LARGE-SCALE RANK AND RIGIDITY OF THE 
WEIL-PETERSSON METRIC

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ABSTRACT. We study the large-scale geometry of Weil-Petersson space, that is, Teichmüller space equipped with the Weil-Petersson metric. We show that this admits a natural coarse median structure of a specific rank. Given that this is equal to the maximal dimension of a quasi-isometrically embedded euclidean space, we recover a result of Eskin, Masur and Rafi which gives the coarse rank of the space. We go on to show that, apart from finitely many cases, the Weil-Petersson spaces are quasi-isometrically distinct, and quasi-isometrically rigid. By a theorem of Brock, Weil-Petersson space is quasi-isometric to the pants graph, so our results apply equally well to that space.

1. Introduction

In this paper, we investigate the large-scale geometry of Teichmüller space in the Weil-Petersson metric. In particular, we give a formula for the coarse rank of this space (the maximal dimension of a quasi-isometrically embedded copy of the euclidean plane), thereby recovering a result of [EMR1]. We go on to prove quasi-isometric rigidity of the Weil-Petersson metric for all but finitely many surfaces. To obtain these results, we show that the Weil-Petersson space admits a natural ternary operation, well defined up to bounded distance, which gives it the structure of a coarse median space, as defined in [Bo1].

The main results only depend on the (equivariant) quasi-isometry type of the spaces involved. Note that, in [Bro], it was shown that the Weil-Petersson space is equivariantly quasi-isometric to the pants graph. Therefore, all the main results stated here are valid also for that space. Indeed we will work mostly with the pants graph in this paper.

Our proofs will rely heavily on results of [Bo8], where relevant ideas are developed, and applied to the case of mapping class group itself (or equivalently to the “marking graph”). Central to this, is the notion of a “coarse median”. (In the case of mapping class group, such a median was constructed in [BeM2].)

Let $\Sigma$ be a compact orientable surface of genus $g = g(\Sigma)$ and with $p = p(\Sigma)$ boundary components. We write $\xi(\Sigma) = 3g + p - 3$ for its complexity. We sometimes denote its topological type by $S_{g,p}$. We will also write $\xi_0(\Sigma) = \lfloor (\xi(\Sigma) + 1)/2 \rfloor$.

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By the *Weil-Petersson space*, $\mathbb{W}(\Sigma)$, of $\Sigma$ (that is, the space of marked complete finite-area hyperbolic structures on the interior of $\Sigma$) equipped with the Weil-Petersson metric. This is a negatively curved riemannian manifold diffeomorphic to $\mathbb{R}^{2\xi(\Sigma)}$. It is not complete, but is geodesically convex and CAT(0) (see [Wo]). Its completion has a natural stratification, where the lower dimensional strata are direct products of lower-complexity Weil-Petersson spaces. From this, one can easily see that the completion contains flats (geodesically embedded euclidean spaces) of dimension $\xi_0(\Sigma)$, but of no higher dimension. In other words, it has “euclidean rank” equal to $\xi_0(\Sigma)$. (Indeed, it does not contain any isometrically embedded euclidean ball of dimension $\xi_0(\Sigma) + 1$.)

Note also that the mapping class group, $\text{Map}(\Sigma)$, acts by isometry on $\mathbb{W}(\Sigma)$. It was shown in [MasW] that $\mathbb{W}(\Sigma)$ is “rigid”, in the sense that every isometry is induced by some element of $\text{Map}(\Sigma)$. Here, we aim to give coarse versions of the above statements.

The following is a central observation for our arguments:

**Theorem 1.1.** There is a ternary operation, $\mu : \mathbb{W}(\Sigma)^3 \rightarrow \mathbb{W}(\Sigma)$, which endows $\mathbb{W}(\Sigma)$ with the structure of a coarse median space of rank $\xi_0(\Sigma)$.

The notion of a “coarse median space” was defined in [Bo1], and will be discussed further in Section 3 here. Roughly speaking, it says that, when dealing with a subset of the space of bounded finite cardinality, the median operation behaves like the standard median operation on a finite CAT(0) cube complex. Moreover, we can take this cube complex to have dimension at most the (coarse median) rank (which is $\xi_0(\Sigma)$ in the case of $\mathbb{W}(\Sigma)$).

Note that if $\xi_0(\Sigma) = 1$ (i.e. $\Sigma$ is $S_{0,5}$ or $S_{1,2}$), then $\mathbb{W}(\Sigma)$ is rank 1, and so (by Theorem 2.1 of [Bo1]) we recover the result of [BroF] (see also [Ar]) that $\mathbb{W}(\Sigma)$ is hyperbolic in this case. (In fact, $\mathbb{W}(S_{0,5})$ is isometric to $\mathbb{W}(S_{1,2})$.)

We remark that it is shown in [BeHS] that Weil-Petersson space is “hierarchically hyperbolic” and that this, in turn, implies that it is coarse median. We comment further on this in Section 3.

The following result of [EMR1] is now an immediate consequence of Theorem 1.1 above together with Corollary 2.4 of [Bo1]:

**Theorem 1.2.** There is a quasi-isometric embedding of euclidean $n$-space into $\mathbb{W}(\Sigma)$ if and only if $n \leq \xi_0(\Sigma)$.

In fact (as in [EMR1]), the same statement holds with “euclidean space” replaced by euclidean half-space. Indeed, one can place an upper bound on the radius of a quasi-isometrically embedded euclidean ball of dimension $\xi_0(\Sigma)$, which depends only on $\xi(\Sigma)$ and the quasi-isometric constants (see Lemma 6.10 of [Bo8]).

An immediate consequence of Theorem 1.2 is that if $\Sigma$ and $\Sigma'$ are two surfaces with $\mathbb{W}(\Sigma)$ quasi-isometric to $\mathbb{W}(\Sigma')$, then $\xi_0(\Sigma) = \xi_0(\Sigma')$. In fact, we will show the following.
Theorem 1.3. Suppose that $\Sigma$ and $\Sigma'$ are compact orientable surfaces with $\mathbb{W}(\Sigma)$ quasi-isometric to $\mathbb{W}(\Sigma')$, then $\xi(\Sigma) = \xi(\Sigma')$. Moreover, if $\xi(\Sigma) = \xi(\Sigma') \geq 6$, then $\Sigma = \Sigma'$.

This gives rise to a complete quasi-isometric classification, except in the cases of complexity 4 and 5, which we leave unresolved here.

The main result of this paper refers to quasi-isometric rigidity. We say that a geodesic metric space with an isometric group action (in this case, $\mathbb{W}(\Sigma)$ by acted upon by $\text{Map}(\Sigma)$) is quasi-isometrically rigid if any self-quasi-isometry is a bounded distance from the isometry induced by some element of the group.

We will show:

Theorem 1.4. If $g(\Sigma) + p(\Sigma) \geq 7$, or if $g(\Sigma) \geq 3$ and $p(\Sigma) \leq 1$, then $\mathbb{W}(\Sigma)$ is quasi-isometrically rigid.

Indeed, the distance bound of the conclusion depends only on the quasi-isometry constants and on $\xi(\Sigma)$. Clearly, the result covers all but finitely many topological types of surfaces.

Theorem 1.4 is a consequence of Theorem 6.8 here, which reduces it to a combinatorial rigidity statement for certain curve graphs. The relevant combinatorial rigidity theorems proven respectively in [BreM], [Ki] and [Bo6]. The result as stated leaves about a dozen cases open (as we discuss further at the end of Section 6).

The proofs of Theorems 1.3 to 1.4 above make use of the asymptotic cone, $\mathbb{W}^\infty(\Sigma)$, of $\mathbb{W}(\Sigma)$ (see [VaW, G]). This is a complete metric space, in this case $\text{CAT}(0)$ (since $\mathbb{W}(\Sigma)$ is). The following is a direct consequence of Theorem 1.1, as shown in [Bo1].

Theorem 1.5. Any asymptotic cone, $\mathbb{W}^\infty(\Sigma)$, of $\mathbb{W}(\Sigma)$ has locally compact dimension $\xi_0(\Sigma)$.

The locally compact dimension of a topological space is the maximal dimension of a locally compact subset (cf. [BeM1]). As described in [Bo1], Theorem 1.5 is easily seen to imply Theorem 1.2.

We also have:

Theorem 1.6. If $\Sigma$ and $\Sigma'$ are compact orientable surfaces with $\mathbb{W}^\infty(\Sigma)$ homeomorphic to $\mathbb{W}^\infty(\Sigma')$, then either $\xi(\Sigma) = \xi(\Sigma') \geq 3$, or $\xi(\Sigma), \xi(\Sigma') \leq 2$.

(In fact, in the above, we could take the asymptotic cones of $\Sigma$ and $\Sigma'$ to be with respect to different scaling factors and ultrafilters.)

In fact, much of this paper will be devoted to understanding the structure of the asymptotic cone. To apply this to quasi-isometric rigidity, we make use of some results of [Bo8]. These were inspired by related arguments in [BeKMM].

As noted above, it is shown in [Bro] that $\mathbb{W}(\Sigma)$ is $\text{Map}(\Sigma)$-equivariantly quasi-isometric to the pants graph, denoted here as $\mathbb{P}(\Sigma)$. (The pants graph will be
discussed in Section 2 here.) In fact, we will mostly work with \( P(\Sigma) \) instead of \( W(\Sigma) \). All the main theorems stated above apply equally well with \( P(\Sigma) \) substituted for \( W(\Sigma) \) (though the asymptotic cone, \( P^\infty(\Sigma) \) will only be be bilipschitz equivalent to a CAT(0) space in this case).

We remark that there are also combinatorial rigidity results for \( P(\Sigma) \). In [Mar] it is shown that any automorphism of \( P(\Sigma) \) is induced by an element of Map(\( \Sigma \)) (with a few qualifications for the low-complexity cases). In [BroMar], this fact was used to give another proof of the result of [MasW] mentioned above.

We note that related results for Teichmüller space in the Teichmüller metric are explored in [Bo7], also using the fact that it admits a coarse median structure (in this case of rank \( \xi(\Sigma) \)). A different approach to quasi-isometric rigidity of the Teichmüller metric has been given independently in [EMR2]. We also remark that the quasi-isometry types of these and various other metrics on Teichmüller space is studied in [Y].

**Notation.** We will be referring to [Bo1] and [Bo8], which deal with more general spaces, or with analogous spaces in a slightly different context. We comment briefly on how these correspond. In all cases, we have two families of spaces, each indexed by a set \( \mathcal{X} \), typically the set of subsurfaces of \( \Sigma \); together with various “projection maps” between them. In the general set up of [Bo1], we have a family of coarse median spaces, denoted \( \Lambda(X) \), as well as a family of hyperbolic spaces, denoted \( \Theta(X) \), where \( X \in \mathcal{X} \). In [Bo8], these are denoted respectively, by \( M(X) \) and \( G(X) \) (and later specialised, respectively, to the marking graph, \( M(X) \), and curve graph, \( G(X) \)). In the present paper, when \( X \) is not an annulus, \( \Lambda(X) \) (or \( M(X) \)) will correspond to the pants graph, \( P(X) \), and \( \Theta(X) \) (or \( G(X) \)) will again correspond to the curve graph, \( G(X) \). When \( X \) is an annulus \( P(X) \) and \( G(X) \) will both be deemed to be singletons. In fact, annular subsurfaces play no essential role in the present paper. (One could alternatively delete annular surfaces from the indexing set, though for consistency with notation elsewhere, we will include them.) The various projection maps are denoted similarly in all cases.

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2. The pants graph

First, we recall some standard general definitions. Throughout this paper, \( N(\cdot; r) \) will denote a closed uniform \( r \)-neighbourhood in a metric space. Suppose that \( \phi : (\Lambda, \rho) \rightarrow (\Lambda', \rho') \) is a map between two geodesic metric spaces. We say that \( \phi \) is **coarsely lipschitz** if

\[
(\exists k_1, k_2 \geq 0)(\forall x, y \in \Lambda)(\rho'(\phi x, \phi y) \leq k_1 \rho(x, y) + k_2).
\]
We say that \( \phi \) is a \textit{quasi-isometric embedding} if it is coarsely lipschitz and

\[
(\exists k_3, k_4 \geq 0)(\forall x, y \in \Lambda)(\rho(x, y) \leq k_3 \rho'(\phi x, \phi y) + k_4).
\]

We say that \( \phi \) is a \textit{quasi-isometry} if it is a quasi-isometric embedding and there is some \( k_5 \geq 0 \) such that \( \Lambda' = N(\phi(\Lambda); k_5) \).

By a \textit{curve} in \( \Sigma \) we mean a free homotopy class of non-trivial non-peripheral simple closed curves in \( \Sigma \). We write \( G^0(\Sigma) \) for the set of curves. Given \( \alpha, \beta \in G^0(\Sigma) \), we write \( \iota(\alpha, \beta) \) for their (geometric) intersection number (that is, the minimum number of intersections among representatives of these classes). As usual, we define the \textit{curve graph}, \( G(\Sigma) \), to be the graph with vertex set \( G^0(\Sigma) \), where \( \alpha, \beta \) are deemed adjacent if they have minimal possible intersection for that surface (in other words, disjoint if \( \xi(\Sigma) \geq 2 \); \( \iota(\alpha, \beta) = 1 \) if \( \Sigma = S_{1,1} \); \( \iota(\alpha, \beta) = 2 \) if \( \Sigma = S_{0,4} \)). We write \( \sigma = \sigma_{\Sigma} \) for the combinatorial metric on \( G(\Sigma) \). Given finite subsets, \( a, b \subseteq G^0(\Sigma) \), we will write \( \iota(a, b) = \max\{\iota(\alpha, \beta) \mid \alpha \in a, \beta \in b\} \).

A \textit{multicurve}, \( a \), is a non-empty set of pairwise disjoint curves in \( \Sigma \). We will sometimes abuse terminology by identifying \( a \) with a realisation thereof as a submanifold, \( a \subseteq \Sigma \). We say that \( a \) is a \textit{complete multicurve} (or “pants decomposition”) if each component of \( \Sigma \setminus a \) is a \( S_{0,3} \).

The \textit{pants graph} is traditionally defined as follows. Let \( \mathbb{P}^0 \) be the set of complete multicurves in \( \Sigma \). Let \( \mathbb{P} = \mathbb{P}(\Sigma) \) be the graph with vertex set \( V(\mathbb{P}) = \mathbb{P}^0 \), and where \( a, b \in \mathbb{P}^0 \) are deemed to be adjacent if there is some \( \gamma \in a \) and \( \delta \in b \) such that \( a \setminus \gamma = b \setminus \delta = c \), say and with \( \iota(\gamma, \delta) \) equal to 1 or 2 depending on whether \( \gamma \) (hence also \( \delta \)) is contained in an \( S_{1,1} \) or a \( S_{0,4} \) component of \( \Sigma \setminus c \). Note that if \( \Sigma \) is an \( S_{1,1} \) or \( S_{0,4} \), then we can identify \( \mathbb{P}(\Sigma) \) with \( G(\Sigma) \). In this case, \( \mathbb{P}(\Sigma) = G(\Sigma) \) is a Farey graph.

The notion is quite robust. For example, if \( q \geq 2 \), let \( \mathbb{P}(\Sigma, q) \) be the graph with vertex set \( \mathbb{P}^0 \), and where \( x, y \) are adjacent if \( \iota(x, y) \leq q \). Thus, \( \mathbb{P} \subseteq \mathbb{P}(\Sigma, q) \), and one sees easily that the inclusion is a quasi-isometry. (We will implicitly make use of this fact later, since various constructions are well defined up to bounded intersection, hence up to bounded distance.)

Note that there is a map, \( \chi_{\Sigma} : \mathbb{P}(\Sigma) \rightarrow G(\Sigma) \), obtained by selecting one curve from the multicurve. It is well defined up to bounded distance (1, in fact). Also, up to bounded distance, it is also characterised by the fact that \( \iota(\tau, \chi_{\Sigma}(\tau)) \) is bounded.

