UNIFORM HYPERBOLICITY OF THE CURVE GRAPHS

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Abstract. We show that there is a universal constant, k, such that the curve graph associated to any compact orientable surface is k-hyperbolic. Independent proofs of this have been given by Aougab, by Hensel, Przytycki and Webb, and by Clay, Rafi and Schleimer.

1. Introduction

Let Σ be a closed orientable surface of genus g, together with a (possibly empty) finite set $\Pi \subseteq \Sigma$. Set $p = |\Pi|$. We assume that $3g + p \geq 5$. Let $\mathcal{G} = \mathcal{G}(g, p)$ be the curve graph associated to (Σ, Π); that is, the 1-skeleton of the curve complex as originally defined in [Ha]. Its vertex set, $V(\mathcal{G})$, is the set of free homotopy classes of non-trivial non-peripheral closed curves in $\Sigma \setminus \Pi$; and two such curves are deemed to be adjacent in $\mathcal{G}$ if they can be realised disjointly in $\Sigma \setminus \Pi$. These, and related, complexes are now central tools in geometric group theory and hyperbolic geometry.

In [MM1], it was shown that, for all g, p, $\mathcal{G}(g, p)$ is hyperbolic in the sense of Gromov [Gr]. In [B], it was shown that the hyperbolicity constant, k, is bounded above by a function that is logarithmic in $g+p$. In fact, we show here that k can be chosen independently of g and p:

Theorem 1.1. *There is a universal constant, $k \in \mathbb{N}$, such that $\mathcal{G}(g, p)$ is k-hyperbolic for all g, p with $3g + p \geq 5$.*

We will give some estimates for k (though certainly not optimal) in Section 4.

Independent proofs of this result have been found by Aougab [A], by Hensel, Przytycki and Webb [HePW], and by Clay, Rafi and Schleimer [CRS]. The proofs in [HePW] and [CRS] are both combinatorial in nature. The proof in [A] is based on broadly similar principles to those decribed here, though the specifics are different. Both this paper and [A] make use of riemannian geometry. The argument of [HePW] seems to give the optimal constants.

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Given Theorem 1.1, one can also obtain uniform bounds for the Bounded Geodesic Image Theorem of [MM2]. For this, one can combine the description of quasigeodesic lines in [B] with an unpublished argument of Leininger. In fact, a more direct approach, just using hyperbolicity, has recently been found by Webb [Web].

We remark that Theorem 1.1 does not imply uniform hyperbolicity of the curve complexes (with simplices realised as regular euclidean simplices) since their 1-skeleta are not uniformly quasi-isometrically embedded — there is an arbitrarily large contraction of distances as the complexity increases.

The proof of Theorem 1.1 consists primarily of going through the arguments of [B] with more careful bookkeeping of constants. This is accomplished in Section 2 here. In Sections 3 and 4 here, we show that much of this can be bypassed. In fact, we only really need a few results from [B], notably Lemmas 1.3, 4.4 and 4.5, together with the construction of singular euclidean structures described in Section 5 thereof.

We were motivated to look again at that paper after reading some estimates in [T] which relate distances to intersection number.

2. Proofs

In this section, we will prove Proposition 2.6, which, together with Proposition 3.1 of [B] implies Theorem 1.1.

We will use the following different measures of the "complexity" of Σ, Π, tailored to different parts of the argument: $\xi_0 = 2g + p - 4$, $\xi_1 = 2g + p - 1$, $\xi_2 = 2g + p + 6$. For $\alpha, \beta \in V(\mathcal{G})$, we write $\iota(\alpha, \beta)$ for the intersection number, and $d(\alpha, \beta)$ for the combinatorial distance in the curve graph.

Lemma 2.1. *If $\gamma, \delta \in V(\mathcal{G})$, with $\iota(\gamma, \delta) \leq \xi_0 + 1$, then $d(\gamma, \delta) \leq 2$.*

Proof. We realise γ, δ in $\Sigma \setminus \Pi$ so that $|\gamma \cap \delta| = \iota(\gamma, \delta) = n$, say. Now, $\gamma \cup \delta$ is a graph with n vertices and $2n$ edges, and hence Euler characteristic $-n$. If $d(\gamma, \delta) > 2$, then $\gamma \cup \delta$ fills $\Sigma \setminus \Pi$ and so this Euler characteristic must be at most that of $\Sigma \setminus \Pi$, namely, $2 - 2g - p$. Thus, $n \geq 2g + p - 2$. Taking the contrapositive, if $n \leq \xi_0 + 1 = 2g + p - 3$, then $d(\gamma, \delta) \leq 2$. $\square$

