H, K, H + K \subset M$ such that for all integers $n$, the surface $H + nK$ is a strongly irreducible Heegaard splitting.

Note that the manifolds of Casson–Gordon have Heegaard genus four and larger. Our examples have genus as low as three. Also, our examples, unlike those of [6] and [10], do not involve twisting around a two-sphere in $S^3$ or require the existence of an incompressible spanning surface.

In the next two sections, we first (7.1) construct our new examples and then (7.2) prove that they have the desired properties.

7.1. Constructing the new examples.

To begin with, we sketch the construction, which has obvious generalizations. Take $V$ a handlebody of genus three or more. Take $\gamma$ to be a “sufficiently complicated” curve in $H = \partial V$. Double $V$ across $H$ and let $W$ be the other copy of $V$. Alter the gluing of $V$ to $W$ by Dehn twisting along $\gamma$ at least six times. This gives $M$, a closed orientable manifold. Now, we will have a properly embedded surface $K' \subset V$ with $K' \cap \gamma = \emptyset$. Thus $K'$ doubles to give a surface $K$ in $M$. Adding copies of $K$ to $H$ will give the desired sequence of Heegaard splittings.

Before giving the details, recall:

**Definition 7.1.** Let $V$ be a handlebody. A simple closed curve $\gamma \subset \partial V$ is **disk-busting** if $\partial V \setminus \gamma$ is incompressible in $V$.

For the remainder of this section, take $V'$ a handlebody of genus two. (Larger genus is also possible.) Let $\gamma' \subset \partial V'$ be a non-separating disk-busting curve. Set $K' = \partial V' \setminus \eta(\gamma')$. For an example of this, see Figure 18.
Heegaard splittings of the form $H + nK$

Take $U$, a solid torus, and fix a subdisk of the boundary $E \subset \partial U$. Let $V'' = (K' \times I) \cup U$ where $K' \times I$ is glued to $U$ via some homeomorphism between a subdisk of $K' \times \{1\}$ and the disk $E$. Thus, $E$ and any meridional disk of $U$ (which is disjoint from $E$) are essential disks in $V''$. Let $\partial_+ V'' = (\{(K' \times \{1\}) \cup \partial U\} \setminus E$. Let $\partial_- V'' = K' \times \{0\}$.

Now, choose $\gamma \subset \partial_+ V''$, a disk-busting curve for $V''$. See Figure 19, for example.

![Figure 18: The curve $\gamma'$ is disk-busting in $V'$](image)

Figure 18: The curve $\gamma'$ is disk-busting in $V'$

Form a genus three handlebody $V$ by gluing $V'$ to $V''$ via the natural map between $K' \subset \partial V'$ and $\partial_- V'' \subset \partial V''$. It is easy to check that $\gamma$ is disk-busting in $V$. As this fact is not needed in what follows, we omit the proof. However, see Figure 20 for a picture.

![Figure 19: The curve $\gamma \subset \partial_+ V''$ is disk-busting for $V''$.](image)

Figure 19: The curve $\gamma \subset \partial_+ V''$ is disk-busting for $V''$.

Now, form a manifold $D(V)$ by doubling $V$—that is, let $W$ be an identical copy of $V$ and glue these two handlebodies by the identity map between their boundaries. Finally, obtain a closed three-manifold $M$ by altering the gluing between $V$ and $W$ by Dehn twisting at least six times along $\gamma$. Again, we do not need the fact that $H$ is a strongly irreducible Heegaard splitting, nor the consequence that $M$ is irreducible.
Let \( K = D(K') \subset M \) be the double of \( K' \). As \( K' \) is connected, so is \( K \). The surface \( K \) is also incompressible in \( M \), but as this fact is not required in the sequel, we omit any direct proof.

Next, choose the Haken sum of \( H \) and \( K \): Label the two curves of \( K \cap H \) by \( \alpha \) and \( \beta \). Recall that \( \gamma' \) was chosen to be disk-busting and non-separating in \( \partial V' \). Note that \( \alpha \) and \( \beta \) cobound an annulus \( A = \eta(\gamma') = \partial V' \setminus K' \subset H \) and that \( \alpha \) and \( \beta \) cut \( K \) into two halves \( K' \subset V \) and \( K'' \subset W \). Also, \( \alpha \) and \( \beta \) cut \( H \) into two connected pieces, \( A \) and \( H \setminus A \cong \partial V'' \). Note that \( K \) and \( H \) are both separating surfaces in \( M \). For a schematic picture, see the left side of Figure 21.