We write \( \mathcal{X} \) for the set of subsurfaces of \( \Sigma \), where a \textit{subsurface} here is required to have no homotopically trivial boundary components, and not be \( S_{0,3} \), or a peripheral annulus. Also, we regard a subsurface as being defined up to homotopy. We will sometimes abuse notion by identifying a subsurface, \( X \), with a realisation thereof in \( \Sigma \), in which case, we take each component of \( \partial X \) to be either a component of \( \partial \Sigma \) or a non-peripheral curve disjoint from \( \partial \Sigma \). We decompose \( \mathcal{X} = \mathcal{X}_A \sqcup \mathcal{X}_N \) into annular and non-annular surfaces. There is a natural identification of \( \mathcal{X}_A \) with \( G^0(\Sigma) \), where an annulus gets identified with a core curve thereof.
Given $X, Y \in \mathcal{X}$, we have the following pentachotomy:

1. $X = Y$.
2. $X \prec Y$: $X \neq Y$, and $X$ can be homotoped into $Y$ but not into $\partial Y$.
3. $Y \prec X$: $Y \neq X$, and $Y$ can be homotoped into $X$ but not into $\partial X$.
4. $X \wedge Y$: $X \neq Y$ and $X, Y$ can be homotoped to be disjoint.
5. $X \bowtie Y$: none of the above.

Given $\gamma \in \mathcal{G}^0(X)$, we write $X(\gamma) \in \mathcal{X}_\lambda$ for the regular neighbourhood of $\gamma$. We write $\gamma \prec X$ to mean that $X(\gamma) \prec X$ etc.

Given $X \in \mathcal{X}_N$, we will write $\mathcal{G}(X)$ and $\mathcal{P}(X)$, respectively, for the curve graph and pants graph defined intrinsically to $X$. Note that we have an intrinsically defined map $\chi_X : \mathcal{P}(X) \longrightarrow \mathcal{G}(X)$.

We will also need to define subsurface projection for complete multicurves. We begin with a more general discussion.

Given $\alpha \in \mathcal{G}^0(\Sigma)$ and $X \in \mathcal{X}_N$, we will generally assume that $\alpha$ and $X$ are realised minimally, that is, in general position so that the number of components of $\alpha \cap X$ is minimal (equivalently $|\alpha \cap \partial X|$ is minimal). In this case, we write $\zeta_X(\alpha)$ for the set of components of $\alpha \cap X$ viewed as defined up to homotopy sliding the endpoints of arcs in the respective boundary components of $X$. This set is well defined (irrespective of the realisation) and has cardinality bounded in terms of $\xi(\Sigma)$.

Given $\alpha, \beta \in \mathcal{G}^0(\Sigma)$, we can define the relative intersection number, $\iota_X(\alpha, \beta)$, to be the maximal intersection number of $\delta \in \zeta_X(\alpha)$ and $\epsilon \in \zeta_X(\beta)$. (Here the intersection number of $\delta, \epsilon$ is the minimal cardinality of $\delta \cap \epsilon$ among realisations in the respective homotopy classes.) We can apply the same definitions to multicurves, $a, b$ in $\Sigma$. In fact $\iota_X(a, b) = \max\{\iota_X(\alpha, \beta) \mid \alpha \in a, \beta \in b\}$. Note that if $Y \in \mathcal{X}_N$, with $Y \preceq X$, then $\iota_Y(a, b) \leq \iota_X(a, b)$. Also, if $a, b$ both lie in $X$, then $\iota_X(a, b) = \iota(a, b)$. In practice, at least one of $a, b$ will always lie in the subsurface $X$.

Now, if $a$ is a multicurve in a $\Sigma$, write $a_X = a \cap \mathcal{G}^0(X) \subseteq \zeta_X(a)$. We also write $a_{\partial X} = a \cap \partial X$. Suppose that $a \in \mathcal{P}^0(\Sigma)$ is a complete multicurve. Then each component of $X \setminus a$ is a disc; an annulus with one boundary component in $a_X$; or an $S_{0,3}$, with all boundary components in $a_X \cup a_{\partial X}$.

Suppose that $Z \in \mathcal{X}_N$ is a component of $X \setminus a_X$ (not an $S_{0,3}$). If $\beta, \gamma \in \mathcal{G}^0(Z)$, then $\iota(\beta, \gamma)$ is bounded above in terms of $\max\{\iota_Z(a, \beta), \iota_Z(a, \gamma)\}$. Moreover, we can always find some $b_Z \in \mathcal{P}^0(Z)$ with $\iota(a, b_Z) \leq 1$. (For us, it would be enough for this to be bounded in terms of $\xi(\Sigma)$.) We choose such a $b_Z$ for each such $Z$, and set $c_X = a_X \cup \bigcup_Z b_Z \in \mathcal{P}^0(X)$. Moreover, it is well defined up to bounded intersection in $X$.

In fact, if $X, Y \in \mathcal{X}_N$, with $Y \preceq X$, we can perform this construction intrinsically to $X$ to give us an element $\psi_{Y,X} \in \mathcal{P}^0(Y)$ for any $a \in \mathcal{P}^0(X)$. We note:
Theorem 2.3. There is some $\psi$ such that the map $\psi(\sigma, \rho)$ bounds about curves in $a_Y$ to the multicurve $b$, we can arrange that $i_Y(b, \psi_Y a)$ is bounded in terms of $i_Y(a, b)$ and $\xi(\Sigma)$. From this the statement follows.

Extending to $\mathbb{P}(X)$ gives us a map $\psi_{YX} : \mathbb{P}^0(X) \to \mathbb{P}^0(Y)$ well defined up to bounded distance, which we think of as subsurface projection in the pants graphs.

Suppose $a, b \in \mathbb{P}^0(X)$. Then it is a simple exercise to find some $c \in \mathbb{P}^0(Y)$, with $\iota_Y(a, c)$ and $\iota_Y(b, c)$ both bounded above in terms of $i(a, b)$ and $\xi(\Sigma)$, by Lemma 2.1, this bounds $\rho_Y(c, \psi_{YX} a)$ and $\rho(c, \psi_{YX} b)$, hence also $\rho_Y(\psi_{YX} a, \psi_{YX} b)$. It now follows that the map $\psi_{YX} : \mathbb{P}^0(X) \to \mathbb{P}^0(Y)$ is coarsely lipschitz, and so extends to a coarsely lipschitz map $\psi_{YX} : \mathbb{P}(X) \to \mathbb{P}(Y)$.

Suppose $Z \in \mathcal{X}_N$ with $Z \leq Y$. Since $\psi_{ZY}$ is also coarsely lipschitz, we have that $\rho_Z(\psi_{ZY} c, \psi_{ZY} \psi_{YX} a)$ is bounded. In fact, we can extend $c$ arbitrarily to a multicurve $d \in \mathbb{P}(X)$, and by construction, we can suppose that $\psi_{ZX} d = \psi_{ZY} c$. Now since $\iota_Z(a, d) = \iota_Z(a, c) = \iota_Y(a, c)$ is bounded, by the previous paragraph again, we see that $\rho_Z(\psi_{ZX} d, \psi_{ZY} a)$ is bounded. It follows that $\rho_Z(\psi_{ZX} a, \psi_{ZY} \psi_{YX} a)$ is bounded. Here all bounds depend only on $\xi(\Sigma)$.

We deduce:

Lemma 2.2. Given $X, Y \in \mathcal{X}_N$ with $Y \leq X$, the map $\psi_{YX} : \mathbb{P}(Y) \to \mathbb{P}(X)$ is coarsely lipschitz. Moreover, if $Z \in \mathcal{X}_N$, with $Z \leq Y$, then $\rho_Z(\psi_{ZX} a, \psi_{ZY} \psi_{YX} a)$ is bounded for all $a \in \mathbb{P}(X)$. Here all bounds depend only on $\xi(\Sigma)$.

Given $X, Y \in \mathcal{X}_N$ with $Y \leq X$, we write $\theta_{YX} = \chi_X \circ \psi_{YX} : \mathbb{P}(X) \to \mathcal{G}(Y)$.

Note that if $a \in \mathbb{P}^0(X)$ and $\gamma \in a$ and then $\theta_{YX} a$ has bounded intersection with any component of $\gamma \cap X$. This means that $\theta_{YX}$ agrees, up to bounded distance, with the usual subsurface projection as defined in [MasM2].

We will often abbreviate $\psi_{YX}$ to $\psi_X$ and $\theta_{YX}$ to $\theta_Y$, where there is no confusion regarding the domain. We will abbreviate $\rho_X(a, b) = \rho_X(\psi_X a, \psi_X b)$ and $\sigma_X(a, b) = \sigma_X(\theta_X a, \theta_X b)$ for $a, b \in \mathbb{P}(\Sigma)$ etc.

The following version of the “distance formula” was described in [MasM2].

Given $a, b \in \mathbb{P}(\Sigma)$ and $r \geq 0$, write

$A_X(a, b; r) = \{ Y \in \mathcal{X}_N \mid Y \leq X, \sigma_Y(a, b) \geq r \}.$

Theorem 2.3. There is some $r_0$ depending only on $\xi(\Sigma)$, such that given any $r \geq r_0$, there exist $k_1 > 0, k_2, k_3, k_4 \geq 0$ such that for all $X \in \mathcal{X}_N$ and all $a, b \in \mathbb{P}(X)$, we have

$k_1 \rho_X(a, b) - k_2 \leq \sum_{Y \in A_X(a, b; r)} \sigma_Y(a, b) \leq k_3 \rho_X(a, b) + k_4.$

Some other facts are also immediate consequences of more general statements about subsurface projection. First note that if $X, Y \in \mathcal{X}_N$ with $Y \prec X$ or $Y \parallel X$,
then we have the standard subsurface projection \( \theta_Y X \in \mathcal{G}^0(Y) \), as usual defined up to bounded distance.

We have the following immediate consequence of the Bounded Geodesic Image Theorem of [MasM2]. If \( X, Y \in \mathcal{X}_N \), \( Y \prec X \), \( a, b \in \mathcal{P}(X) \), then \( \min\{\sigma_X(a, Y) + \sigma_X(b, Y) - \sigma_X(a, b), \sigma_Y(a, b)\} \leq r_0 \). Also the following is a consequence of Behrstock’s Lemma [Be] (see also [Man]). If \( X, Y, Z \in \mathcal{X}_N \), with \( Y \prec X \), \( Z \prec X \), \( Y \preccurlyeq Z \) and \( x \in \mathcal{P}_0(X) \cup \mathcal{X}_N \), then \( \min\{\sigma_Y(x, Z), \sigma_Z(x, Y)\} \leq r_0 \). Here, \( r_0 \geq 0 \) depends only on \( \xi(\Sigma) \).

For later reference, we recall a few facts regarding the low complexity cases. In the cases where \( \xi(\Sigma) \leq 2 \), we have already noted that \( \mathcal{P}(\Sigma) \) (or equivalently \( \mathcal{W}(\Sigma) \)) is Gromov hyperbolic. When \( \xi(\Sigma) = 1 \), \( \mathcal{P}(\Sigma) \) is the Farey graph, hence a quasitree, and its Gromov boundary, \( \partial \mathcal{P}(\Sigma) \cong \partial \mathcal{W}(\Sigma) \), is homeomorphic to the set of irrational numbers. When \( \xi(\Sigma) = 2 \), however, \( \partial \mathcal{W}(\Sigma) \) is connected. In fact, it follows from the description in [BroMas], that it naturally contains the boundary of the corresponding curve graph as a dense subspace. This was, in turn, shown to be homeomorphic to the Nöbeling curve in [HP]. In particular, \( \mathcal{W}(\Sigma) \) is not a quasitree.

### 3. Coarse median spaces

In this section, we recall the notion of a coarse median space. We show that \( \mathcal{W}(\Sigma) \) is naturally such a space. We give proofs of Theorems 1.1, 1.2 and 1.3, and derive some further results.

Recall that a median algebra is a set, \( M \), equipped with a ternary operation \( \mu : M^3 \to M \), satisfying \( \mu(a, b, c) = \mu(b, c, a) = \mu(b, a, c) \), \( \mu(a, a, b) = a \) and \( \mu(a, b, \mu(c, d, e)) = \mu(\mu(a, b, c), \mu(a, b, d), e) \) for all \( a, b, c, d, e \in M \). (See for example, [Is, BaH, Ve, R, Bo1] for further discussion.) A subalgebra is a subset closed under \( \mu \). An n-cube is a (sub)algebra isomorphic to the direct product \( \{-1, 1\}^n \).

The rank of \( M \) is the maximal \( n \) such that \( M \) contains an n-cube. (This will always be finite for the median algebras we deal with in this paper.) Given \( a, b \in M \), write \( [a, b] = \{x \in M \mid \mu(a, b, x) = x\} \) for the median interval from \( a \) to \( b \). A subset \( C \subseteq M \) is convex if \( [a, b] \subseteq C \) for all \( a, b \in C \). Intervals are always convex.

A median metric space is (equivalent to) a median algebra, \( M \), equipped with a metric, \( \rho \), such that if \( a, b, c \in M \), then \( c \in [a, b] \) if and only if \( \rho(a, b) = \rho(a, c) + \rho(c, b) \). (See, for example, [Ve, ChaDH, Bo4].) The vertex set of a finite CAT(0) cube complex has a natural structure as a median algebra. Indeed it is a median metric space with the metric induced by the combinatorial metric on the 1-skeleton. It turns out that every finite median algebra arises in this way [Che]. In this case, the rank of the median algebra equals the dimension of the cube complex.

If \( Q \subseteq M \) is an n-cube in a median metric space, \( M \), then its convex hull, \( \text{hull}(Q) \), is a median direct product of \( n \) real intervals. In fact, it is isometric to such in the \( l^1 \)-metric. Clearly, \( Q \) itself corresponds to the “corners” of \( \text{hull}(Q) \).
Suppose now that \((\Lambda, \rho)\) is a geodesic metric space. As in [Bo1], we define a \textit{coarse median} on \(\Lambda\) to be a ternary operation \(\mu : \Lambda^3 \rightarrow \Lambda\), satisfying:

\begin{enumerate}[(C1)]
    \item There are constants, \(k, h(0)\), such that for all \(a, b, c, a', b', c' \in \Lambda\) we have \(\rho(\mu(a, b, c), \mu(a', b', c')) \leq k(\rho(a, a') + \rho(b, b') + \rho(c, c')) + h(0)\), and
    \item There is a function, \(h : \mathbb{N} \rightarrow [0, \infty)\), with the following property. Suppose that \(A \subseteq \Lambda\) with \(1 \leq |A| \leq p < \infty\), then there is a finite median algebra, \((\Pi, \mu_\Pi)\) and maps \(\pi : A \rightarrow \Pi\) and \(\lambda : \Pi \rightarrow \Lambda\) such that for all \(x, y, z \in \Pi\) we have \(\rho(\lambda\mu_\Pi(x, y, z), \mu(\lambda x, \lambda y, \lambda z)) \leq h(p)\) and for all \(a \in A\), we have \(\rho(a, \lambda\pi a) \leq h(p)\).
\end{enumerate}

We say that \(\Lambda\) has rank at most \(n\) if we can always take \(\Pi\) to have rank at most \(n\) (as a median algebra).

We refer to \((\Lambda, \rho, \mu)\) as a \textit{coarse median space}. We refer to \(k, h\) as the parameters of \(\Lambda\).

A map, \(\phi : \Lambda \rightarrow \Lambda'\) between two coarse median spaces is an \(h\)-\textit{quasimorphism} if \(\rho'(\phi\mu(a, b, c), \mu'(\phi a, \phi b, \phi c)) \leq h\) for all \(a, b, c \in \Lambda\). We will use the same terminology when the domain is a median algebra.

We recall notion of an asymptotic cone [VaW, G]. Here, we use the notation of [Bo1, Bo8]. Given a countable set, \(\mathcal{Z}\), equipped with a non-principal ultrafilter, and a \(\mathcal{Z}\)-sequence, \((\Lambda_\zeta, \rho_\zeta)_\zeta\), of metric spaces, we obtain a limiting space, \((\Lambda^\infty, \rho^\infty)\), which is a complete metric space. In particular, if \(\Lambda_\zeta = \Lambda\) is constant, \(\rho\) is a fixed metric on \(\Lambda\), and \((t_\zeta)_\zeta\) is a \(\mathcal{Z}\)-sequence of positive numbers with \(t_\zeta \rightarrow 0\) (with respect to the ultrafilter) then we can set \(\rho_\zeta = t_\zeta \rho\), to get a limiting space, \((\Lambda^\infty, \rho^\infty)\). This is referred to as an \textit{asymptotic cone} of \((\Lambda, \rho)\) [VaW, G]. If \(\mu\) is a coarse median on \(\Lambda\), then we get a limiting lipschitz ternary operation, \(\mu^\infty : (\Lambda^\infty)^3 \rightarrow \Lambda^\infty\), so that \((\Lambda^\infty, \mu^\infty)\) is a median algebra.