Now, Lemma 1.3 of [B] shows that if $\alpha, \beta \in V(\mathcal{G})$ with $2\iota(\alpha, \beta) \leq ab$ for $a, b \in \mathbb{N}$, then there is some $\gamma \in V(\mathcal{G})$ with $\iota(\alpha, \gamma) \leq a$ and $\iota(\beta, \gamma) \leq b$. Applying this q times, together with Lemma 2.1, we get:

Corollary 2.2. *If $q \in \mathbb{N}$ and $\alpha, \beta \in V(\mathcal{G})$ with $2^q \iota(\alpha, \beta) \leq \xi_0^{q+1}$, then $d(\alpha, \beta) \leq 2(q + 1)$.*

Definition. By a *region* in Σ, we mean a subsurface, $H \subseteq \Sigma$, with $\partial H \cap \Pi = \emptyset$. A region is *trivial* if it is a topological disc containing at most one point of Π. An *annulus* in Σ is a region $A \subseteq \Sigma \setminus \Pi$ homeomorphic to $S^1 \times [0,1]$ such that no component of $\Sigma \setminus A$ is trivial.

The core curve of an annulus therefore determines an element of $V(\mathcal{G})$.

Suppose that ρ is a riemannian metric on Σ. We allow for a finite number of cone singularities (which need bear no relation to Π). We define the *width* of an annulus $A \subseteq \Sigma$ to be the length of a shortest path in A connecting its two boundary components.

The following lemma is a slight variation of Lemma 5.1 of [B]. We follow a similar argument, but taking more care with constants.

The proof will make use of the following notion. Let α be an essential non-peripheral closed curve in $\Sigma \setminus \Pi$.

Definition. A *bridge* (across α) is an arc, $\delta \subseteq \Sigma \setminus \Pi$, with $\partial\delta = \delta \cap \alpha$ such that no component of $\Sigma \setminus (\alpha \cup \delta)$ is a disc not meeting Π.

In other words, $\alpha \cup \delta$ is an embedded π_1-injective theta-curve in $\Sigma \setminus \Pi$, i.e. it is the union of three arcs which meet precisely in their endpoints and are pairwise non-homotopic relative to their endpoints.

Lemma 2.3. *Suppose that ρ is a (singular) riemannian metric on Σ, with* $\operatorname{area}(\Sigma) = 1$. *Suppose that $3g + p \geq 5$. Suppose that there is a constant $h > 0$ such that for any trivial region $\Delta \subseteq \Sigma$ we have* $\operatorname{area}(\Delta) \leq h(\operatorname{length}(\partial\Delta))^2$. *Then Σ contains an annulus of width at least $\eta = 1/4\xi_1\xi_2\sqrt{h}$.*

Proof. To avoid technical details obscuring the exposition, we will relax inequalities so that they are assumed to hold up to an arbitrarily small additive constant $\epsilon > 0$. Thus, for example, a "shortest" curve will assumed to be shortest to within ϵ etc. This will allow us, for example, to adjust paths so that they can be assumed to avoid Π. Finally, we can allow $\epsilon \to 0$. In what follows any "curve" in $\Sigma \setminus \Pi$ will be assumed to be essential and non-peripheral, i.e. it does not bound a trivial region in Σ.

Let $\eta_0 = 1/4\xi_2\sqrt{h}$. We claim that there are curves, $\alpha, \beta \subseteq \Sigma \setminus \Pi$ with $\rho(\alpha, \beta) \geq \eta_0$. Given this, we let $\phi : \Sigma \longrightarrow [0, \eta_0] = [0, \xi_1\eta]$ be a 1-lipschitz map with $\alpha \subseteq \phi^{-1}(0)$ and $\beta \subseteq \phi^{-1}(\xi_1\eta)$. Given any $i \in \{1, \dots, \xi_1 - 1\}$, we can find a multicurve, $\gamma_i \subseteq \phi^{-1}(i\eta)$, which separates Σ into exactly two components, S_i^α, S_i^β, containing α and β respectively. We can assume $\gamma_i \cap \Pi = \emptyset$, and that $S_i^\alpha \subseteq S_{i+1}^\alpha$ for all i. These multicurves cut Σ into ξ_1 regions $M_i = S_i^\alpha \cap S_{i-1}^\beta$ (where $M_0 =$

S_1^α and $M_{\xi_1} = S^\beta_{\xi_1 - 1}$). At least one of these must have a component which is an annulus (otherwise each $M_i \setminus \Pi$ would have negative Euler characteristic, giving the contradiction that the Euler characteristic of $\Sigma \setminus \Pi$ is at most $-\xi_1 < 2 - 2g - p$). This annulus must have width at least η as required.

To find α, β, we take α to be a shortest curve in $\Sigma \setminus \Pi$. We suppose, for contradiction, that if $\beta \subseteq \Sigma \setminus \Pi$ is any curve, then $\rho(\alpha, \beta) < \eta_0$. Let $\lambda = 2\eta_0$.