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Figure 21: Picture showing (schematically) the relative positions of \( H, K \), and \( H + K \).
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So choose the Haken sum of \( H \) and \( K \) as indicated by the right side of Figure 21. To be precise, let \( \mathcal{H} : M \times I \to M \) be an ambient isotopy of \( M \) which is fixed pointwise outside of \( \eta(A) \), moves \( \alpha \) across \( A \) to \( \beta \), sends the solid torus \( \eta(\alpha) \) to \( \eta(\beta) \), takes \( K \cap \eta(\alpha) \) to \( K \cap \eta(\beta) \), and takes \( H \cap \eta(\alpha) \) to \( H \cap \eta(\beta) \). Now choose any Haken sum of \( H \) and \( K \) along \( \alpha \) and use \( \mathcal{H} \) to transfer this choice to \( \beta \). Again, see Figure 21. This defines the Haken sum \( H + K \) and thus defines \( H + nK \).
7.2. Demonstrating the desired properties.

We now can state:

**Theorem 7.2.** Given $V$ and $\gamma$ as above, the surface $H + nK$ is a strongly irreducible Heegaard splitting of $M$, for any even $n > 0$.

**Remark 7.3.** In fact $H + nK$ is a strongly irreducible Heegaard splitting for any integer $n$. We restrict $n$ to positive and even, in order to simplify the proof.

**Remark 7.4.** The curve $\gamma$ in Figure 20 does not give a hyperbolic manifold because the resulting $M$ contains a pair of Klein bottles. See Figure 22 for a more complicated curve $\gamma$. This curve does yield a hyperbolic manifold with the desired sequence of Heegaard splittings.

![Figure 22: Doubling the handlebody and twisting along the curve shown gives a hyperbolic manifold satisfying the hypotheses of Theorem 7.2](image)

The proof of Theorem 7.2 divides naturally into two pieces.

**Claim 7.5.** For positive, even $n$ the surface, $H + nK$ is a Heegaard splitting.

**Proof.** Recall that $M \setminus \eta(H \cup K)$ is homeomorphic to the disjoint union of $V'$, $V''$, $W'$, and $W''$. Also, the curves $K \cap H$ are denoted by $\alpha$ and $\beta$.

Let $nK$ be $n$ evenly spaced parallel copies of $K$ in $\eta(K)$. That $H + nK$ is connected follows from our choice of Haken sum along $\alpha$ and $\beta$. $H + nK$ is separating because $H$ and $K$ are separating. See Figure 23.

Label the closures of the two components of $M \setminus (H + nK)$ by $V_n$ and $W_n$ where $V_n$ contains $V \setminus \eta(K)$ and $W_n$ contains $W \setminus \eta(K)$. (This is where
Figure 23: Adding $nK$ to $H$ yields a connected, separating surface.

“$n$ positive and even” is used. Again, see the right half of Figure 23 for a picture with $n = 4$.

Consider now the collection of closed annuli $H \cap \text{interior}(V_n)$. Cutting $V_n$ along all of these gives several components: The first, $V'_n$, contains $V'_n \setminus \eta(K)$ while the second, $V''_n$, contains $V''_n \setminus \eta(K)$ and the rest are isotopic to $\eta(K')$ or $\eta(K'')$. Let $V_n^P$ be the submanifold of $V_n$ obtained by taking the union of all the latter (thus, not $V'_n$ or $V''_n$). Here the “$P$” in the superscript stands for “product”.

Let $A_n \cup B_n$ be the two annuli in $H \cap \text{interior}(V_n)$ which are also in $\partial V_n^P$. Here we assign labels so that $A_n$ meets the component of $H \cap \eta(K)$ which contains $\alpha$. Thus, as $n$ is even, $B_n$ meets the component of $H \cap \eta(K)$ which contains $\beta$. We have realized $V_n$ as the union of three pieces $V'_n$, $V''_n$, and $V_n^P$, glued to each other along the annuli $A_n$ and $B_n$.