If \((\Lambda, \rho, \mu)\) has rank at most \(n\), then \((\Lambda^\infty, \rho^\infty)\) has rank at most \(n\) (as a median algebra). It has locally compact dimension at most \(n\) (that is, any locally compact subset has dimension at most \(n\)). From this, it follows that \(\Lambda\) does not admit any quasi-isometric embedding of \(\mathbb{R}^{n+1}\) [Bo1]. Moreover, \(\rho^\infty\) is bilipschitz equivalent to a median metric, inducing the same median structure [Bo2, Bo8] (though this construction is not canonical). In fact, \(\rho^\infty\) is also bilipschitz equivalent to a CAT(0) metric [Bo4] (though this information is redundant in the case where \(\Lambda = \mathcal{W}(\Sigma)\), since \(\mathcal{W}(\Sigma)\) is already CAT(0), and it is easily verified that this property is preserved after taking asymptotic cones).

It is easily seen that the existence of a coarse median on a geodesic space is invariant under quasi-isometry [Bo1]. We will show that \(\mathcal{P}(\Sigma)\) admits such a structure, from which it will follow that \(\mathcal{W}(\Sigma)\) does too. This will be a consequence of Theorem 7.2 of [Bo8], which gives hypotheses (A1)--(A10) which imply the existence of medians.

Recall that we have a collection of maps, \(\psi_{YX} : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)\), for \(X, Y \in \mathcal{X}_N\) with \(Y \preceq X\). To be precise, the hypotheses laid out in [Bo8] require these to be defined for all \(X \in \mathcal{X}\). Rather than reformulate the hypotheses to take care
of this (which would be easy to do), we will simply set \( P(X) = G(X) \) to be a singleton for all \( X \in \mathcal{X}_A \). In this way, the relevant hypotheses relating to \( \mathcal{X}_A \) become trivial, so we simply ignore them.

We now go briefly through the hypotheses. First, (A1) says that \( G(X) \) is uniformly hyperbolic, which is certainly true here, by [MasM1]. (There are only finitely many topological types of \( X \) in any given \( \Sigma \).) Properties (A2) and (A3) require that the maps \( \chi_X \) and \( \psi_{YX} \) are uniformly coarsely Lipschitz. The case of \( \chi_X \) is trivial, and that of \( \psi_{YX} \) is a consequence of Lemma 2.2 here. Property (A2) also requires that \( \chi_X \) is cobounded, which is immediate here. Property (A4) is the second clause of Lemma 2.2. Property (A5) is a simple consequence of subsurface projection. As noted in [Bo8], Properties (A6) and (A7) are both immediate from Property (B1) there, which is a formulation of the distance formula given as Theorem 2.3 here. Properties (A8) and (A9) are respectively consequences of the Bounded Geodesic Image Theorem and Behrstock’s Lemma, as laid out at the end of Section 2. Finally, (A10) just says that we can combine complete multicurves on disjoint subsurfaces and then extend them to give a complete multicurve on the whole surface.

We have now verified the hypotheses of Theorem 7.2 of [Bo8]. We therefore get:

**Theorem 3.1.** There is a coarse median, \( \mu : \mathbb{P}(\Sigma)^3 \to \mathbb{P}(\Sigma) \), with the property that for all \( X \in \mathcal{X}_N \), the maps \( \theta_X : \mathbb{P}(X) \to G(X) \) are uniform quasimorphisms. Here, the constants depend only on \( \xi(\Sigma) \).

Here, the median on \( G(X) \) is the usual “centroid” map on a hyperbolic space (which gives it the structure of a rank-1 coarse median space). The fact that the maps \( \theta_X \) are uniform quasimorphisms determine the median on \( \mathbb{P}(\Sigma) \) up to bounded distance, so in particular, it is necessarily Map(\( \Sigma \))-equivariant up to bounded distance. Note that we can perform this construction intrinsically on subsurfaces, to give a median, \( \mu_X \), on each \( \mathbb{P}(X) \). Moreover the maps \( \psi_{YX} : \mathbb{P}(X) \to \mathbb{P}(Y) \) are necessarily all coarse median. (See [Bo8] for details.)

The results of [Bo8] immediately imply that \( \mathbb{P}(\Sigma) \) has rank at most \( \xi(\Sigma) \). This used Proposition 10.2 of [Bo1]. In fact, in the present situation, we see by that result that \( \mathbb{P}(\Sigma) \) has rank at most \( \xi_0(\Sigma) \). For this, note that the hypotheses (P1)–(P4) of Proposition 10.2 of [Bo1] are all satisfied. (For the purposes of this discussion, we could interpret the indexing set, \( X \), used in [Bo1], to be \( \mathcal{X}_N \) here. Alternatively, we could just deem any space associated with \( X \in \mathcal{X}_A \) to be a singleton.) In fact, (P3) calls for an upper bound, \( \nu \), on the number of disjoint elements of \( \mathcal{X}_V \) we can embed disjointly in \( \Sigma \). A simple topological argument gives \( \nu = \xi_0(\Sigma) \) in this case (see Lemma 4.2). Proposition 10.2 now tells us that \( \mathbb{P}(\Sigma) \) has rank at most \( \xi_0(\Sigma) \). In fact, as observed in Section 1, \( \mathbb{W}(\Sigma) \) (hence \( \mathbb{P}(\Sigma) \)) has rank at least \( \xi_0(\Sigma) \) (since it admits a (quasi)isometric embedding of \( \mathbb{R}^{\xi_0(\Sigma)} \)). Since these properties are quasi-isometrically invariant, they apply equally well to \( \mathbb{W}(\Sigma) \).
As observed above, from the general results of [Bo1], we can now deduce Theorems 1.2 and 1.5.

As remarked in the introduction, it is shown in [BeHS], that \( \mathbb{P}(\Sigma) \) satisfies a stronger set of axioms, namely those of a “hierarchically hyperbolic space”. They also verify that these in turn imply the properties (A1)–(A10) of [Bo8], and therefore recover the fact that \( \mathbb{P}(\Sigma) \) is coarse median. In order to obtain the rank bound, one needs to make the additional observation regarding (P3) of [Bo1] as explained above. (As the statement is formulated in [BeHS], the rank bound there is weaker, though it seems that their argument also yields a rank bound of \( \xi_0 \).)

One can derive some further results. As in Section 7 of [Bo8], we see that \(\mathcal{W}(\Sigma)\), hence \(\mathcal{W}_\infty(\Sigma)\), is “finitely colourable”, as defined in Section 12 of [Bo1]. As a consequence [Bo2], we see:

**Theorem 3.2.** \(\mathcal{W}_\infty(\Sigma)\) admits a bilipschitz embedding into a finite product of \(\mathbb{R}\)-trees. Moreover, we can take this embedding to be a median homomorphism with respect to the respective median structures.

An immediate consequence (see [Bo2]) is that \(\mathcal{W}_\infty(\Sigma)\) admits a bilipschitz equivalent median metric with the same median structure. (In fact, we only need to know that \(\mathcal{W}_\infty(\Sigma)\) is of finite rank for this statement, see Theorem 6.9 of [Bo8].)

In addition to the asymptotic cone, \(\mathcal{W}_\infty(\Sigma)\), we will also need to consider the “extended asymptotic cone” \(\mathcal{W}^*(\Sigma)\) (or equivalently \(\mathbb{P}^*(\Sigma)\)). This is defined in Section 5 of [Bo8]. This definition is the same as for the asymptotic cone, except we do not require distances to be finite. Formally, we can think of \(\mathcal{W}^*(\Sigma)\) as an \(\mathbb{R}^*\)-metric space, that is taking non-negative values in the extended reals, \(\mathbb{R}^*\), (that is the ultraproduct of the reals factored out by infinitesimals). For our purposes here, we can simply think of it as taking values in \([0, \infty]\) (i.e. where we identify all positive unlimited numbers, and denote them all by \(\infty\)). Two points of \(\mathcal{W}^*(\Sigma)\) lie in the same component if the distance between them is finite. Thus, \(\mathcal{W}_\infty(\Sigma)\) is a component, and (since \(\mathcal{W}(\Sigma)\) is quasi-homogeneous — \(\text{Map}(\Sigma)\) acts coboundedly), every component of \(\mathcal{W}^*(\Sigma)\) is an isometric copy of \(\mathcal{W}_\infty(\Sigma)\). We also note that \(\mathcal{W}^*(\Sigma)\) has a natural ternary operation, \(\mu^* : (\mathcal{W}^*(\Sigma))^3 \rightarrow \mathcal{W}^*(\Sigma)\), so that \(\mathcal{W}^*(\Sigma)\) is a topological median algebra (that is, the median operation is continuous). As a topological (or metric) space, \(\mathcal{W}^*(\Sigma)\) is just an uncountable disjoint union of copies of \(\mathcal{W}_\infty(\Sigma)\). The median endows it with additional structure which interrelates these components.

4. MULTICURVES AND PRODUCT STRUCTURE

To go further, we need to discuss the local “product structure” in \(\mathbb{P}(\Sigma)\). Much of this fits into the more general picture (applicable to the mapping class group and to the Teichmüller metric) as described in [Bo8]. We briefly outline this.

Let \(\tau \subseteq \mathcal{G}^0(\Sigma)\) be a multicurve. Let \(\mathcal{X}_A(\tau) = \{X(\gamma) \in \mathcal{X}_A \mid \gamma \in \tau\}\) be the corresponding annular neighbourhoods, and let \(\mathcal{X}_N(\tau) \subseteq \mathcal{X}_N\) be the set of non-\(S_{0,3}\)
components of \( \Sigma \setminus \tau \). Let \( \mathcal{X}(\tau) = \mathcal{X}_A(\tau) \sqcup \mathcal{X}_N(\tau) \). If \( Y \in \mathcal{X}_N \), write \( \tau \cap Y \) to mean that there is some \( \gamma \in \tau \) with either \( \gamma \cap Y \) or \( \gamma \prec Y \).

We write \( T(\tau) = \{ a \in \mathbb{P}(\Sigma) | \tau \subseteq a \} \), and \( \mathcal{T}(\tau) = \prod_{X \in \mathcal{X}_N(\tau)} \mathbb{P}(X) \). Here, we give \( \mathcal{T}(\tau) \) the \( l^1 \) product metric. Note that it also has a product ternary operation, which gives it the structure of a coarse median space. We write \( T_\lambda(\tau) = \) (cube complex). We can clearly identify \( T \) components multicurves together with \( \tau \) and \( \nu \) mean that there is some \( \gamma \). Write \( \omega \) subsurface projections \( \psi \) : \( T(\tau) \longrightarrow T^0(\tau) \) and \( \nu : T^0(\tau) \longrightarrow T(\tau) \) for the inverse bijections. In fact, by combining the subsurface projections \( \psi_X \) for \( X \in \mathcal{X}_N(\tau) \), we get a map \( \psi_{\tau} : \mathbb{P}(\Sigma) \longrightarrow T^0(\tau) \). Write \( \omega_{\tau} = \nu_{\tau} \circ \psi_{\tau} : \mathbb{P}(\Sigma) \longrightarrow T(\tau) \). (We can assume that \( \omega_{\tau}|T(\tau) \) is just inclusion.) All the above maps are uniform quasimorphisms.

The definition of \( T(\tau) \) is quite robust. Note that, up to bounded Hausdorff distance, we could have defined \( T(\tau) \) to be the set of \( a \in \mathbb{P}(\Sigma) \) with \( \ell(a, \tau) \) bounded by some constant. In fact, to tie this in with the discussion in [Bo8], we set \( T(\tau; r) \) be the set of \( a \in \mathbb{P}(\Sigma) \) such that \( \sigma_Y(a, \tau) \leq r \) for all those \( Y \in \mathcal{X} \) satisfying \( \tau \cap Y \). (Note that there is a well defined projection \( \theta_{\tau} \tau \in G(Y) \) up to bounded distance.) We claim:

**Lemma 4.1.** There is some \( r_0 \geq 0 \) such that for all \( r \geq r_0 \), the Hausdorff distance \( \text{hd}(T(\tau), T(\tau; r)) \) is finite and bounded above in terms of \( r \) and \( \xi(\Sigma) \).

**Proof.** Clearly, if \( a \in T(\tau) \), then \( \theta_Y(a, \tau) \) is bounded for all \( Y \in \mathcal{X}_N \).

Conversely, suppose \( a \in T(\tau; r) \). By construction, \( \omega_\tau a \in T(\tau) \). We claim that \( \sigma_Y(a, \omega_\tau a) \) is bounded for all \( Y \in \mathcal{X}_N \). The statement then follows from the distance formula (given as Theorem 2.3 here). If \( \tau \cap Y \), this claim is immediate given that \( \sigma_Y(a, \tau) \) and \( \sigma_Y(\tau, \omega_\tau a) \) are bounded. Otherwise, \( Y \preceq X \) for some \( X \in \mathcal{X}_N(\tau) \). By definition of \( \omega_\tau \), we have \( \rho_X(a, \omega_\tau a) \) is bounded, so \( \sigma_Y(a, \omega_\tau a) \) is bounded as required.

Note also that the constructions of \( \nu_\tau, \psi_\tau \) and \( \omega_\tau \) agree, up to bounded distance, with those described in Section 9 of [Bo8], so the following are direct consequences of the more general discussion there.

First, \( \omega_\tau : \mathbb{P}(\Sigma) \longrightarrow T(\tau) \) is a “coarse gate map”. This means that \( \rho(\omega_\tau a, \mu(a, \omega_\tau a, c)) \) is uniformly bounded for all \( a \in \mathbb{P}(\Sigma) \) and \( c \in T(\tau) \). It follows that \( T(\tau) \) is uniformly quasiconvex in \( \mathbb{P}(\Sigma) \); that is, if \( a, b \in T(\tau) \) and \( c \in \mathbb{P}(\Sigma) \), then \( \rho(c, T(\tau)) \) is bounded above in terms of \( \rho(c, \mu(a, b, c)) \). From this, it follows that the map \( \nu_\tau : T(\tau) \longrightarrow T(\tau) \subseteq \mathbb{P}(\Sigma) \) is a quasi-isometric embedding of \( T(\tau) \) into \( \mathbb{P}(\Sigma) \).

To proceed we make a few topological observations about \( \Sigma \).

We first note that we can embed at most \( \xi(\Sigma) \) disjoint surfaces of complexity at least 1 in \( \Sigma \):

**Lemma 4.2.** Suppose that \( Y \subseteq \mathcal{X}_N \) satisfies \( X \cap Y \) for all distinct \( X, Y \in \mathcal{Y} \). Then \( |\mathcal{Y}| \leq \xi(\Sigma) \).
Proof. We write $e(X)$ for minus the Euler characteristic of $X$, so that $e(\Sigma) = 2g+p-2$. It is easily seen that $\sum_{Y \in \mathcal{Y}} e(Y) \leq e(\Sigma)$. Write $\mathcal{Y}_1 \subseteq \mathcal{Y}$ for the set of $S_{1,1}$ elements of $\mathcal{Y}$, and write $\mathcal{Y}_2 = \mathcal{Y} \setminus \mathcal{Y}_1$. Thus, $e(Y) = 1$ for all $Y \in \mathcal{Y}_1$, and $e(Y) \geq 2$ for all $Y \in \mathcal{Y}_2$. Write $A = |\mathcal{Y}_1|$ and $B = |\mathcal{Y}_2|$. We have $A \leq g$ and $A + 2B \leq e(\Sigma)$, so $2(A + B) \leq g + e(\Sigma) = \xi(\Sigma) + 1$, and so $|\mathcal{Y}| = A + B \leq \lfloor (\xi(\Sigma) + 1)/2 \rfloor = \xi_0(\Sigma)$ as required.