We first claim that there is a collection disjoint bridges, $\delta_1, \ldots, \delta_n$, across α with length$(\delta_i) < \lambda$ for all i and with each component of $\Sigma \setminus (\alpha \cup \delta_1 \cup \cdots \cup \delta_n)$ trivial. (An example is shown in Figure 1.)

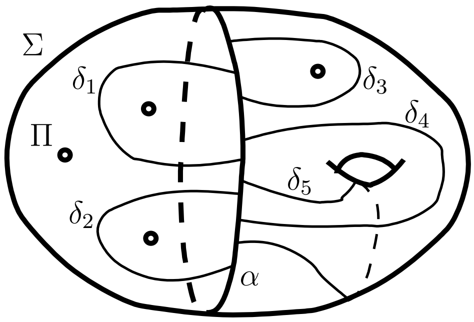

FIGURE 1. Example of a curve with bridges, $(g, p) = (1, 4)$

To prove this claim, let $N(\alpha, t)$ be the metric t-neighbourhood of α in Σ. Let $G(t)$ be the image of $\pi_1(N(\alpha, t) \setminus \Pi)$ in $\pi_1(\Sigma \setminus \Pi)$. Note that $G(0)$ is infinite cyclic, and $G(\eta_0) = \pi_1(\Sigma \setminus \Pi)$. As t increases from 0 to η_0, $G(t)$ gets bigger at certain critical times, $t_1, \ldots, t_n$. At these times, we can suppose we have added another generator, which we can represent as a bridge, δ_i, of length at most $2t_i < 2\eta_0 = \lambda$. Thus, inductively, $G(t_i)$ is supported on $\alpha \cup \delta_1 \cup \cdots \cup \delta_i$. It follows that $\alpha \cup \delta_1 \cup \cdots \cup \delta_n$ must fill $\Sigma \setminus \Pi$ (that is, carries all of $\pi_1(\Sigma \setminus \Pi)$), otherwise we could find a curve, β, with $\rho(\alpha, \beta) \geq \eta_0$. This gives us our collection of bridges as claimed.

Let $l = \text{length}(\alpha)$. We now claim that $l \leq 6\lambda$. So, suppose, to the contrary, that $l > 6\lambda$.

Given any i, write $\alpha = \alpha_i \cup \alpha_i'$, where α_i and α_i' are respectively the shorter and longer arcs with endpoints at $\partial\delta_i$. Thus, length$(\alpha_i) \leq l/2$, so length$(\alpha_i \cup \delta_i) \leq (l/2) + \lambda < l$, and so, by minimality of α, $\alpha_i \cup \delta_i$ must be trivial or peripheral, i.e. it bounds a trivial region in Σ. This region must be a disc containing exactly one point of Π. Since this is true of all bridges δ_i, we already get a contradiction if $g > 0$ (and we can deduce that $l \leq 3\lambda$ in this case). So can assume that $g = 0$, and so α cuts Σ into two discs, H_0, H_1. We have $|\Pi \cap H_i| \geq 2$, and we can assume that $|\Pi \cap H_0| \geq 3$.

Note also, if $\alpha'_i \cup \delta_i$ is non-trivial, then $\operatorname{length}(\alpha'_i \cup \delta_i) \geq \operatorname{length}(\alpha)$ and so $\operatorname{length}(\alpha_i) \leq \operatorname{length}(\delta_i) < \lambda$.

Now H_0 must contain at least two bridges from our collection. We can assume these are δ_1 and δ_2. Recall that $\delta_1 \cap \delta_2 = \emptyset$. From the above, it follows that $\operatorname{length}(\alpha_1) < \lambda$ and $\operatorname{length}(\alpha_2) < \lambda$. Since δ_1 and δ_2 cannot cross, we must have $\alpha_1 \cap \alpha_2 = \emptyset$.

Now let δ_3 be a bridge in H_1. As before, $\operatorname{length}(\alpha_3) \leq l/2$, and so $\operatorname{length}(\alpha_i \cup \alpha_3 \cup \delta_i \cup \delta_3) \leq 3\lambda + (l/2)$ for $i = 1, 2$. Now $\alpha_1 \cap \alpha_3 = \emptyset$ (otherwise, $\alpha_1 \cup \alpha_3 \cup \delta_1 \cup \delta_3$ would contain a curve of length at most $3\lambda + (l/2) < l$). Similarly, $\alpha_2 \cap \alpha_3 = \emptyset$. Now, given $i, j \in \{1, 2, 3\}$, let α_{ij} be the component of $\alpha \setminus (\alpha_1 \cup \alpha_2 \cup \alpha_3)$ between α_i and α_j. (See Figure 2.) Let θ_{ij} be the curve in Σ with image $\alpha_{ij} \cup \alpha_i \cup \alpha_j \cup \delta_i \cup \delta_j$, which passes through α_{ij} exactly twice. Together, the curves θ_{12}, θ_{23} and θ_{31} pass twice through each edge of $\alpha \cup \delta_1 \cup \delta_2 \cup \delta_3$, and so their lengths sum to at most $2l + 6\lambda$. We arrive at the contradiction that the length of at least one of the θ_{ij} is at most $\frac{1}{3}(2l + 6\lambda) < l$.