Recall now that $V'_n \cong V'$, $V''_n \cong V''$, and thus both are handlebodies. Also, the annulus $B_n$ is primitive in $V''_n$. There is a disk in $V''_n$ meeting $B_n$ in a single co-core arc. See Figure 19 and notice that $B_n$ is parallel to $\beta \times I \subset \partial K' \times I \subset V''$.

Since $V_n^P$ and $V''_n$ are handlebodies it follows that $V_n^P \cup B_n V''_n$ is also a handlebody. Also, as $V_n^P$ is a product, the annulus $A_n$ is primitive on $V_n^P \cup B_n V''_n$. So, since $V'_n$ is a handlebody, we finally have $V_n = V'_n \cup A_n V_n^P \cup B_n V''_n$ is a handlebody and applying similar arguments to $W_n$ the surface $H + nK$ is a Heegaard splitting of $M$. \hfill $\square$

**Claim 7.6.** For positive, even $n$ the surface $H + nK$ is strongly irreducible.

**Proof.** Recall that $\gamma$ was a curve in $\partial_n V''$ and thus also a curve in $H + nK$. Recall that $M$ was obtained by doubling $V$ and then twisting at least six times along $\gamma$. 
We will show that \( \gamma \) is disk-busting for \( V_n \) and thus for \( W_n \). The proof of the claim will then conclude with a Theorem of Casson [11] proving that \( H + nK \) is strongly irreducible.

Choose \( D \), any essential disk in \( V_n \). Choose a hyperbolic metric on \( H + nK \). Tighten \( \partial D, \partial A_n, \partial B_n, \gamma \) to be geodesics. Perform a further isotopy of \( D \) relative to \( \partial D \) to minimize the intersection of \( D \) with \( A_n \cup B_n \).

Now note that \( A_n \) and \( B_n \) are incompressible in \( V_n \). If not, then some boundary component of \( A_n \) bounds a disk in \( V_n' \) or some boundary component of \( B_n \) bounds a disk in \( V_n'' \). (None of these curves bound disks in \( V_n' \) because neither \( K' \) nor \( K'' \) is a planar surface.) The first is impossible because \( \partial A_n \) is parallel to \( \gamma \subset V_n' \) which is disk-busting. The second is impossible because \( \partial_- V'' \) is \( \pi_1 \)-injective into \( V_n'' \).

So no component of \( D \cap (A_n \cup B_n) \) is a simple closed curve. Let \( D' \) be an outermost disk of \( D \setminus (A_n \cup B_n) \): That is, \( D' \) is the closure of a disk component of \( D \setminus (A_n \cup B_n) \) and \( D' \) meets \( A_n \cup B_n \) in at most one arc. It follows that \( D' \) is an essential disk in \( V_n' \), \( V_n'' \), or \( V_n''' \). (If not we could decrease \( |\partial D \cap (A_n \cup B_n)| \), an impossibility.)

There are three cases: \( D' \) lies in \( V_n', V_n'' \), or \( V_n''' \).

Suppose first that \( D' \subset V_n' \). If \( D' = D \) is disjoint from \( A_n \) then, as \( A_n \) is parallel to \( \gamma ' \) in \( \partial V_n' \), we may isotope \( D \) to be disjoint from \( \gamma ' \). This contradicts our choice of \( \gamma ' \) being disk-busting in \( V_n' \). If \( D' \subset D \) is a strict inclusion then \( D' \cap A_n \) is a single arc. Then, \( D' \) may be isotoped either to lie disjoint from \( \gamma ' (D' \cap A_n \) is inessential in \( A_n \)) or to meet \( \gamma ' \) in a single point (\( D' \cap A_n \) is essential in \( A_n \)). Again, this is because \( \gamma ' \) and \( A_n \) are parallel on the boundary on \( V_n \). The former contradicts \( \gamma ' \) being disk-busting. For the latter take two parallel copies of \( D' \) in \( V_n' \) and band these together along \( \gamma ' \setminus \eta(D') \) to obtain an essential disk disjoint from \( \gamma ' \). This is again a contradiction.

The next possibility is that \( D' \) lies in \( V_n'' \). However, this cannot happen as \( V_n'' \) is the trivial \( I \)-bundle over a surface.