Note that it follows that if $\tau$ is any multicurve in $\Sigma$ then $|\mathcal{X}_N(\tau)| \leq \xi_0(\Sigma)$. (Recall that the notation $\mathcal{X}_N$ excludes all $S_{0,3}$'s.)

Definition. We say that a multicurve $\tau$ is good if $\Sigma \setminus \tau$ has exactly $\xi_0(\Sigma)$ components, none of which is an $S_{0,3}$.

For much of the discussion in Section 5, we will need to split into odd and even cases. (The odd case being somewhat simpler.)

Definition. We say that $\Sigma$ is odd (respectively even) if $\xi(\Sigma)$ is odd (respectively even).

In other words, $S_{g,p}$ is odd precisely when $g + p$ is even.

We distinguish the following types of multicurves. In the following, $\tau$ is assumed to be a good multicurve.

(T0): $\Sigma$ is odd and each component of $\Sigma \setminus \tau$ has complexity 1.

(T1): $\Sigma$ is even and $\Sigma \setminus \tau$ has exactly one component of complexity 2, and all other components have complexity 1.

(T2): $\Sigma$ is even and each component of $\Sigma \setminus \tau$ has complexity 1.

(Recall that complexity-1 corresponds to $S_{1,1}$ or $S_{0,4}$, and complexity-2 corresponds to $S_{1,2}$ or $S_{0,5}$.)

Lemma 4.3. Any good multicurve is one of the types (T0), (T1) or (T2) described above.

Proof. Let $\mathcal{Y} = \mathcal{X}_N(\tau)$. We use the notation of the proof of Lemma 4.2. Recall that $A \leq g$ and $A + 2B \leq 2g + p - 2$. Let $Z = \Sigma \setminus \bigcup \mathcal{Y}_1$. This is a connected non-empty surface (provided $\Sigma$ is not $S_{2,0}$).

If $\Sigma$ is odd, then $2(A + B) \leq 3g + p - 2$. We see that $A + B$ is maximised when $A = g$, and then $Z$ is an $S_{0,g+p}$. We can cut $Z$ into $(g + p - 2)/2$ $S_{0,4}$'s, and so the maximum is realised with $B = (g + p - 2)/2$. This is type (T0).

If $\Sigma$ is even, then $2(A + B) \leq 3g + p - 3$. In this case, the maximum is realised when $A \geq g - 1$. If $A = g$, then $Z$ is an $S_{0,g+p}$ which we can cut into $(g + p - 5)/2$ $S_{0,4}$'s together with an $S_{0,5}$, giving the maximal $B = (g + p - 3)/2$. This is type (T1). If $A = g - 1$, then $Z$ is an $S_{1,g+p-1}$ which we can cut either into $(g + p - 1)/2$ $S_{0,4}$'s (type (T2)), or else into $(g + p - 3)$ $S_{0,4}$'s and one $S_{1,2}$ (type (T1) again). Either way, we again get a maximal $B = (g + p - 1)/2$. □
We note further that if $\tau$ has type (T0) or (T1) then all curves of $\tau$ separate $\Sigma$. If $\tau$ is of type (T2), then any pair of non-separating curves will together separate. In fact, in (T0) or (T1) the complement of the union of the $S_{1,1}$ components of $\Sigma \setminus \tau$ is a planar (genus 0) surface (or empty in the case where $\Sigma$ is an $S_{2,0}$). In case (T2) this complement has genus 1. The set of non-separating curves are cylically arranged in this subsurface.

In case (T1), we will write $W(\tau) \in \mathcal{X}_N(\tau)$ for the complexity-2 component of $\Sigma \setminus \tau$.

Now if $\tau$ is a good multicurve, then all the factors of $T(\tau)$ are hyperbolic spaces. In fact, each is a bushy hyperbolic space in the sense that every point is (a bounded distance from) the centroid of three ideal points. We see that $T(\tau)$ is a quasi-isometrically embedded copy of a direct product of $\xi_0(\Sigma)$ bushy hyperbolic spaces. (We only really need that all components of $\Sigma \setminus \tau$ have complexity at most 2 for the above observation.)

The following will be needed later (see the proof of Lemma 5.10). Analogous statements can be found in Section 9 of [Bo8].

**Lemma 4.4.** Suppose that $\tau, \tau'$ are good multicurves with $\tau \neq \tau'$. Then there are constants, $k, t \geq 0$, depending only on $\xi(\Sigma)$, such that if $a \in T(\tau')$ and $r \geq 0$, then there is some $b \in T(\tau')$ with $\rho(b, T(\tau)) \geq r$ and $\rho(a, b) \leq kr + t$.

**Proof.** In fact, we show that for all $l \geq 0$, we can find $b \in T(\tau')$ with $l = \rho(a, b)$ and with $l \leq k\rho(b, c) + t$ for all $c \in T(\tau)$. The statement then follows setting $l = kr + t$.

In what follows, $\sim$ will denote “up to an additive constant” or “up to bounded distance”, where the bound only depends on $\xi(\Sigma)$. Also all “linear bounds” referred to are assumed to depend only on $\xi(\Sigma)$.

First note that there is some complexity-1 subsurface, $X \in \mathcal{X}_N(\tau')$, which intersects $\tau$ non-trivially. (Otherwise $\tau \subseteq \tau'$, so $\tau = \tau'$.) In particular, $\theta_X \tau \in \mathcal{G}(X)$ is defined. Moreover, $\theta_X c \sim \theta_X \tau$ for all $c \in T(\tau)$.

Let $\alpha \subseteq a$ be the component of $\tau'$ lying in the interior of $X$ (so $\theta_X a = \alpha$). Now $\mathcal{G}(X)$ is a Farey graph. In particular, we can find a geodesic ray, $\pi \subseteq \mathcal{G}(X)$, emanating from $\alpha$, such that for all vertices, $\beta \in \pi$, we have $\sigma_X(\theta_X \tau, \beta) \sim \sigma_X(\theta_X \tau, \alpha) + \sigma_X(\alpha, \beta)$.

Given such $\beta$, let $b = (\tau' \setminus \alpha) \cup \beta \in T(\tau')$ (so $\theta_X b = \beta$). Note that for all $Y \in \mathcal{X}_N \setminus \{X\}$, $\theta_Y b \sim \theta_Y a \in \mathcal{G}(Y)$. Therefore, by the distance formula (given as Theorem 2.3 here) $\rho(a, b)$ agrees with $\sigma_X(a, b) = \sigma_X(\alpha, \beta)$, up to linear bounds.

Moreover, again by the distance formula, $\sigma_X(b, c)$, is bounded above by a linear function of $\rho(b, c)$. Since $\theta_X c \sim \theta_X \tau$, it follows that $\sigma_X(a, b)$ is also. It follows that $\sigma(a, b) \leq k\rho(b, c) + t$, where $k, t$ depend only on $\xi(\Sigma)$.

Now, by continuity, we can arrange that $\sigma(a, b)$ takes whatever positive (integer) value we want (since $\pi$ also gives us a path in $T(\tau') \subseteq \mathcal{P}(\Sigma)$, and we have seen that $\sigma_X(a, b)$ hence also $\rho(a, b)$ can be made arbitrarily large). Therefore, we can take $\rho(a, b) = l$, so $l \leq k\rho(b, c) + t$ as claimed. \qed
We will also need:

**Lemma 4.5.** Suppose that \( \tau, \tau' \) are good multicurves which together fill \( \Sigma \). Then \( T(\tau) \) and \( T(\tau') \) uniformly diverge.

The hypothesis means that \( \tau \cup \tau' \) cuts \( \Sigma \) into discs and peripheral annuli, and the conclusion means that, for any \( r \geq 0 \), the diameter of \( N(T(\tau), r) \cap N(T(\tau'), r) \) is bounded above in terms of \( r \) and \( \xi(S) \). This follows exactly as in Lemma 9.6 of [Bo8].

We now move on to discuss the extended asymptotic cone, \( \mathbb{P}^*(\Sigma) \), and asymptotic cone, \( \mathbb{P}^\infty(\Sigma) \subseteq \mathbb{P}^*(\Sigma) \). The asymptotic cone is a well established notion (see [VaW, G]). The "extended" version thereof is used in [Bo8] (see the discussion at the end of Section 3 of the present paper).

As in [Bo8], we write \( \mathcal{U}G^0 \supseteq G^0 \) and \( \mathcal{U}\mathcal{X} \supseteq \mathcal{X} \) for the ultrapoducts of \( G^0 \) and \( \mathcal{X} \), etc. We have \( \mathcal{U}\mathcal{X} = \mathcal{U}\mathcal{X}_N \sqcup \mathcal{U}\mathcal{X}_A \). Given \( \gamma \in \mathcal{U}G^0 \), we will generally write \( (\gamma_\zeta)_\zeta \) for the corresponding \( \mathbb{Z} \)-sequence in \( G^0 \), etc. We extend the notation, \( X \sqsubseteq Y \), \( X \wedge Y \), \( X \cap Y \) etc. to \( \mathcal{U}\mathcal{X} \). If \( X \in \mathcal{U}\mathcal{X} \), we have limiting spaces, \( \mathbb{P}^*(X) \), \( \mathbb{G}^*(X) \) etc.

In what follows we will refer to elements of \( \mathcal{U}G^0 \), \( \mathcal{U}\mathcal{X} \) as "curves" and "subsurfaces", and to elements of \( G^0 \) and \( \mathcal{X} \) as "standard curves" and "standard subsurfaces". We can also speak of "multicurves" and "standard multicurves" etc.

Note that it makes sense to talk of the "topological type" and "complexity" of (non-standard) subsurfaces. Since there are only finitely many topological types possible in \( \Sigma \), these will always be standard. (A \( \mathbb{Z} \)-sequence taking finitely many values is almost always constant.) We can also refer to complementary components of multicurves, and so on. These will again be (non-standard) subsurfaces.

Now suppose that \( \tau \subseteq \mathcal{U}G^0 \) is a multicurve (in the above sense). We get a limiting closed subset, \( T^*(\tau) \subseteq \mathbb{P}^*(\Sigma) \), and uniformly lipschitz maps, \( \psi^*_x : \mathbb{P}^*(\Sigma) \rightarrow T^*(\tau) \), \( \nu^*_x : T^*(\tau) \rightarrow T^*(\tau) \), and \( \omega^*_x = \nu^*_x \circ \psi^*_x : \mathbb{P}^*(\Sigma) \rightarrow T^*(\tau) \). Recall that the spaces \( T(\zeta) \subseteq \mathbb{P}(\Sigma) \) are uniformly quasiconvex, and that the maps \( \omega^*_x : \mathbb{P}(\Sigma) \rightarrow T(\zeta) \) are uniform gate maps. It follows that \( T^*(\tau) \) is convex in \( \mathbb{P}^*(\Sigma) \), and that \( \omega^*_x \) is a gate map (that is, \( \omega^*_x x \in [x, c] \) for all \( x \in \mathbb{P}^*(\Sigma) \) and all \( c \in T^*(\tau) \)). We also note that \( T^*(\tau) \) is naturally median isomorphic to the direct product, \( \prod_{x \in \mathcal{U}\mathcal{X}_N(\tau)} \mathbb{P}^*(X) \). Indeed the isomorphism is bilipschitz with respect to the \( l^1 \) metric on the product. (Here \( \mathcal{U}\mathcal{X}_N(\tau) \) denotes the set of complexity-1 complementary components — in the "non-standard" sense alluded to above.)

Restricting to \( \mathbb{P}^\infty(\Sigma) \), we write \( T^\infty(\tau) = T^*(\tau) \cap \mathbb{P}^\infty(\Sigma) \). If this is non-empty, then \( T^\infty(\tau) = \prod_{x \in \mathcal{U}\mathcal{X}_N(\tau)} \mathbb{P}^\infty(X) \). Now, if \( \tau \) is good, then each \( \mathbb{P}(X_\zeta) \) is (almost always) a bushy hyperbolic space, and it follows that \( \mathbb{P}^\infty(X) \) is the complete \( 2^{\aleph_0} \)-regular \( \mathbb{R} \)-tree. In particular, it is furry (that is, each point has valence at least 3). Thus, \( T^\infty(\tau) \) is a median direct product of \( \xi_0(\Sigma) \) furry \( \mathbb{R} \)-trees.
We recall the following fact from [Bo8]. We need to assume that $\xi_0(\Sigma) \geq 2$ for this.

**Proposition 4.6.** Suppose that $D$ is a direct product of $\xi_0(\Sigma)$ furry $\mathbb{R}$-trees, and that $f : D \rightarrow \mathbb{P}^\infty(\Sigma)$ is continuous injective map with closed image. Then $f$ is a median homomorphism, and $f(D) \subseteq \mathbb{P}^\infty(\Sigma)$ is convex.

**Proof.** Noting that $\mathbb{P}^\infty(\Sigma)$ is bilipschitz equivalent to a median metric space of rank $\xi_0(\Sigma) \geq 2$, this is an immediate consequence of Proposition 4.7 of [Bo8]. $\square$

We also remark that, by the results of [KaKL] (see also [Bo5]) the product structure of $D$, hence that of $f(D)$, are also determined by its topology.

5. **Rigidity**

In this section, we discuss the quasi-isometric rigidity of the pants graphs. We give a proof of Theorem 1.3, and reduce the question of quasi-isometric rigidity to the combinatorial rigidity of certain curve graphs. Combinatorial rigidity will be discussed further in Section 6.

In this section, we assume that $\xi(\Sigma) \geq 3$, unless otherwise stated. We first consider quasicubes in $\mathbb{P}(\Sigma)$.

Let $Q = \{-1,1\}^{\xi_0}$ be a $\xi_0$-cube. (In fact, all “cubes” in this section will be $\xi_0$-cubes.) By an *ith side* of $Q$ we mean a pair of elements which differ precisely in their $i$th coordinates. Suppose that $\phi : Q \rightarrow \mathbb{P}(\Sigma)$ is an $h$-quasimorphism.

**Lemma 5.1.** There is some $k_0$ depending only on $h$ and $\xi(\Sigma)$ such that if $c,d$ and $c',d'$ are respectively $i$th and $j$th sides of $Q$, and if $X \in \mathcal{A}(\phi c, \phi d; k_0)$ and $Y \in \mathcal{A}(\phi c, \phi d; k_0)$, then either $i = j$ or $X \land Y$.

(Recall, from Section 2, that if $a,b \in \mathbb{P}(\Sigma)$, then $\mathcal{A}(a,b;r) = \{X \in \mathcal{X}_N \mid \sigma_X(a,b) \geq r\}$.)

**Proof.** This is just Lemma 10.4 of [Bo8] applied to the particular case of $\mathbb{P}(\Sigma)$. $\square$

We will fix $h = h_0$ (as determined below), and hence $k_0$, depending only on $\xi(\Sigma)$. We then abbreviate $\mathcal{A}(a,b) = \mathcal{A}(a,b;k_0)$. We say that $\phi : Q \rightarrow \mathbb{P}(\Sigma)$ is non-degenerate if $\mathcal{A}(\phi c, \phi d) \neq \emptyset$ for every side, $c,d$, of $Q$.

We also want to discuss cubes in $\mathbb{P}^*(\Sigma)$.

Suppose that is a $\xi_0$-cube in $\mathbb{P}^*$. Given $i \in \{1, \ldots, \xi_0\}$, let $a,b$ be an $i$th side of $Q$. We write

$$A_i(Q) = \{X \in \mathcal{U}\mathcal{X}_N \mid \theta_X^i a \neq \theta_X^i b\},$$

$$B_i(Q) = \{X \in \mathcal{U}\mathcal{X}_N \mid \psi_X^i a \neq \psi_X^i b\},$$

$$C_i(Q) = \{X \in \mathcal{U}\mathcal{X}_N \mid \theta_X^i[|a,b| \text{ is injective}]\},$$

$$D_i(Q) = \{X \in \mathcal{U}\mathcal{X}_N \mid \psi_X^i[|a,b| \text{ is injective}]\}.$$ 

Note that any two such sides are parallel, so the above are well defined independently of the choice of $a,b$. Since $\theta_X^i$ factors through $\psi_X^i$, we clearly have
$C_i(Q) \subseteq A_i(Q) \subseteq B_i(Q)$, and $C_i(Q) \subseteq D_i(Q) \subseteq B_i(Q)$. We also note that any complexity-$1$ surface in $D_i(Q)$ lies in $C_i(Q)$.