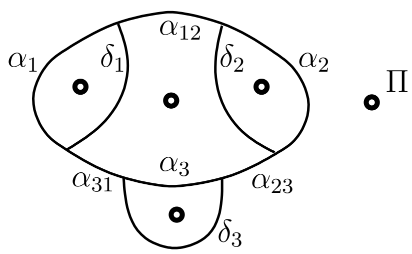

FIGURE 2. Picture of three bridges, $(g, p) = (0, 5)$

This shows that $l \leq 6\lambda$ as claimed.

After removing some of the bridges if necessary, we can assume that at most two of the complementary components are discs not meeting Π, and so $n \leq 2g + p$. Let $\sigma = \alpha \cup \delta_1 \cup \cdots \cup \delta_n$. Thus $\operatorname{length}(\sigma) < 6\lambda + n\lambda = (n + 6)\lambda \leq (2g + p + 6)\lambda = \xi_2\lambda$.

Since each component of $\Sigma \setminus \sigma$ is trivial, we must have $\operatorname{area}(\Sigma) \leq h(2 \operatorname{length} \sigma)^2$ (the worst case being when $\Sigma \setminus \sigma$ is connected). But we have assumed that $\operatorname{area}(\Sigma) = 1$ and so $1 < h(2\xi_2\lambda)^2$. Now, $\lambda = 2\eta_0 = 2(1/4\xi_2\sqrt{h}) = 1/2\xi_2\sqrt{h}$, so we arrive at the contradiction that $1 < 1$.

This shows that there must be a curve, β, in $\Sigma \setminus \Pi$ with $\rho(\alpha, \beta) \geq \eta_0$ as claimed. $\square$

In fact, the argument also applies if $(g, p) = (1, 1)$. If $(g, p) = (0, 4)$, we will only need to consider a special case, namely, the quotient of a euclidean torus by an involution with four fixed points. In that case, we can set $\eta = 1/2$.

We will now set $h = 1/2\pi$. This gives $\eta = 1/4\xi_1\xi_2\sqrt{1/2\pi} = \sqrt{2\pi}/4\xi_1\xi_2$. As in Section 5 of [B], we define $R = \sqrt{2}/\eta$. In this case therefore, $R = (4/\sqrt{\pi})\xi_1\xi_2$.

Now suppose that α, β are weighted multicurves in the sense defined in [B]. (In other words, each is a measured lamination whose support is a disjoint union of curves.)

Definition. The *weighted intersection number, $\iota(\alpha, \beta)$, of α and β is the sum $\sum_{i,j} \lambda_i \lambda_j \iota(\alpha_i, \beta_j)$, where α_i and β_j vary over the components of the support of α and β, where λ_i and λ_j are the respective weighting on them, and where $\iota(\alpha_i, \beta_j) \in \mathbb{N}$ is the usual geometric intersection number.*

We write $d(\alpha, \beta) = \min_{i,j}\{d(\alpha_i, \beta_j)\}$, again where α_i and β_j vary over the components of α, β.

Given $\gamma \in V(\mathcal{G})$ we set $l(\gamma) = l_{\alpha\beta}(\gamma) = \max\{\iota(\alpha, \gamma), \iota(\beta, \gamma)\}$ (interpreting γ as a one-component multicurve of unit weight). One can think of $l(\gamma)$ as describing a "length" in a singular euclidean structure arising from α and β (cf. Section 5 of [B]).

Lemma 2.4. *Suppose that α, β are weighted multicurves with $\iota(\alpha, \beta) = 1$ and $d(\alpha, \beta) \geq 2$. Then there is some $\delta \in V(\mathcal{G})$ with $l(\delta) \leq R$ and such that $\iota(\gamma, \delta) \leq Rl(\gamma)$ for all $\gamma \in V(\mathcal{G})$ (where R is defined as above).*

Note that this is just a restating of Lemma 4.1 of [B] for this particular definition of R.