We conclude that \( D' \) is an essential disk in \( V_n''' \). It follows that \( D' \) intersects \( \gamma \) because, \( \gamma \) was chosen to be disk-busting for \( V'' \cong V_n''' \). Thus, \( D \) has a non-trivial geometric intersection with \( \gamma \). As our choice of \( D \) is arbitrary, we conclude that \( \gamma \subset H + nK \) is disk-busting for both \( V_n \) and \( W_n \).

Note that \( D(V) \), the double of \( V \), is reducible. To obtain \( M \) from \( D(V) \), we cut open along a neighborhood of \( \gamma \) in \( \partial_+ V'' \) and Dehn twisted at least six times. It follows that \( H + nK \) gives a Heegaard splitting of \( D(V) \) and all of these are reducible in \( D(V) \). (To see this, recall that the disk \( E \) cut the solid torus \( U \) from \( V'' \). Thus, the double \( D(E) \) is a reducing sphere for all of the \( H + nK \) in \( D(V) \).) Thus, we are in a position to apply the following Theorem of Casson (see the appendix of [11]):
Theorem 7.7. Suppose \( \gamma \subset H \subset N \) is a curve on a weakly reducible Heegaard splitting surface of a closed orientable manifold \( N \), and that \( H \setminus \gamma \) is incompressible in \( N \). Cutting \( N \) open along a neighborhood of \( \gamma \) in \( H \) and Dehn-twisting at least six times gives a strongly irreducible splitting \( H' \) of the new manifold \( N' \).

It follows that for all positive, even \( n \) the splittings \( H + nK \) are strongly irreducible. We are done. \( \Box \)

Claim 7.5 and Claim 7.6 together prove Theorem 7.2. \( \Box \)

Remark 7.8. There is a well-known relationship, due to H. Rubinstein [14] and M. Stocking [18], between strongly irreducible splittings and almost normal surfaces. In particular, strongly irreducible surfaces should contain a single place (or “site” in Rubinstein’s terminology) where the curvature is highly negative. This supposedly corresponds to the almost normal octagon or annulus of the almost normal surface. In our examples, we find that the subsurface \( \partial_+V'' \) is the distinguished subsurface of \( H + nK \) which presumably contains this special site.

8. Questions.

Recall that Theorem 5.1 was originally conjectured by Sedgwick along with the much stronger:

Conjecture 1.4. Let \( M \) be a closed, orientable 3-manifold which contains infinitely many irreducible Heegaard splittings that are pairwise non-isotopic. Then, \( M \) is Haken.

This conjecture may be split, roughly, into two parts. First we have the so-called “Generalized Waldhausen Conjecture”:

Conjecture 8.1. Let \( M \) be a closed, orientable 3-manifold which contains infinitely many Heegaard splittings, pairwise non-isotopic, all of the same genus. Then \( M \) is toroidal.

Note that this has been claimed by W. Jaco and Rubinstein. However, no manuscript is available as of the writing of this paper.

The other half of Sedgwick’s conjecture deals with splittings of increasing genus and is the inspiration for our current work:

Conjecture 8.2. Let \( M \) be a closed, orientable 3-manifold which contains irreducible Heegaard splittings of arbitrarily large genus. Then \( M \) is Haken.
We now turn to questions about examples. In all of the manifolds listed above, which contain splittings of arbitrarily large genus, the three-manifold has had Heegaard genus three or higher. Kobayashi asks:

**Question 8.3.** Is there an example of a Heegaard genus two-manifold which admits strongly irreducible splittings of arbitrarily large genus?

**Remark 8.4.** Note that there are examples of toroidal manifolds containing infinitely many strongly irreducible splittings all of the form $H+nT$. Here, $H$ is a genus two Heegaard splitting and $T$ is an incompressible torus; see [12].

Sedgwick, in [17], has shown that the Casson–Gordon examples satisfy the so-called “Stabilization Conjecture [16]”. That is, for any two splittings $H$ and $H'$ obtained from the same pretzel knot, after stabilizing the higher genus splitting once, we may destabilize to find the lower genus splitting. Sedgwick’s techniques apply to all of the splittings discussed in Section 3. Kobayashi suggests that the examples of $H+nK$ given in this paper, after stabilizing twice, should destabilize about $2^n$ times.

**Question 8.5.** Does one stabilization suffice?

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**References.**


Heegaard splittings of the form $H + nK$


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