**Definition.** Given two $\xi_0$-cubes $Q, Q' \subseteq \mathbb{P}^*(\Sigma)$, we say that $Q'$ is smaller than $Q$ (or that $Q$ is bigger than $Q'$) if $Q' \subseteq \text{hull}(Q)$.

Note that, in this case, we can label the sides of $Q$ and $Q'$ so that any $i$th side of $Q'$ is parallel to a pair of points lying in the interval given by any $i$th side of $Q$. We also have $B_i(Q') \subseteq B_i(Q)$, $C_i(Q') \supseteq C_i(Q)$ and $D_i(Q') \supseteq D_i(Q)$.

Note that by Lemma 6.7 of [Bo8], there is some $h_0$, depending only on $\xi(\Sigma)$, such that for any cube, $Q$, there are $h_0$-quasimorphisms, $\phi_\xi : Q \to \mathbb{P}(\Sigma)$, such that $\phi_\xi x \to x$ for all $x \in Q$ (in the sense that the $\mathcal{Z}$-sequence $(\phi_\xi x)_\xi$ corresponds to the point $x$). Note that $\phi_\xi$ will be non-degenerate for almost all $\xi$.

The following is now a simple consequence of Lemma 5.1:

**Lemma 5.2.** If $X \in A_i(Q)$ and $Y \in A_j(Q)$, then either $i = j$, or $X \wedge Y$.

In what follows, we will split the discussion into “odd” and “even” cases. The odd case is somewhat simpler to describe, and we will deal mainly with that case first. For the moment, we will assume that $\Sigma$ is odd, unless otherwise stated. In statements of lemmas (in particular Lemmas 5.7 to 5.11) we will omit this hypothesis where it is not necessary. However, we will first only give proofs in the odd case, and describe later how to prove them in the even case.

**Lemma 5.3.** Suppose that $\Sigma$ is odd, and that $\phi : Q \to \mathbb{P}(\Sigma)$, is a non-degenerate quasimorphism. Then there is a good multicurve, $\tau$, and for each $i \in \{1, \ldots, \xi_0\}$, there is some $Y_i \in \mathcal{X}_N(\tau)$ such that if $c, d$ is any $i$th side of $Q$, then $A(\phi c, \phi d) = \{Y_i\}$.

**Proof.** By the definition of non-degeneracy, we can find, for each $i$, some $Y_i \in A(\phi c, \phi d)$, where $c, d$ is an $i$th side of $Q$. By Lemma 5.1, the $Y_i$ are pairwise disjoint. Since they all have complexity at least 1, they all have complexity exactly 1, and we see that there is a good multicurve, $\tau$, with $\mathcal{X}_N(\tau) = \{Y_1, \ldots, Y_{\xi_0}\}$. It now follows by Lemma 5.1 again that if $c', d'$ is any $i$th side of $Q$, then $A(\phi c', \phi d') \subseteq \{Y_i\}$, so $A(\phi c', \phi d') = \{Y_i\}$.

Note that by the distance formula (given as Theorem 2.3 here), we see that if $c, d$ is an $i$th side of $Q$, then $\rho(\phi c, \phi d)$ agrees to within linear bounds with $\sigma_{Y_i}(\phi a, \phi c)$ (since the only contribution to the formula comes from $Y_i$).

Given $i \in \{1, \ldots, \xi_0\}$, choose any $i$th side, $c, d$, of $Q$, and set $A(\phi, i) = A(\phi c, \phi d)$. (The choice of side is not important.)

We revert to the terminology of “curves” and “standard curves” etc. as at the end of Section 4.

**Lemma 5.4.** Suppose that $\Sigma$ is odd, and that $Q \subseteq \mathbb{P}^*(\Sigma)$ is a $\xi_0$-cube. Then there is a (non-standard) good multicurve, $\tau$, such that for each $i \in \{1, \ldots, \xi_0\}$, there is some $Y_i \in \mathcal{U} \mathcal{X}_N(\tau)$ such that $A_i(Q) = C_i(Q) = \{Y_i\}$. 
Note that \( \tau \) is determined by \( Q \), and we write \( \tau = \tau(Q) \).

Proof. By Lemma 6.7 of [Bo8], we have a \( \mathcal{Z} \)-sequence of \( h_0 \)-quasimorphisms, \( \phi_\xi : Q \to \mathbb{P}(\Sigma) \), with \( \phi_\xi x \to x \) for all \( x \in Q \). By Lemma 5.3 here, we have standard good multicurves, \( \tau_\xi \), with \( \mathcal{X}_N(\tau) = \{Y_1, \ldots, Y_{\xi_N}\} \) and with \( \mathcal{A}(\phi_\xi, i) = \{Y_i\} \).

This gives us a multicurve \( \tau \) and surfaces \( Y_i \) with \( \tau_\xi \to \tau \), \( Y_i \to Y_\xi \) and with \( \mathcal{U}\mathcal{X}_N(\tau) = \{Y_1, \ldots, Y_{\xi_0}\} \). As observed after Lemma 5.3, if \( c, d \) is an \( i \)-th side of \( Q \), then \( \rho_\xi(\phi_\xi c, \phi_\xi d) \) agrees with \( \sigma_{Y_i}(\phi_\xi c, \phi_\xi d) \) to within linear bounds. Passing to the limit, we see that since \( \rho^*(c, d) \neq 0 \), we have \( \sigma_{Y_i}^*(c, d) \neq 0 \), i.e. \( \theta_{Y_i}c \neq \theta_{Y_i}d \), so \( Y_i \in A_i(Q) \). It now follows that \( A_i(Q) = \{Y_i\} \). To see that \( Y_i \in C_i(Q) \), note that if \( Q' \subseteq Q \) is a smaller \( \xi_0 \)-cube, then \( A_i(Q') = A_i(Q) \), so applying the above to \( Q' \), we must have \( A_i(Q') = A_i(Q) = \{Y_i\} \), and the result follows easily. \( \square \)

Note that if \( Q, Q' \) are \( \xi_0 \)-cubes, with \( Q \) bigger than \( Q' \), then \( \tau(Q) = \tau(Q') \).

Recall that if \( \tau \subseteq \mathcal{U}\mathcal{G}_0^0(\Sigma) \) is a good multicurve, we have associated a closed convex subset \( T^*(\tau) \subseteq \mathbb{P}^*(\Sigma) \).

Lemma 5.5. Suppose that \( \xi(\Sigma) \) is odd. Suppose that \( \tau \subseteq \mathcal{U}\mathcal{G}_0^0(\Sigma) \) is a good multicurve, and that \( Q \subseteq T^*(\tau) \) is a \( \xi_0 \)-cube. Then \( \tau(Q) = \tau \).

Proof. Note that \( \text{hull}(Q) \subseteq T^*(\tau) \), and we can identify this set with \( \prod_i \mathbb{P}^*(Y_i) \) via \( v^* \), where \( Y_1, \ldots, Y_{\xi_0} \) are the complementary components of \( \tau \). Thus, \( \text{hull}(Q) \equiv \prod_i Y_i \), where each \( Y_i \) is a non-trivial interval in \( \mathbb{P}^*(Y_i) \). In particular, we see that \( \theta_{Y_i} \mid I_i \) is injective, and so \( Y_i \in C_i(Q) \). It follows by Lemma 5.4 that \( C_i(Q) = \{Y_i\} \), and so \( \tau(Q) = \tau \). \( \square \)

Note that it follows for an odd surface that if \( \tau \) and \( \tau' \) are good multicurves and \( T^*(\tau) \cap T^*(\tau') \) contains a \( \xi_0 \)-cube, then \( \tau = \tau' \).

Lemma 5.6. If \( \Sigma \) is odd, and \( Q \subseteq \mathbb{P}^*(\Sigma) \) is a \( \xi_0 \)-cube, then \( Q \subseteq T^*(\tau(Q)) \).

Proof. The proof follows exactly as with Lemma 12.6 of [Bo8], given that \( \omega_{\tau(Q)}^*: \text{hull}(Q) \to T^*(\tau) \) is injective. \( \square \)

We now restrict to \( \mathbb{P}^\infty(\Sigma) \subseteq \mathbb{P}^*(\Sigma) \).

We recall the following definition from Section 4 of [Bo8].

Definition. A tree-product, \( T \), in \( \mathbb{P}^*(\Sigma) \) is a convex subset which is median isomorphic to a direct product of \( \xi_0 \) non-trivial rank-1 median algebras. It is maximal if it is not contained in any strictly larger tree-product.

Note that \( T \) is bilipschitz equivalent to a direct product of \( \mathbb{R} \)-trees. (Since \( \mathbb{P}(\Sigma) \) is bilipschitz equivalent to a median metric space, and there it is isometric to a direct product of \( \mathbb{R} \)-trees in the \( l^1 \)-metric.) It is easily seen that the closure of a tree-product is a tree-product, and so any maximal tree-product is closed.

Note that in the above terminology, if \( \xi_0 \geq 2 \), then any closed subset of \( \mathbb{P}^\infty(\Sigma) \) homeomorphic to a direct product of \( \xi_0 \) furry \( \mathbb{R} \)-trees is a tree-product (by Proposition 4.4).
The following few statements (5.7 to 5.11) will be valid for both odd and even surfaces. We restrict proofs for the moment to the odd case.

**Lemma 5.7.** Suppose that $T \subseteq \mathbb{P}^\infty(\Sigma)$ is a tree-product. Then there is a good multicurve $\tau \subseteq \mathcal{U}\mathcal{G}^0(\Sigma)$ such that $T \subseteq T^\infty(\tau)$.

**Proof.** (For $\Sigma$ odd.) Let $Q \subseteq T$ be any $\xi_0$-cube in $T$, and set $\tau = \tau(Q)$. If $x \in T$, then there are $\xi_0$-cubes $P$ and $Q'$, with $Q, Q'$ both bigger than $P$, and with $x \in Q'$. Now $\tau(Q') = \tau(P) = \tau(Q) = \tau$, and so by Lemma 5.6, $x \in Q' \subseteq T^\infty(\tau)$. Thus $T \subseteq T^\infty(\tau)$. \qed

Note that (by Lemma 5.5), if $\Sigma$ is odd, then $\tau$ is unique.

**Corollary 5.8.** If $\tau$ is a good multicurve, then $T^\infty(\tau)$ is either empty or a maximal tree-product.

**Proof.** (For $\Sigma$ odd.) Suppose $T^\infty(\tau) \neq \emptyset$. Let $T \supseteq T^\infty(\tau)$ be a tree-product. By Lemma 5.7, $T \subseteq T^\infty(\tau')$ for some $\tau'$, since $T^\infty(\tau) \neq \emptyset$. It follows that $\tau = \tau'$ and that $T = T^\infty(\tau)$. \qed

Putting the above together, we see that a (closed) subset of $\mathbb{P}^\infty(\Sigma)$ is a maximal tree-product if and only if it has the form $T^\infty(\tau)$ for some good multicurve $\tau$ such that $T^\infty(\tau) \neq \emptyset$. In particular, each factor must be a furry tree. Moreover, by Proposition 4.6, a closed subset of $\mathbb{P}^\infty(\Sigma)$, homeomorphic to a product of $\xi_0$ furry trees is a tree-product. Therefore the collection of maximal tree-products is determined topologically (as the collection of maximal products of furry trees topologically embedded in $\mathbb{P}^\infty(\Sigma)$). We deduce (in the odd case):

**Lemma 5.9.** Suppose that $\Sigma$ and $\Sigma'$ are compact orientable surfaces with $\xi_0(\Sigma) = \xi_0(\Sigma')$ and that $f : \mathbb{P}^\infty(\Sigma) \rightarrow \mathbb{P}^\infty(\Sigma')$ is a homeomorphism. Then if $\tau \subseteq \mathcal{U}\mathcal{G}^0(\Sigma)$ is a good multicurve, there is a unique good multicurve, $\tau' \subseteq \mathcal{U}\mathcal{G}^0(\Sigma')$, such that $f(T^\infty(\tau)) = T^\infty(\tau')$.

To recover information back in $\mathbb{P}(\Sigma)$, we use the results of Section 14 of [Bo8]. The general argument follows almost exactly as with the proof of Lemma 14.5 of that paper (where a similar statement was proven for flats in the marking complex).

**Lemma 5.10.** Suppose that $\Sigma$ and $\Sigma'$ are compact orientable surfaces with $\xi_0(\Sigma) = \xi_0(\Sigma') \geq 2$. Suppose that $\phi : \mathbb{P}(\Sigma) \rightarrow \mathbb{P}(\Sigma')$ is a quasi-isometry. Then, given any good multicurve, $\tau$, in $\Sigma$, there is a good multicurve $\tau'$ in $\Sigma'$ such that $\text{hd}(T(\tau'), \phi T(\tau))$ is bounded above by some constant depending only on $\xi(\Sigma)$ and the parameters of $\phi$.

**Proof.** (If $\Sigma$ and $\Sigma'$ are both odd.) To simplify the account, we first show that there is such a bound for a given map $\phi$. To this end, $\mathcal{E}$ be the set of subsets of the form $T(\tau)$ as $\tau$ ranges over all good multicurves, and let $\mathcal{F} = \{\phi(E) \mid E \in \mathcal{E}\}$.
We need to verify that these collections satisfy the conditions (S1)–(S3) laid out in Section 14 of [Bo8].

Now (S1) simply says that each $E \in \mathcal{E}$ is coarsely connected (that is some uniform neighbourhood is connected). But this is clear, since it is the quasi-isometric image of a connected graph.

By Lemma 14.4 of [Bo8], Property (S2) holds provided the family $\mathcal{E}$ is “linearly divergent”. In our case, linear divergence is precisely the conclusion of Lemma 4.4 here.

As in Section 14 of [Bo8] Property (S3) is an immediate consequence of the fact that if $\tau \subseteq U\mathcal{G}^0$ is a good multicurve, then there is some good multicurve, $\tau' \subseteq U\mathcal{G}^0$, with $\phi^\infty(T^\infty(\tau)) \subseteq T^\infty(\tau')$. (See the discussion before Lemma 14.5 of [Bo8].)

We can thus apply Lemma 14.3 of [Bo8]. This tells us that for each $F \in \mathcal{F}$, there is some $E \in \mathcal{E}$ with $\text{hd}(E, F)$ bounded. In other words, this shows that the conclusion holds for a particular to quasi-isometry $\phi$.

To show that $k$ only depends on $\xi(\Sigma)$ and the quasi-isometry parameters, we consider simultaneously all quasi-isometries, $\phi$, of $\mathbb{P}(\Sigma)$ to $\mathbb{P}(\Sigma')$ with given parameter bounds, and let $\mathcal{F}$ be the set of all images $\phi(E)$ now allowing $\phi$ also to vary among such maps. This now gives a uniform bound as required. □

Recall that $T(\tau)$ is, up to bounded distance, the image of a quasi-isometric embedding, $\nu_\tau : T(\tau) \rightarrow \mathcal{E}(\Sigma)$, where $T(\tau) = \prod_{X \in X_N(\tau)} \mathbb{P}(X)$. We can elaborate on Lemma 5.10.

**Proposition 5.11.** Suppose that $\Sigma, \Sigma', \phi$ are as in Lemma 5.10. Then, given any good multicurve, $\tau$, in $\Sigma$, there is a good multicurve $\tau'$ in $\Sigma'$, a bijection, $\pi : X_N(\tau) \rightarrow X_N(\tau')$, and a quasi-isometry, $\phi_X : \mathbb{P}(X) \rightarrow \mathbb{P}(\pi(X))$ for each $X \in X_N(\tau)$, such that the maps $\nu_\tau \circ (\prod_{X} \phi_X)$ and $\phi \circ \nu_{\tau'} : T(\tau) \rightarrow \mathbb{P}(\Sigma')$ agree up to bounded distance. The bound, and the parameters of the maps $\phi_X$ depend only on $\xi$ and the parameters of $\phi$.

(Here, of course, $X_N(\tau')$ denotes a collection of subsurfaces of $\Sigma'$.)