Proof. The proof is the same as that of Lemma 4.1 of [B]. Suppose first that $\alpha \cup \beta$ fills $\Sigma \setminus \Pi$. As in Section 5 of that paper, we construct a singular euclidean surface, tiled by rectangles, dual to $\alpha \cup \beta$. The cone angles are all multiples of π, and all cone singularities of angle π lie in Π. Thus, any trivial region, $\Delta \subseteq \Pi$, contains at most one cone point of angle less than 2π. Passing to a branched double cover over this cone point (if it exists) we are reduced to considering the case where all cone angles are at least 2π. But then the worst case is a round circle in the euclidean plane [Wei] which would give $\text{area}(\Delta) = \text{length}(\partial\Delta)^2/4\pi$. We can therefore set $h = 2(1/4\pi) = 1/2\pi$. Now apply Lemma 2.3, and set δ to be a core curve of that annulus. The statement then follows exactly as in [B] (at the end of Section 5 thereof). (In [B], h was given inaccurately as $\pi/2$.)

If $\alpha \cup \beta$ does not fill $\Sigma \setminus \Pi$, we get instead a singular euclidean structure on a "smaller" surface, namely a region of Σ with each boundary component collapsed to a point. However, this process can only decrease ξ_1 and ξ_2, so we again get an annulus of width at least η. (This case is the reason we needed a version of Lemma 2.3 when $3g + p = 4$. In the case where $(g, p) = (0, 4)$, note that $1/2$ is certainly greater than the required $\sqrt{2\pi}/120$.) □

Given $r \geq 0$, set $L(\alpha, \beta, r) = \{\gamma \in V(\mathcal{G}) \mid l(\gamma) \leq r\}$. Note that the curve δ given by Lemma 2.4 lies in $L(\alpha, \beta, R)$.

Lemma 2.5. *Suppose that* $2g+p \geq 195$. *Suppose that* α, β *are weighted multicurves with* $\iota(\alpha, \beta) = 1$ *and* $d(\alpha, \beta) \geq 2$. *Then, the diameter of* $L(\alpha, \beta, 2R)$ *in* $\mathcal{G}$ *is at most* 20.

Proof. Let δ be as given by Lemma 2.4. If $\gamma \in L(\alpha, \beta, 2R)$, then $l(\gamma) \leq 2R$, so $\iota(\gamma, \delta) \leq 2R^2$. If we knew that $16\iota(\gamma, \delta) \leq \xi_0^5$, then Corollary 2.2 with $q = 4$ would give $d(\gamma, \delta) \leq 10$ and the result would follow.

It is therefore sufficient that $16(2R^2) \leq \xi_0^5$. Recall that $R = (4/\sqrt{\pi})\xi_1\xi_2$, so this reduces to $32(4/\sqrt{\pi})^2\xi_1^2\xi_2^2 \leq \xi_0^5$, that is, $512\xi_1^2\xi_2^2 \leq \pi\xi_0^5$. In other words, we want

$$(*) \qquad 512(2g + p - 1)^2(2g + p + 6)^2 \leq \pi(2g + p - 4)^5$$

which holds whenever $2g + p \geq 195$. □

We now assume that $2g + p \geq 195$.

Recall that Lemma 4.3 of [B] states that $L(\alpha, \beta, R)$ has diameter bounded by some constant D (which there, depended on R). Since $L(\alpha, \beta, R) \subseteq L(\alpha, \beta, 2R)$, we have now verified Lemma 4.3 of [B] with $D = 20$. Recall that Lemma 4.2 of [B], more generally, placed a bound on the diameter of $L(\alpha, \beta, r)$ depending on r and R (specifically, $\operatorname{diam} L(\alpha, \beta, r) \leq 2Rr + 2$). This was used in the proof of Lemma 4.12 of [B]. We can now use Lemma 2.5 above, in place of Lemma 4.2 of [B], to give a proof of Lemma 4.12 of [B] with the constant $4D$ now replaced by 40. We can now proceed as in [B] to prove Lemma 4.13 and Proposition 4.11 of that paper. In fact, the improvement on Lemma 4.12 allows us, respectively, to replace the constants $14D$ by $10D$ and $18D$ by $14D$, where $D = 20$. Thus, the original diameter bound of $18D$ of Proposition 4.11 of [B] now becomes 280.

Recall that Proposition 3.1 of [B] gives a criterion for hyperbolicity depending on a constant, K, in the hypotheses. The three clauses (1), (2) and (3) of those hypotheses were verified respectively by Lemma 4.10, Proposition 4.11 and Lemma 4.9. These respectively gave K bounded by $4D$, $18D$ and $2D$, which we can now replace by 80, 280 and 40. In particular, we have shown:

Proposition 2.6. *If $2g+p \geq 195$, then the curve graph $\mathcal{G}(g,p)$ satisfies the hypotheses of Proposition 3.1 of* [B] *with $K = 280$.*

For $2g + p \geq 195$, one can now explicitly estimate k from the proof of Proposition 3.1 of [B]. In fact, one can do better.