**Proof.** (If $\Sigma$ and $\Sigma'$ are both odd.) By Lemma 5.10, we see that $\phi|T(\tau)$ is a bounded distance from quasi-isometry from $T(\tau)$ to $T(\tau')$. Thus, via the quasi-isometric embedding $\nu_\tau$ and $\nu_{\tau'}$, we get a quasi-isometry $\hat{\phi} : T(\tau) \rightarrow T(\tau')$. Now each of the factors of $T(\tau)$ and $T(\tau')$ is a bushy hyperbolic space (in this case, a quasitree). It therefore follows from [KaKL] (see also [Bo5]) that, up to bounded distance and permutation of factors, $\hat{\phi}$ splits as a product of quasi-isometries of the factors. (To apply the result of [KaKL] as stated one would need to observe, in addition, that each of the factors admits a cobounded isometric action. In any case, bushy is all that is really required for their argument. See [Bo5] for further discussion of this.) □

We move on to consider the even case.
Recall that in this case, a “good multicurve”, $\tau$, is of one of the types (T1) or (T2) described in Section 4. In case (T2), we denote the complexity-2 component of $\Sigma \setminus \tau$ by $W(\tau)$.

We will also need to refer to another class of multicurve:

(T1'): $\Sigma \setminus \tau$ has exactly one $S_{0.3}$ component, and exactly $\xi_0$ other components, all of complexity-1.

Again, $|X_N(\tau)| = \xi_0$ in this case. As with (T1), all curves of $\tau$ are separating.

(Expanding on Lemma 4.3, it’s not hard to see that any multicurve $\tau$, with $|X_N(\tau)| = \xi_0$ is of exactly one of the types, (T0), (T1), (T1') or (T2).)

Again, we begin by describing quasicubes. Suppose that $Q$ is a $\xi_0$-cube, and that $\phi : Q \to P(\Sigma)$ is a non-degenerate $h_0$-quasimorphism. Recall that, given $i \in \{1, \ldots, \xi_0\}$, we choose any $i$th side, $c, d$, of $Q$, and set $A(\phi, i) = A(\phi, \phi d)$.

We suppose that $\Sigma$ is even, and $Q$ is a $\xi_0$-cube.

**Lemma 5.12.** Suppose that $\phi : Q \to P(\Sigma)$ is a non-degenerate quasimorphism. Then there is a uniquely determined multicurve, $\tau = \tau(\phi)$ such that we can write $X_N(\tau) = \{Y_1, \ldots, Y_{\xi_0}\}$, so that exactly one of the following hold.

(U1): $\tau$ is of type (T1), and for all $i \in \{1, \ldots, \xi_0\} \setminus \{i_0\}$, $A(\phi, i) = \{Y_i\}$, where $i_0$ is the index for which $Y_{i_0} = W(\tau)$. Moreover, the elements of $A(\phi, i_0)$ together fill $W(\tau)$.

(U1'): $\tau$ is of type (T1'), and $A(\phi, i) = \{Y_i\}$ for all $i \in \{1, \ldots, \xi_0\}$.

(U2): $\tau$ is of type (T2), and $A(\phi, i) = \{Y_i\}$ for all $i \in \{1, \ldots, \xi_0\}$.

**Proof.** By non-degeneracy, $A(\phi, i) \neq \emptyset$ for all $i$. Also, by Lemma 5.2, all elements of $A(\phi, i)$ are disjoint from all elements of $A(\phi, j)$, whenever $i \neq j$. Since all these surfaces have complexity at least 1, there can be at most one index, say $i_0$, for which $A(\phi, i_0)$ does not consist of a single complexity-1 surface. In this situation, the surfaces in $A(\phi, i_0)$ fill some complexity-2 surface, $Y_{i_0}$. Write $Y_i$ for the unique element of $A(\phi, i) = \{Y_i\}$ when $i \neq i_0$. Let $\tau$ be the union of all relative boundary components of all the $Y_i$ in $\Sigma$. Then, $X_N(\tau) = \{Y_1, \ldots, Y_{\xi_0}\}$, and we see we are in situation (U1).

If there is no such index, $i_0$, then we can write $A(\phi, i) = \{Y_i\}$ for all $i \in \{1, \ldots, \xi_0\}$, where each $Y_i$ has complexity-1. We again write $\tau$ for the union of all relative boundary curves, and we see that in this case, we are either in case (U1') or case (U2).

Note that (as in the odd case) if $c, d$ is an $i$th side of $Q$ with $A(\phi, i) = Y_i$, then $\rho(\phi c, \phi d)$ agrees up to linear bounds with $H_Y(\phi c, \phi d)$. In the case where $i = i_0$ in (U1), we get instead that $\rho(\phi c, \phi d)$ agrees to within linear bounds with $\rho(W(\tau))(\phi c, \phi d)$. This follows by a similar argument. All contributions to the distance formula (as given as Theorem 2.3 here) in $P(\Sigma)$ come from subsurfaces of $X$, and so agree up to linear bounds, with the same formula in $P(W(\tau))$.

We now again pass to $P^*(\Sigma)$. We need the following general facts.
Lemma 5.13. Let $X, Y \in \mathcal{X}_N$. Suppose $a, b, c, d \in \mathbb{P}^*(\Sigma)$, with $c \in [a, d]$ and $b \in [a, c]$, with $\theta_X a \neq \theta_X b$, $\theta_X c \neq \theta_X d$ and $\theta_Y b \neq \theta_Y c$. Then either $X = Y$ or $X \wedge Y$.

Proof. This is an immediate consequence of Lemma 11.2 of [Bo8]. \hfill \square

Corollary 5.14. Suppose $Q \subseteq \mathbb{P}^*(\Sigma)$ is a $\xi_0$-cube, and that $i \in \{1, \ldots, \xi_0\}$. If $X \in C_i(Q)$, $Y \in A_i(Q)$, then $X = Y$ or $X \wedge Y$.

Proof. Let $a, d$ be an $i$th side of $Q$. By the definition of $A_i(Q)$, $\theta_Y a \neq \theta_Y d$. Since $\theta_X ||[a, d]$ is injective, $\theta_X a, \theta_X b, \theta_X c, \theta_X d$ are all distinct. By continuity of $\theta_Y$, we can find points $b, c \in [a, d] \setminus \{a, d\}$ with $\theta_Y b \neq \theta_Y c$. We now apply Lemma 5.13. \hfill \square

Lemma 5.15. Suppose that $Q \subseteq \mathbb{P}^*(\Sigma)$ is a $\xi_0$-cube. Then there is a uniquely determined multicurve, $\tau = \tau(Q)$, such that we can write $U\mathcal{X}_N(\tau) = \{Y_1, \ldots, Y_{\xi_0}\}$, so that exactly one of the following hold.

(V1): $\tau$ is of type (T1), and for all $i \in \{1, \ldots, \xi_0\} \setminus \{i_0\}$, $A_i(Q) = C_i(Q) = \{Y_i\}$, where $i_0$ is the index for which $Y_{i_0} = W(\tau)$. Moreover, $C_{i_0}(Q) \subseteq \{W(\tau)\}$ and $D_{i_0}(Q) \cap U\mathcal{X}_N(\tau) = \{W(\tau)\}$.

(V1'): $\tau$ is of type $(T1')$, and $A_i(Q) = C_i(Q) = \{Y_i\}$ for all $i \in \{1, \ldots, \xi_0\}$.

(V2): $\tau$ is of type (T2), and $A_i(Q) = C_i(Q) = \{Y_i\}$ for all $i \in \{1, \ldots, \xi_0\}$.

Proof. We proceed similarly as with the proof Lemma 5.4. Let $\phi_{\xi} : Q \rightarrow \mathbb{P}(\Sigma)$ be a sequence of non-degenerate $h_0$-quasimorphism converging on $Q \subseteq \mathbb{P}^*(\Sigma)$. We can assume that the type of $\tau(\phi_{\xi})$ is constant. By Lemma 5.12, this means that each $\phi_{\xi}$ is exactly as described by (U1), (U2) or (U1'), according to this type.

Suppose first that the type each $\tau(\phi_{\xi})$ is (T2). Write $\mathcal{X}_N(\tau) = \{Y_{1,\xi}, \ldots, Y_{\xi_0,\xi}\}$. Let $\tau \subseteq \mathcal{X}_N(\tau)$ be the limiting multicurve, and let $Y_i$ be the limit of $(Y_{i,\xi}).$ Thus, $U\mathcal{X}_N(\tau) = \{Y_1, \ldots, Y_{\xi_0}\}$. As with Lemma 5.4, we see that $A_i(Q) = C_i(Q) = \{Y_i\}$, and we are in case (V2).

If the type is (T1'), we similarly get the a limiting $\tau$, with $\mathcal{X}_N(\tau) = \{Y_1, \ldots, Y_{\xi_0}\}$, and with $Y_i \in C_i(Q)$. Now, if $Y \in A_i(Q) \setminus \{Y_i\}$, then by Lemma 5.2, it must be disjoint from each $Y_j$, with $j \neq i$. Therefore $Y \not\sqsubseteq Y_i$ giving a contradiction to Corollary 5.14. We conclude that $A_i(Q) = C_i(Q) = \{Y_i\}$.

Now suppose that the type is (T1). Let $\tau_0$ be the limiting multicurve. We can write $\mathcal{X}_N(\tau_0) = \{Y_1, \ldots, Y_{\xi_0}\}$, where $Y_i$ is the limit of $(Y_{i,\xi})$. Let $c, d$ be an $i_0$th side of $Q$. Now, $\rho(\phi_{\xi}c, \phi_{\xi}d)$ agrees to within linear bounds with $\rho_{W(\tau_0)}(\phi_{\xi}c, \phi_{\xi}d)$, and so it follows that $\psi_{W(\tau_0)}^* c \neq \psi_{W(\tau_0)}^* d$.

In fact, we claim that $\psi_{W(\tau_0)}^* [c, d]$ is injective. To see this, suppose that $e, f \in [c, d]$, with $e \neq f$. Let $Q' \subseteq \mathbb{P}^*(\Sigma)$ be the cube with $i_0$th side parallel to $e, f$, and all other sides parallel to those of $Q$. Thus, $Q'$ is a smaller cube. Now $A_i(Q') = C_i(Q') = A_i(Q) = C_i(Q) = \{Y_i\}$ for all $i \neq i_0$. By Lemma 5.2 applied to $Q'$, we see that all elements of $A_{i_0}(Q')$ are subsurfaces of $W(\tau_0)$. Applying what we already know to $Q'$ in place of $Q$, we see we must either be in the situation of
the previous paragraph, so that there is some complexity-1 subsurface of \(W(\tau_0)\) lying in \(C_{i_0}(Q)\); or else we are in the situation described in the present paragraph, so that \(\psi^{*}_{W(\tau_0)}e \neq \psi^{*}_{W(\tau_0)}f\). Either way, we must have \(\psi^{*}_{W(\tau_0)}e \neq \psi^{*}_{W(\tau_0)}f\). This proves the claim. In other words, we have shown that \(W(\tau_0) \in D_4(Q)\).

As observed above, each element of \(C_{i_0}(Q)\) is a subsurface of \(W(\tau_0)\). Suppose that \(C_{i_0}(Q)\) does not contain any complexity-1 surface. Then \(C_{i_0} \subseteq \{W(\tau_0)\}\). Moreover, since any complexity-1 surface in \(D_{i_0}(Q)\) would also lie in \(C_{i_0}(Q)\), we get that \(D_{i_0}(Q) = \{W(\tau_0)\}\). Setting \(\tau = \tau_0\), and \(Y_{i_0} = W(\tau)\), we see that we are in case (V1).

Suppose instead that there is some complexity-1 subsurface, \(Y \in C_{i_0}(Q)\). Applying Lemma 5.2 and Corollary 5.14 (similarly as for the type \((T1')\) case above) we see that \(C_{i_0}(Q) = \{Y\}\). Let \(\beta\) be the relative boundary component of \(Y\) in \(W(\tau_0)\), and let \(\tau = \tau_0 \cup \beta\). Set \(Y_{i_0} = Y \in U\mathcal{X}_N(\tau)\). We are now back in case (V1').

**Definition.** We will say that a \(\xi_0\)-cube, \(Q\), is of **type (1)** if \(\tau(Q)\) is of type \((T1)\) or of type \((T1')\) (i.e. \(Q\) is of type \((U1)\) or \((U1')\)). We will say that it is of **type (2)** if \(\tau(Q)\) is of type \((T2)\).

**Lemma 5.16.** Suppose that \(\Sigma\) is even and that \(Q \subseteq \mathbb{P}^*(\Sigma)\) is a \(\xi_0\)-cube. Then \(Q \subseteq T^*(\tau(Q))\).

**Proof.** This now follows exactly as with Lemma 5.6. 

Suppose that \(Q' \subseteq Q\) is a smaller cube. As noted earlier, \(C_i(Q) \subseteq C_i(Q')\). In fact, we now see that the only way that these can differ is if \(\tau(Q')\) is of type \((T1')\) and \(\tau(Q)\) is of type \((T1)\), obtained by deleting a single curve. It follows that \(\tau(Q') = \tau(Q)\) except in the above situation. In particular, \(Q, Q'\) are either both of type \((1)\), or else they are both of type \((2)\).

We now restrict attention to \(\mathbb{P}^\infty(\Sigma) \subseteq \mathbb{P}^*(\Sigma)\).

Suppose that \(T \subseteq \mathbb{P}^\infty(\Sigma)\) is a tree-product. Write \(T = \prod_{i=1}^{\xi_0} \Delta_i\), where \(\Delta_i\) are the \(\mathbb{R}\)-tree factors of \(T\), as a median algebra. If \(X \in U\mathcal{X}_N\), then we have well defined maps \(\theta^X_{\xi_0}|_{\Delta_i} : \Delta_i \longrightarrow \mathbb{G}^\infty(X)\) and \(\psi^X_{\xi_0}|_{\Delta_i} : \Delta_i \longrightarrow \mathbb{P}^\infty(X)\).

**Lemma 5.17.** Suppose \(T \subseteq \mathbb{P}^\infty(\Sigma)\) is a tree-product. Then either all \(\xi_0\)-cubes in \(T\) are of type \((1)\) or else they are all of type \((2)\).

**Proof.** Suppose \(Q_0, Q_1 \subseteq T\) are \(\xi_0\)-cubes. Then there are \(\xi_0\)-cubes \(Q'_0, Q'_1, Q''_0\), with \(Q_0, Q''_0\) bigger than \(Q'_0\), and with \(Q_1, Q''_0\) bigger than \(Q'_1\). It follows that they all have the same type. 

We refer to a tree-product as **type \((1)\)** or **type \((2)\)** accordingly.

The type \((2)\) case is simpler, and similar to the situation for an odd surface. In particular, following the same argument as with Lemma 5.5, we see that if \(\tau\) is of type \((T2)\), and \(Q \subseteq T^*(\tau)\), then \(\tau(Q) = \tau\). Moreover, similarly as with Lemma 5.7, we see that if \(T \subseteq \mathbb{P}^\infty(\Sigma)\) is a type \((2)\) tree-product, then there is a type \((T2)\) multicurve, \(\tau\), with \(T \subseteq T^\infty(\tau)\). As with Corollary 5.8, we conclude:
Lemma 5.18. If $\tau$ is a type (2) multicurve, then $T^\infty(\tau)$ is a either empty or a tree-product. Moreover, every maximal tree-product of type (2) has this form.

The type (1) case is a bit more complicated. We have:

Lemma 5.19. Suppose that $\Sigma$ is even, and that $T \subseteq \mathbb{P}^\infty(\Sigma)$ is a type (1) tree-product. Then there is a type (T1) multicurve, $\tau$, and some $i_0 \in \{1, \ldots, \xi_0\}$ such that for all $i \neq i_0$, there is a (unique) $Y_i \in \mathcal{U}N(\tau) \setminus \{W(\tau)\}$ such that $\theta_{Y_i}|\Delta_i$ is injective, and such that $\psi_{W(\tau)}|\Delta_{i_0}$ is injective.

(Note that we are allowing the possibility that there may be some (unique) complexity-1 surface $Y \prec W(\tau)$ with $\theta_Y|\Delta_{i_0}$ injective.)

Proof. Suppose first that all of the $\xi_0$-cubes in $T$ are of type (V1') (as in Lemma 5.17). If $Q_0, Q$ are $\xi_0$-cubes in $T$ with $Q$ bigger than $Q_0$, then we see that $\tau(Q) = \tau(Q_0)$, as in Lemma 5.18. It follows that $\tau(Q) = \tau$, say, is constant for all $\xi_0$-cubes, $Q \subseteq T$. (For if $Q, Q'$ are such, then we can find $\xi_0$-cubes, $Q_0, Q_0', Q'' \subseteq T$ with $Q, Q''$ bigger than $Q_0$, and with $Q', Q''$ bigger than $Q_0'$. More, we can index the elements of $\mathcal{U}N(\tau)$ consistently as $Y_1, \ldots, Y_{\xi_0}$, so that $Y_1$ the $i$th side of any $\xi_0$-cube $Q \subseteq T$ is (parallel to) an interval in $\Delta_i$. Now, if $a, b \in \Delta_i$ are distinct, let $Q$ be any $\xi_0$-cube with $i$th side $\{a, b\}$, say. Since $Y_i \in C_i(Q)$, we get $\theta_{Y_i}a \neq \theta_{Y_i}b$. In other words, $\theta_{Y_i}|\Delta_i$ is injective. We can now (arbitrarily and somewhat artificially) remove one of the non-peripheral boundary curves of $Z(\tau_0)$ to give $\tau$. If $Y_{i_0}$ is the element lying in $W(\tau)$, then since $\theta_{Y_{i_0}}|\Delta_{i_0}$ is injective, so is $\psi_{W(\tau)}|\Delta_{i_0}$.

We can therefore assume that there is a type (V1) cube, $Q \subseteq T$. Let $\tau = \tau(Q)$. We claim that if $i \neq i_0$, then $\theta_{Y_i}|\Delta_i$ is injective. For suppose $a, b \in \Delta_i$ are distinct. Let $Q'$ be the $\xi_0$-cube with $i$th side parallel to $\{a, b\}$, and all other sides parallel to the corresponding sides of $Q$. Thus, for all $j \neq i$, we have $C_j(Q') = C_j(Q)$, so $C_j(Q) = \{Y_j\}$ for $j \neq i, i_0$ and $C_{i_0}(Q) \subseteq \{W(\tau)\}$. It therefore follows that $Q'$ must also be of type (T1) with $\tau(Q') = \tau$, and so $C_i(Q') = \{Y_i\}$. In particular, $\theta_{Y_i}a \neq \theta_{Y_i}b$ as claimed. Finally, we claim that $\psi_{W(\tau)}|\Delta_{i_0}$ is injective. For suppose that $c, d \in \Delta_{i_0}$ are distinct. Let $Q''$ be any $\xi_0$-cube with $i_0$th side parallel to $\{c, d\}$. If $i \neq i_0$, then (since $\theta_{Y_i}|\Delta_i$ is injective) we have $Y_i \in C_i(Q'')$. It follows that $Q''$ is either of Type (U1) with $\tau(Q'') = \tau$, or Type (U2) with $\tau(Q'') \supseteq \tau$. Either way we get $\psi_{W(\tau)}c = \psi_{W(\tau)}d$ as required. \qed

It follows that any cube $Q \subseteq T$ has $\tau(Q) \supseteq \tau$. Thus, by Lemma 5.16, we have $Q \subseteq T^\infty(\tau(Q)) \subseteq T^\infty(\tau)$. Since any point of $T$ lies in such a cube, we have $T \subseteq T^\infty(\tau)$. We deduce:

Lemma 5.20. If $\tau$ is a type (T2) multicurve, then $T^\infty(\tau)$ is a either empty or a tree-product. Moreover, every maximal tree-product of type (2) has this form.

Since good multicurves are precisely those of type (T1) or (T2), we deduce Corollary 5.8 in the even case.
We can now proceed to prove Lemma 5.10 and Proposition 5.11 in the even case, hence in general.

In fact we can say more. Note that if \( \tau \) is a type (T1) multicurve, we can always find another type (T1) multicurve, \( \tau' \), such that \( \tau \cup \tau' \) is a type \( (T_1') \) multicurve. (To see this, let \( \beta \in \tau \) be a boundary component of \( W(\tau) \), and let \( \gamma \prec W(\tau) \) be any curve which cuts off an \( S_{0,3} \) with \( \beta \) as another boundary curve. We set \( \tau' = (\tau \setminus \beta) \cup \gamma \). Note that this uses our assumption that \( \xi(\Sigma) \geq 3 \), hence at least 4 in the even case.) In particular, \( T^\infty(\tau \cup \tau') \subseteq T^\infty(\tau) \cap T^\infty(\tau') \) is a tree-product. On the other hand, if \( \tau \) is of type (T2), and \( \tau' \) is any good multicurve different from \( \tau \), then \( T^\infty(\tau) \cap T^\infty(\tau') \) cannot contain any cube. Now any topological embedding of a \( \xi_0 \)-ball into \( P(\Sigma) \) must contain a \( \xi_0 \)-cube. (This is because any compact subset thereof lies inside a “cubulated” set: see Proposition 4.3 of [Bo8]. The statement there assumed that the image was closed and concluded that the whole disc was cubulated. But as noted at the end of Section 3 of that paper, one does not need to assume closed for the weaker statement given here to hold. A more direct argument when the disc is contained in a tree-product is given in [Bo5].) It follows that \( T^\infty(\tau) \cap T^\infty(\tau') \) it cannot contain any topological \( \xi_0 \)-ball.

The distinction between type (1) and type (2) is therefore detectable from the topology of \( P^\infty(\Sigma) \).

More formally, in either the odd or even case, we say that a maximal tree-product is isolated if its intersection with any other maximal tree-product does not contain any topological \( \xi_0 \)-ball. We see that \( \Sigma \) is odd if and only if all tree-products in \( P^\infty(\Sigma) \) are isolated. The topology of \( P^\infty(\Sigma) \) therefore determines the parity of \( \xi(\Sigma) \). Also \( \xi_0(\Sigma) \) is determined (for example, as the locally compact dimension of \( P^\infty(\Sigma) \), or as the maximal rank of a tree-product). These facts together determine \( \xi(\Sigma) \).

This proves Theorem 1.6.

For the following statement, we can drop the assumption that \( \xi(\Sigma) \geq 3 \).

**Proposition 5.21.** Suppose that \( \Sigma, \Sigma' \) are compact surfaces with \( P(\Sigma) \) quasi-isometric to \( P(\Sigma') \), then \( \xi(\Sigma) = \xi(\Sigma') \).

**Proof.** This follows by the above, if \( \xi(\Sigma), \xi(\Sigma') \geq 3 \). Note that such spaces are never hyperbolic. If \( \xi(\Sigma) = 2 \), then \( P(\Sigma) \) is hyperbolic but not a quasitree (see the remarks at the end of Section 2). If \( \xi(\Sigma) = 1 \), then \( P(\Sigma) \) is a quasitree. This deals with all cases. \( \square \)

Note that this is equivalent to Theorem 1.3. We will give a stronger statement later, see Theorem 6.7.

We also note that we can refine Lemma 5.10 and Proposition 5.11 so as to distinguish the type of a good multicurve, \( \tau \), in terms of the coarse structure of \( T(\tau) \) in the quasi-isometry type \( P(\Sigma) \). Thus, in Lemma 5.10, \( \tau' \) must have the same type as \( \tau \). This follows from the above discussion of \( P^\infty(\Sigma) \). (In the proof of Lemma 5.10 in the even case, we can take \( \mathcal{E} \) be the set of all tree-products of
type (T1), and ignore those of type (T2).) In fact, it also follows more directly, since \( \tau \) is of type (T0) or (T2) if and only if all factors of \( T(\tau) \) are quasitrees.

**Lemma 5.22.** In Lemma 5.10 and Proposition 5.11, if \( \tau \) is of type (T0) or (T1), then so is \( \tau' \) is also of type (T0) or (T1).

(Of course, we will eventually show that the parity of \( \Sigma \) and \( \Sigma' \) are equal, and so, retrospectively we see that the type of \( \tau' \) is preserved.)

We now apply this to identify certain subsurfaces of \( \Sigma \) in terms of the coarse geometry of \( \mathbb{P}(\Sigma) \). The discussion applies to both the odd and even cases, though with a few differences.

Given a multicurve, \( \tau \), write \( G(\tau) \leq \text{Map}(\Sigma) \) for the subgroup which preserves setwise each component of \( \tau \) and each component of \( \Sigma \setminus \tau \). If \( X \in \mathcal{X} \), write \( G(X) \leq \text{Map}(\Sigma) \) for the subgroup supported on \( X \). (Note that \( G(\tau) \) is a direct product of the groups \( G(X) \) as \( X \) ranges over \( \mathcal{X}_N(\tau) \).)

We recall the notation used in Section 15 of [Bo8] as follows.

We write \( \mathcal{B} = B(\mathbb{P}(\Sigma)) \) for the set of subsets of \( \mathbb{P}(\Sigma) \) defined up to finite Hausdorff distance. If \( A, B \in \mathcal{B} \), we write \( A \leq B \) to mean that some representative of \( A \) is contained in some representative of \( B \). This “coarse inclusion” defines a partial order on \( \mathcal{B} \).

Any subgroup, \( G \leq \text{Map}(\Sigma) \), determines an element \( B(G) \in \mathcal{B} \), namely the class of any orbit of \( G \) in \( \mathbb{P}(\Sigma) \). We will abbreviate \( B(\tau) = B(G(\tau)) \) and \( B(X) = B(G(X)) \). Note that \( G(\tau) \) acts coboundedly on \( T(\tau) \), and so \( B(\tau) \) is just the class of \( T(\tau) \). Also, if \( X \) is an annulus, the \( B(X) \) is just the class of bounded subsets. (Of course, the groups \( G(X) \) do not act discretely here, but that does not affect our discussion.)

Clearly if \( X \preceq Y \), then \( G(X) \leq G(Y) \), so \( B(X) \leq B(Y) \). If \( X \in \mathcal{X}_N(\tau) \), then \( G(X) \leq G(\tau) \) and so \( B(X) \leq B(\tau) \).

**Lemma 5.23.** If \( X, Y \in \mathcal{X}_N \) with \( B(X) \leq B(Y) \), then \( X \preceq Y \).

**Proof.** Let \( a \in \mathbb{P}(\Sigma) \) be any complete multicurve containing the relative boundary of \( Y \) in \( \Sigma \). If \( X \) is not contained in \( Y \), then there is some component, \( \alpha \), of \( a \), disjoint from \( Y \) (or peripheral in \( Y \)) which crosses or is contained in \( X \). If \( h \in G(Y) \), then \( ha = \alpha \). Write \( \beta = \theta_X \alpha \in G^0(\Sigma) \). Let \( g \in G(X) \) be any pseudoanosov in \( X \). Then \( \sigma_X(\beta, g^n \beta) \to \infty \) as \( n \to \infty \). Thus, \( \sigma_X(a, g^n a) \to \infty \).

On the other hand, since \( \alpha \) is a component of both \( a \) and \( ha \), we have \( \sigma_X(a, ha) \) bounded for all \( h \in G(X) \). Thus, \( \sigma_X(g^n a, G(Y)a) \to \infty \). But \( \rho(g^n a, G(Y)a) \) is linearly bounded below by its projection in \( G(X) \), and so \( \rho(g^n a, G(Y)a) \to \infty \), contradicting the assumption that \( B(X) \leq B(Y) \).

In particular, it follows that if \( B(X) = B(Y) \), then \( X = Y \).

**Definition.** We say that a non-empty subset, \( \mathcal{Y} \subseteq \mathcal{X}_N \), is compatible if there is a multicurve, \( \tau \), of either type (T0) or type (T1) such that \( \mathcal{Y} \subseteq \mathcal{X}_N(\tau) \). A subsurface, \( X \in \mathcal{X}_N \) is admissible if \( \{X\} \) is compatible.
If $\Sigma$ is odd, then $X$ is admissible if and only if either it is an $S_{1,1}$ or it is an $S_{0,4}$ and each component of the complement is even and meets $X$ in exactly one curve. If $\Sigma$ is even, then $X$ is admissible if and only if it is one of the following four possibilities: (1) an $S_{1,1}$; (2) an $S_{0,4}$ with all but one of the complementary components odd and all meeting $X$ in a single curve; (3) an $S_{0,4}$ with all complementary components even, one meeting $X$ in two curves, and each of the others meeting $X$ in a single curve; or (4) an $S_{1,2}$ or $S_{0,5}$ with all complementary components odd and meeting $X$ in a single curve. One can give a similar description of compatibility (though we won't need this here).

Note that maximal compatible sets are in bijective correspondence to type (T0) or (T1) multicurves — they are precisely those of the form $X_N(\tau)$.

We claim that we can recognise compatibility in terms of the coarse geometry of $\mathbb{P}(\Sigma)$.

**Lemma 5.24.** Suppose that $\Sigma, \Sigma'$ are compact surfaces with $\xi(\Sigma) = \xi(\Sigma) \geq 3$. Suppose that $\phi : \mathbb{P}(\Sigma) \to \mathbb{P}(\Sigma')$ is a quasi-isometry. If $X$ is an admissible subsurface of $\Sigma$, then there is a unique admissible subsurface, $\pi X$, in $\Sigma'$ such that $B(\pi X) = \phi B(X)$. Moreover, $\pi$ is a bijection between the set of admissible subsurfaces of $\Sigma$ and the set of admissible subsurfaces of $\Sigma'$. It preserves the complexity of the surface. Moreover, a set, $\mathcal{Y}$, of admissible surfaces of $\Sigma$ is compatible, if and only if their images, $\pi \mathcal{Y}$, are compatible in $\Sigma'$.

**Proof.** Let $X \subseteq \Sigma$ be an admissible subsurface of $\Sigma$, i.e. $X \in X_N(\tau)$ for some multicurve, $\tau$, of type (T0) or (T1). By Proposition 5.11, there is some multicurve, $\tau'$, in $\Sigma'$ with $\text{hd}(T(\tau'), \phi T(\tau)) < \infty$. Moreover, there is some $X' \in X_N(\tau')$ so that the factor of $T(\tau')$ corresponding to $X'$ is a finite Hausdorff distance from the $\phi$-image of the factor corresponding to $X$ in $T(\tau)$. In other words, $B(X') = \phi(B(X))$. By Lemma 5.22, $\tau'$ is also of type (T0) or (T1), and so $X'$ is an admissible subsurface of $\Sigma'$. By Lemma 5.23, $X'$ is uniquely determined, and we set $\pi X = X'$. Also, by Lemma 5.23, $\pi$ is injective. If $X, Y$ are compatible, then we can take $X, Y \in X_N(\tau)$, and so $\pi X, \pi Y$ are distinct elements of $X_N(\tau')$, hence compatible. Clearly this construction is invertible, so $\pi$ is a bijection, as claimed. Note that an admissible subsurface, $X$, has complexity 1 if and only if it does not properly contain another admissible surface, and this is detectable, again using Lemma 5.23. (Alternatively, $X$ has complexity 1 if and only if $B(X)$ is a quasitree.) Therefore $\pi$ preserves complexity. \qed

Note that, if $\tau$ is a (T0) or (T1) multicurve, then by construction, the map $\pi$ of Lemma 5.24, when restricted to $X_N(\tau)$, agrees with the map $\pi$ given by Proposition 5.11.

**Definition.** A terminal subsurface is a subsurface $X$, of $\Sigma$ which is either an $S_{1,1}$ or else an $S_{0,4}$ with all but one boundary components peripheral in $\Sigma$.
In other words, it is a complexity-1 surface cut off by a single curve in $\Sigma$. We will refer to such a curve as 1-separating. (We elaborate on this terminology in Section 6.)

Note that any terminal surface is admissible, and that two terminal surfaces (or equivalently the corresponding 1-separating curves) are disjoint, if and only if they are compatible.

**Lemma 5.25.** With the hypotheses of Lemma 5.24, $X$ is terminal in $\Sigma$ if and only if $\pi X$ is terminal in $\Sigma'$.

**Proof.** We show that we can recognise terminal surfaces among admissible surfaces from the properties already verified for $\pi$.