3. A criterion for hyperbolicity

We give a self-contained account of a criterion for hyperbolicity which is related to, but simpler than, that used in [B]. In particular, it does not require the condition on moving centres (clause (2) of Proposition 3.1 of [B]) which complicated the argument there. Essentially the same statement can be found in Section 3.13 of [MS], though without a specific estimate for the hyperbolicity constant arising (or the final clause about Hausdorff distance). Our proof uses an idea to be found in [Gi], but bypasses use of the isoperimetric inequality. Since this criterion has many applications, this may be of some independent interest. For definiteness, we say that a space is *k-hyperbolic* if, in every geodesic triangle, each side lies in a k-neighbourhood of the union of the other two.

Proposition 3.1. *Given $h \geq 0$, there is some $k \geq 0$ with the following property. Suppose that G is a connected graph, and that for each $x, y \in V(G)$, we have associated a connected subgraph, $\mathcal{L}(x,y) \subseteq G$, with $x, y \in \mathcal{L}(x,y)$. Suppose that:*
(1) for all $x, y, z \in V(G)$,

$$\mathcal{L}(x,y) \subseteq N(\mathcal{L}(x,z) \cup \mathcal{L}(z,y), h),$$

and
(2) for any $x, y \in V(G)$ with $d(x,y) \leq 1$, the diameter of $\mathcal{L}(x,y)$ in G is at most h.
Then G is k-hyperbolic.
In fact, we can take any

$$k \geq (3m - 10h)/2,$$

where m is any positive real number satisfying

$$2h(6 + \log_2(m+2)) \leq m.$$

Moreover, for all $x, y \in V(G)$, the Hausdorff distance between $\mathcal{L}(x, y)$ and any geodesic from x to y is bounded above by $m - 4h$.

Here, d is the combinatorial metric on G, and $N(., h)$ denotes h-neighbourhood. Note that we can assume that $\mathcal{L}(x, y) = \mathcal{L}(y, x)$ (on replacing $\mathcal{L}(x, y)$ with $\mathcal{L}(x, y) \cup \mathcal{L}(y, x)$). Note that the condition on m is monotonic: if it holds for m, it holds strictly for any $m' > m$.

Proof. Given any $x, y \in V(G)$, let $\mathcal{I}(x, y)$ be the set of all geodesics from x to y. Given any $n \in \mathbb{N}$, write

$$f(n) = \max\{d(w, \alpha) \mid (\exists x, y \in V(G))\, d(x, y) \leq n,\ \alpha \in \mathcal{I}(x, y),\ w \in \mathcal{L}(x, y)\}.$$

In other words, $f(n)$ is the minimal $f \geq 0$ such that $\mathcal{L}(x, y) \subseteq N(\alpha, f)$ for any geodesic, α, connecting any two vertices x, y a distance at most n apart.

We first claim that $f(n) \leq (2 + [\log_2 n])h$ (cf [Gi]). To see this, write $l = d(x, y) \leq n$ and $p = [\log_2 l] + 2$. Let $z \in V(G)$ be a "near midpoint" of α, that is, it cuts α into two subpaths, α^- and α^+ whose lengths differ by at most 1. By (1), $\mathcal{L}(x, y) \subseteq N(\mathcal{L}(x, z) \cup \mathcal{L}(z, y), h)$. We now choose near midpoints of each of the paths α^+ and α^- and then continue inductively. After at most $p - 1$ steps, we see that $\mathcal{L}(x, y) \subseteq N(\bigcup_{i=0}^{l-1} \mathcal{L}(x_i, x_{i+1}), (p - 1)h)$ where $x = x_0, x_1, \ldots, x_l = y$ is the sequence of vertices along α. Applying (2) now gives $\mathcal{L}(x, y) \subseteq N(\alpha, ph)$, and so $f(n) \leq ph$ as claimed.

In fact, we aim to show that $f(n)$ is bounded purely in terms of h. We proceed as follows.

Let $t = f(n) + 2h + 1$. Choose any $w \in \mathcal{L}(x, y)$. Let $l_0 = \max\{0, d(w, x) - t\}$ and $l_1 = \max\{0, d(w, y) - t\}$. Since $l = d(x, y)$, we have $l \leq l_0 + l_1 + 2t$, and so we can find vertices x', y' in α cutting it into subpaths $\alpha = \alpha_0 \cup \delta \cup \alpha_1$, where $d(x, x') \leq l_0$, $d(x', y') \leq 2t$ and $d(y', y) \leq l_1$. If $x = x'$ we leave out α_0, and/or if $y = y'$ we leave out α_1. (We can always assume that $x' \neq y'$.)

Note that $d(w, \alpha_0) \geq d(w, x) - d(x, x') \geq d(w, x) - l_0$. Therefore, if $x \neq x'$, then $l_0 = d(w, x) - t$, and so $d(w, \alpha_0) \geq t$. But $d(x, x') \leq d(x, y) \leq n$ and so $\mathcal{L}(x, x') \subseteq N(\alpha_0, f(n))$. It follows that $d(w, \mathcal{L}(x, x')) \geq t - f(n) = 2h + 1$. In other words, if $x \neq x'$, then $d(w, \mathcal{L}(x, x')) \geq 2h + 1$. Similarly, if $y \neq y'$, then $d(w, \mathcal{L}(y', y)) \geq 2h + 1$. But

$$w \in \mathcal{L}(x, y) \subseteq N(\mathcal{L}(x, x') \cup \mathcal{L}(x', y') \cup \mathcal{L}(y', y), 2h)$$

and so $d(w, \mathcal{L}(x', y')) \leq 2h$. Now $d(x', y') \leq 2t$ and so $\mathcal{L}(x', y') \subseteq N(\delta, f(2t))$. Thus, $w \in N(\delta, f(2t) + 2h) \subseteq N(\alpha, f(2t) + 2h)$. Since w

was an arbitrary point of $\mathcal{L}(x,y)$, it follows that

$$f(n) \leq f(2t) + 2h = f(2f(n) + 4h + 2) + 2h,$$

Writing $F(n) = 2f(n)+4h+2$, we have shown that $F(n) \leq F(F(n)) + 4h$ for all n.

Now, from the earlier claim,

$$F(n) \leq 2((2 + \log_2 n)h) + 4h + 2 = 2h(4 + \log_2 n) + 2.$$

Suppose m is as in the statement of the theorem. Writing $r = m + 2$, we have $2h(6+\log r)+2 \leq r$, and so $F(n)+4h \leq 2h(6+\log_2 n)+2 < n$ for any $n > r$.

In summary, we have shown that

$$F(n) \leq F(F(n)) + 4h$$

for all n, and that

$$F(n) + 4h < n$$

for all $n > r$. It follows that $F(n) \leq r$ for all n (otherwise, we have the contradiction $F(n) \leq F(F(n)) + 4h < F(n)$). It now follows that $f(n) \leq s$, where $s = \frac{r}{2} - 2h - 1 = \frac{m}{2} - 2h$.

We have shown that for all $x, y \in V(G)$ and $\alpha \in \mathcal{I}(x,y)$, we have $\mathcal{L}(x,y) \subseteq N(\alpha, s)$. It now follows also that $\alpha \subseteq N(\mathcal{L}(x,y), 2s)$. Since if $w \in \alpha$, then w cuts α into two subpaths, α^- and α^+. Since $\mathcal{L}(x,y)$ is connected and contains x, y, we can find some $v \in \mathcal{L}(x,y)$ and $v^\pm \in \alpha^\pm$ with $d(v, v^\pm) \leq s$. Now $d(w, \{v^-, v^+\}) \leq s$, so $d(v,w) \leq 2s$. We deduce that $d(w, \mathcal{L}(x,y)) \leq 2s$ as required.

Now suppose that $x, y, z \in V(G)$ and that $\alpha \in \mathcal{I}(x,y)$, $\beta \in \mathcal{I}(x,z)$ and $\gamma \in \mathcal{I}(y,z)$. We have

$$\alpha \subseteq N(\mathcal{L}(x,y), 2s) \subseteq N(\mathcal{L}(x,z) \cup \mathcal{L}(z,y), 2s + h) \subseteq N(\beta \cup \gamma, k),$$

where

$$k = 3s + h \leq 3((r/2) - 2h - 1) + h = (3m - 10h)/2.$$

Thus, G is k-hyperbolic. □

4. Estimation of constants

Given Proposition 3.1 of this paper, we can extract information more efficiently from [B], and bypass much of the proof of Theorem 1.1. Given, $\alpha, \beta \in V(\mathcal{G}(g,p))$ with $d(\alpha, \beta) \geq 2$ and $t \in \mathbb{R}$, let $\Lambda_{\alpha\beta}(t) = L((e^t/\iota)\alpha, (e^{-t}/\iota)\beta, R)$, where $\iota = \iota(\alpha, \beta) > 0$.

Now, $\iota((e^t/\iota)\alpha, (e^{-t}/\iota)\beta) = 1$. Therefore if $2g + p \geq 195$, then by Lemma 2.4, $\Lambda_{\alpha\beta}(t) \neq \emptyset$. Let $\mathcal{L}(\alpha,\beta)(t)$ be the full subgraph of $\mathcal{G}$ with vertex set $\Lambda_{\alpha\beta}(t)$. It is not hard to see that $\mathcal{L}(\alpha,\beta)(t)$ is connected.

(For example, the standard argument, going back to work of Lickorish, for showing that $\mathcal{G}$ itself is connected effectively does this. This involves interpolating between two curves by a series of surgery operations, cf. Lemma 1.3 of [B] for example. These can only decrease the intersection number with any fixed curve.) It follows easily that $\mathcal{L}(\alpha,\beta) = \bigcup_{t\in\mathbb{R}} \mathcal{L}(\alpha,\beta)(t)$ is connected. Note that the vertex set of $\mathcal{L}(\alpha,\beta)$ is the "line" $\Lambda_{\alpha\beta} = \bigcup_{t\in\mathbb{R}} \Lambda_{\alpha\beta}(t)$ as defined in [B]. Note also that $\alpha, \beta \in \Lambda_{\alpha\beta}$. If $d(\alpha,\beta) \leq 1$, we set $\Lambda_{\alpha\beta} = \{\alpha,\beta\}$, so that $\mathcal{L}(\alpha,\beta)$ is a single vertex or edge.

We can now verify that the collection $(\mathcal{L}(\alpha,\beta))_{\alpha,\beta\in V(\mathcal{G})}$ satisfies the hypotheses of Proposition 3.1 here with $h = 40$. Condition (2) is immediate. For condition (1), let $\alpha,\beta,\gamma \in V(G)$. If these three curves all pairwise intersect, then we set $\tau = \frac{1}{2}\log_e(\iota(\alpha,\beta)\iota(\alpha,\gamma)/\iota(\beta,\gamma))$. As in Lemma 4.5 of [B], we see that if $t \leq \tau$, the diameter of $\mathcal{L}(\alpha,\beta)(t) \cup \mathcal{L}(\alpha,\gamma)(t)$ is at most 40 (since we can set $D = 20$). Similarly, if $t \geq \tau$ then $\mathcal{L}(\alpha,\beta)(t) \cup \mathcal{L}(\beta,\gamma)(t)$ has diameter at most 40. Thus, $\mathcal{L}(\alpha,\beta) \subseteq N(\mathcal{L}(\alpha,\gamma) \cup \mathcal{L}(\gamma,\beta), h)$ with $h = 40$. The cases where at least two of the curves α,β,γ are disjoint follow from a slight modification of this argument, as in [B]. This now gives $m \leq 1320$ and $k \leq 1780$. This shows that if $2g + p \geq 195$, then $\mathcal{G}(p,q)$ is 1780-hyperbolic.

In fact, since we are now only using Lemma 4.3 of [B], we can replace $2R$ by R in Lemma 2.5 here, so that the requirement $16(2R^2) \leq \xi_0^5$ becomes $16R^2 \leq \xi_0^5$, and so we can replace the resulting factor of 512 in $(*)$ by 256. It is therefore sufficient that $2g + p \geq 107$. We have therefore shown that if $2g + p \geq 107$, then $\mathcal{G}(g,p)$ is 1780-hyperbolic.

We can deal with lower complexity surfaces using larger values of q from Corollary 2.2. In general, we require that $2^{q+4}(2g+p-1)^2(2g+p+6)^2 \leq \pi(2g+p-4)^{q+1}$. For example, with $q = 5$, this is satisfied for $2g + p \geq 26$. This gives $D = 4(q+1) = 24$, $h = 2D = 48$, $m \leq 1584$ and $k \leq 2136$. In other words, if $2g + p \geq 26$, then $\mathcal{G}(g,p)$ is 2064-hyperbolic. Similarly (with $q = 6$), if $2g + p \geq 14$, then $\mathcal{G}(g,p)$ is 2492-hyperbolic etc.

For the cases where $2g + p \leq 6$, we need to revert to previous arguments. The estimates and methods in [T] might give improvements for some of the lower complexities.

There is scope for other improvements in various directions. For the bounds on complexity for example, suppose $p = 0$. In the proof of Lemma 2.3 we don't have to worry about trivial regions, so we can easily obtain $l \leq 2\lambda$, allowing us to reset $\xi_2 = 2g+2$. We can also reset $\xi_1 = 2g$. For Lemma 2.2, we could set $h = 1/4\pi$, further decreasing R by a factor of $\sqrt{2}$. In Lemma 1.3 of [B], we can eliminate the factor of

2 in the hypotheses, and thereby weaken those of Corollary 2.2 here to saying that $\iota(\alpha,\beta) \le x_0^q$. The fact that we have replaced $2R$ by R, also gives us another factor of 2, so that our requirement, when $q = 4$, now becomes $R^2 \le \xi_0^5$. Together these now give $8(2g)^2(2g+2)^2 \le \pi(2g-4)^5$, that is, $4g^2(g+1)^2 \le \pi(g-2)^5$, which holds for $g \ge 8$. In other words, $\mathcal{G}(g,0)$ is 1780-hyperbolic for $g \ge 8$.

We remark that in [HePW], it is shown that every curve graph is "17-hyperbolic" in the sense that, for every geodesic triangle, there is a vertex a distance at most 17 from each of its sides. From this, one can easily derive a uniform hyperbolicity constant in the sense we have defined it.

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