Suppose that $X$ is admissible. We have already noted that we can distinguish complexity, so we can assume $X$ to be complexity-1. If $X$ is terminal, then we can find another admissible surface, $Y$, compatible with $X$, with the property that if $Y$ is any maximal compatible family containing $Y$, then $Y$ also contains $X$. (We take $Y$ to be any admissible surface meeting $X$ in its boundary.) Conversely, if $X$ is admissible and there is such a $Y$, then $X$ must be terminal. For if not, it must be an $S_{0,4}$ with $\Sigma \setminus X$ disconnected. Suppose $Y$ and $Y'$ are as given. Let $Z$ be the component of $\Sigma \setminus Y$ containing $X$. Now $X$ must be strictly contained in $Z$, so after applying some element of $G(Z)$ to $Y$ if necessary, we can certainly arrange that $Y'$ does not contain $X$. This gives a contradiction.

Note that the above criterion makes reference only to complexity, admissibility and compatibility of subsurfaces, and is hence preserved by $\pi$. \qed

We can express the conclusion of Lemma 5.25 as follows. Let $G_1(\Sigma)$ be the full subgraph of the curve graph $G(\Sigma)$ with vertex set the set of all 1-separating curves in $\Sigma$, then $\pi$ gives rise to an isomorphism from $G_1(\Sigma)$ to $G_1(\Sigma')$.

We will elaborate on this in Section 6, and explain how, for most surfaces at least, it can be used to deduce quasi-isometric rigidity of the pants graph.

### 6. Variations on the curve graph

Most of the content of this section is combinatorial. We will reduce the combinatorial rigidity of the graph, $G_1(\Sigma)$, introduced at the end of Section 5, to that of the “strongly separating curve graph” (here denoted $G_{1+}(\Sigma)$) as investigated in [Bo6]. The application to quasi-isometric rigidity of the pants graphs will be discussed at the end (see Theorems 6.7 and 6.8).

Let $\Sigma$ be a compact surface with $\xi(\Sigma) \geq 3$. Recall that $\text{Map}(\Sigma)$ acts cofinitely on the curve graph, $G(\Sigma)$. Given a subset, $C \subseteq G^0(\Sigma)$, we write $G(\Sigma, C)$ for the full subgraph of $G(\Sigma)$ with vertex set $C$. Clearly, if $C$ is $\text{Map}(\Sigma)$-invariant, then $\text{Map}(\Sigma)$ acts on $G(\Sigma, C)$.

**Definition.** We say that $G(\Sigma, C)$ is rigid if any automorphism of $G(\Sigma, C)$ is induced by an element of $\text{Map}(\Sigma)$.
(This is often expressed in terms of the curve complex, rather than the curve graph, but since the curve complex is the flag complex with 1-skeleton $\mathcal{G}(\Sigma)$, these notions are equivalent.)

It is natural to ask for which $\text{Map}(\Sigma)$-invariant subsets, $C$, the graph is $\mathcal{G}(\Sigma, C)$ rigid. It was shown in [Iv, Ko, L] that $\mathcal{G}(\Sigma)$ itself is rigid for all but finitely many $\Sigma$ (see also [S]). In fact, $\mathcal{G}(S_{g,p})$ is rigid if $g \geq 2$ or $(g = 1$ and $p \geq 2)$ or $(g = 0$ and $p \geq 6)$.

Various other cases are known. For example the rigidity of (most of) the non-separating curve graphs was established in [Ir] (though that is not directly relevant to present paper). Of particular interest here is when $C$ is the set of all separating curves. In this case, we write $\mathcal{G}_s(\Sigma) = \mathcal{G}(\Sigma, C)$. It follows from [BreM, Ki] that this is also rigid for all but finitely many surfaces (if $g \geq 1$). Clearly, if $g = 0$, then $\mathcal{G}_s(\Sigma) = \mathcal{G}(\Sigma)$, and this case was dealt with independently in [Ko] and [L]. Therefore, combining these we get:

**Theorem 6.1.** [Ko, L, BreM, Ki] $\mathcal{G}_s(S_{g,p})$ is rigid if $g \geq 3$ or $(g = 2$ and $p \geq 2)$ or $(g = 1$ and $p \geq 3)$ or $(g = 0$ and $p \geq 6)$.

Note that, under the above conditions, we see that the isomorphism class of the graph $\mathcal{G}(\Sigma, C)$ determines $\Sigma$ — since it determines $\text{Map}(\Sigma)$ up to isomorphism, hence $\Sigma$.

One can reduce certain other cases to this.

Given a separating curve $\gamma$, write $X^-(\gamma), X^+(\gamma)$ for the complementary components (as usual defined up to isotopy). Let $\kappa(\gamma) = \max\{\xi(X^-(\gamma)), \xi(X^+(\gamma))\}$.

**Definition.** We say that a curve $\gamma$ is $n$-separating if it is separating and $\kappa(\gamma) = n$. We say that $\gamma$ is $(n+)$-separating if it is separating and $\kappa(\gamma) \geq n$.

Thus, for example, a curve is 0-separating if cuts off an $S_{0,3}$. Similarly it is 1-separating if it cuts off an $S_{1,1}$ or $S_{0,4}$. (This accords with the terminology in Section 5.) In this case, we write $F(\gamma)$ for the surface cut off by $\gamma$. (If $\xi(\Sigma) \geq 4$, this is well defined.)

We write $C_n$ and $C_{n+}$, respectively, for the sets of $n$-separating and $(n+)$-separating curves, and write $\mathcal{G}_n(\Sigma) = \mathcal{G}(\Sigma, C_n)$ and $\mathcal{G}_{n+}(\Sigma) = \mathcal{G}(\Sigma, C_{n+})$. (We could write $\mathcal{G}_s(\Sigma) = \mathcal{G}_{0+}(\Sigma)$ in this notation.)

Recall that $\mathcal{G}_1(\Sigma)$ was the graph introduced at the end of Section 5, where we saw that it is determined by the coarse geometry of $\mathcal{P}(\Sigma)$. We can partition $C_1$ as $C_{1HT} \sqcup C_{1HS}$, depending on whether the curve bounds an $S_{1,1}$ or an $S_{0,4}$. Note that this partition is not a-priori deemed part of the structure of $\mathcal{G}_1(\Sigma)$. It can however be recovered, at least in most cases, as we show in Lemma 6.3.

In what follows we write $\mathcal{G}^c(\Sigma)$ for the complementary graph of $\mathcal{G}(\Sigma)$; that is, with the same vertex set and complementary edge set. If $C$ is any set of curves, we write $\mathcal{G}^c(\Sigma, C)$ for the full subgraph of $\mathcal{G}^c(\Sigma)$ with vertex set $C$. Clearly this is complementary to $\mathcal{G}(\Sigma, C)$. We write $\mathcal{G}^c_1(\Sigma) = \mathcal{G}^c(\Sigma, C_1)$.
Given a subsurface $X$ of $\Sigma$, we write $P(X) = \{ \gamma \in C_1 \mid \gamma \prec X \}$. Clearly, $P(X)$ is invariant under the subgroup, $G(X)$, of $\text{Map}(\Sigma)$, supported on $X$.

**Definition.** We say that $X$ is **big** if $\xi(X) \geq 2$ and $P(X) \neq \emptyset$.

(Note that the latter condition is redundant if $X$ has only one relative boundary component in $\Sigma$.)

In other words, $X$ is big precisely if it properly contains a terminal subsurface (as defined in Section 5) — note that the terminal subsurface cut off by any element of $P(X)$ is properly contained in $X$.

One sees easily (applying the group, $G(X)$), that if $X$ is big, then $P(X)$ is infinite, and the elements of $P(X)$ fill $X$. Moreover, $G_c(\Sigma, P(X))$ is connected (of diameter 2).

**Definition.** By a **division** of $\Sigma$, we mean an ordered pair, $X = (X^-, X^+)$, where $X^-$ and $X^+$ are big subsurfaces of $\Sigma$ which can be realised disjointly so that $\Sigma \setminus (X^- \cup X^+)$ is a disjoint union of (non-peripheral) annuli.

In other words, it is equivalent to a transversely oriented multicurve in $\Sigma$ which separates $\Sigma$ into two big subsurfaces. (Here the transverse orientation points from $X^-$ to $X^+$.)

Given a subset $P \subseteq C_1$, write $L(P)$ for the set of elements of $C_1 \setminus P$ which are adjacent in $G_c(\Sigma)$ to some element of $C_1$. By a **division** of $G_c(\Sigma)$, we mean an ordered pair, $P = (P^-, P^+)$, of disjoint infinite subsets, $P^-, P^+ \subseteq C_1$, such that $G_c(\Sigma, P^-)$ and $G_c(\Sigma, P^+)$ are connected, and $C_1 \setminus (P^- \cup P^+) = L(P^-) = L(P^+)$. In particular, this implies that every curve in $P^-$ is disjoint from every curve in $P^+$, and that every curve of $C_1 \setminus P^\pm$ crosses some curve of $P^\mp$. (We should imagine the curves in $P^-$ and $P^+$ as lying in two complementary surfaces, $X^+$ and $X^-$, as we describe in the next lemma.)

Given a division $X = (X^-, X^+)$ of $\Sigma$, write $P = P(X) = (P^-, P^+)$, where $P^\pm = P(X^\pm)$.

**Lemma 6.2.** The map $[X \mapsto P(X)]$ is a bijection between divisions of $\Sigma$ and divisions of $G_c(\Sigma)$.

**Proof.** Let $X$ be a division of $\Sigma$. We have already observed that $P^\pm$ is infinite and that $G_c(\Sigma, P^\pm)$ is connected. Every curve of $P^-$ is disjoint from every curve of $P^+$. Any curve in of $C_1 \setminus P^\pm$ must cross $X^\mp$ and hence some curve of $P^\mp$. We see that $P(X)$ is a division of $G_c(\Sigma)$.

Conversely, suppose that $P$ is a division of $G_c(\Sigma)$. Let $X^\pm$ be the subsurface of $\Sigma$ filled by the curves of $P^\pm$. Since $G_c(\Sigma, P^\pm)$ is connected, so is $X^\pm$. By definition, $X^\pm$ is big. Also $X^-$ and $X^+$ are homotopically disjoint (otherwise some element of $P^-$ would cross some element of $P^+$). We need to show that these are complementary, in the sense that $\Sigma \setminus (X^- \cup X^+)$ consists only of annuli.

Suppose that $Z$ were some non-annular component of $\Sigma \setminus (X^- \cup X^+)$. We can assume that $Z$ meets $X^-$. Let $Y$ be the subsurface $X^- \cup Z$. This is big, and is
filled by elements of $C_1$. In particular, there must be some $\gamma \in C_1$, contained in $Y$ but not contained in $X^-$. Since $\gamma \notin P^-$, by hypothesis, it must cross some element of $P^+$, giving a contradiction. This shows that $\Sigma \setminus (X^- \cup X^+)$ is a disjoint union of annuli. We see that $\Sigma = (X^-, X^+)$ is a division of $\Sigma$. Now clearly, $P^\pm \subseteq P(X^\pm)$. In fact, $P^\pm = P(X^\pm)$, since any curve in $P(X^\pm) \setminus P^\pm$ would again have to cross some element of $P^\pm$, giving a contradiction. This shows that $P = P(X)$.

It is clear from the construction that this gives a bijection as claimed. \qed

We have therefore shown that we can “see” divisions of $\Sigma$ in terms of the structure of $G_1(\Sigma)$.

Given two divisions, $X, Y$, of $\Sigma$, write $X \leq Y$ to mean that $X^- \leq Y^-$, or equivalently, that $Y^+ \leq X^+$. We write $X < Y$ to mean that $X \leq Y$ and $X \neq Y$.

Clearly this defines a partial order on the set of divisions. Similarly, if $P, Q$, are divisions of $G_1(\Sigma)$, we write $P \leq Q$ to mean that $P^- \subseteq Q^-$, or equivalently that $Q^+ \subseteq P^+$. These notions are equivalent under the bijection defined by Lemma 6.2.

By definition, we are only allowed to “divide” $\Sigma$ into big subsurfaces. However, we will also want to include terminal subsurfaces in this. To this end we we make the following definition.

**Definition.** A slice of $\Sigma$ is an ordered pair, $X = (X^-, X^+)$, where each of $X^-$ and $X^+$ is either a terminal subsurface of $\Sigma$, or a big subsurface of $\Sigma$, and such that these can be realised disjointly so that $\Sigma \setminus (X^- \cup X^+)$ is a disjoint union of (non-peripheral) annuli.

In other words, it is the same as a division except that we are allowing $X^-$ or $X^+$ to be an $S_{1,1}$ or a terminal $S_{0,4}$.

Again, this can be recognised from $G_1(\Sigma)$. We can identify a complexity-1 slice, $X$, with an element, $\gamma$, of $C_1$ (the separating curve), together with a sign, $\pm$, indicating whether the complexity-1 surface is $X^-$ or $X^+$. Note that if $Y$ is a division, then $X < Y$ corresponds to saying that $\gamma \prec Y^-$ and the sign is $\pm$. Similarly $Y < X$ says that $\gamma \prec Y^+$ with sign $\pm$.

Using slices, we can now distinguish elements of $C_{1HT}$ and $C_{AHS}$, at least if $\xi(\Sigma) \geq 6$. Consider the following statement about an element $\gamma \in C_1$:

\((\ast)\): Suppose that $X, Y$ are slices of $\Sigma$ with $\gamma \prec X^+$ and $\gamma \prec Y^-$. Then there exist slices $Z, W$ of $\Sigma$ with $X < Z < W < Y$.

Note that this is detectable in terms of $G_1(\Sigma)$. For example, the statement that $\gamma \prec X^+$ is the same as saying that either $X$ corresponds to $P$ with $\gamma \in P^+$, or else $X$ corresponds to $(\beta, -)$, where $\beta \in C_1$ is a curve disjoint from $\gamma$.

**Lemma 6.3.** Suppose $\xi(\Sigma) \geq 6$, and $\gamma \in C_1$. Then $\gamma \in C_{AHS}$ if and only if it satisfies \((\ast)\).
Proof. Suppose first that $\gamma \in C_{4HS}$. Let $X, Y$ be as given. We can realise these so that $X^-, Y^+$ and $\gamma$ are pairwise disjoint. Let $U$ be the (possibly disconnected) subsurface $X^+ \cap Y^-$. This contains $\gamma$, and so $F(\gamma) \subseteq U$. Let $\beta_1, \beta_2, \beta_3 \subseteq U \cap \partial \Sigma$ be the other boundary components of $F(\gamma)$. Let $\alpha_1, \alpha_2$ be disjoint arcs in $U$ respectively connecting the relative boundary of $X^-$ to $\beta_1$ and to $\beta_2$. Let $Z^- \subseteq W^-$ be subsurfaces respectively obtained by taking regular neighbourhoods of $X^- \cup \alpha_1 \cup \beta_1$ and $X^- \cup \alpha_1 \cup \beta_1 \cup \alpha_2 \cup \beta_2$. (We can take both of these disjoint from $Y^+$. ) Let $Z^+, W^+$ be the closures of the complements. This gives slices, $Z, W$ with $X < Z < W < Y$ as required, thereby verifying $(\ast)$.

Suppose instead that $\gamma \in C_{1HT}$. Since $\xi(\Sigma) \geq 6$, $\Sigma \setminus F(\gamma)$ has complexity at least 4. Therefore we can find an arc, $\beta$, in $\Sigma \setminus F(\gamma)$ meeting $\gamma$ precisely in its endpoints, which cuts $\Sigma \setminus F(\gamma)$ into two surfaces each of complexity at least 1, and with connected relative boundary. Let $H$ be a regular neighbourhood of $F(\gamma) \cup \beta$ in $\Sigma$. Thus $H$ is an $S_{1,2}$. Let $X^-, Y^+$ be the components $\Sigma \setminus H$, and let $X^+, Y^-$ be their respective complements in $\Sigma$. These give us slices, $X, Y$, with $X < Y$ and $\gamma < X^+$ and $\gamma < Y^-$. Suppose that $Z, W$ are slices with $X < Z < W < Y$, as required by $(\ast)$. Then $H$ is a union of the subsurfaces $X^+ \cap Z^-, Z^+ \cap W^-$ and $W^+ \cap Y^-$, all containing at least an $S_{0,3}$. But clearly there is no room for this in an $S_{1,2}$, thereby giving a contradiction. $\square$

Note that this implies that we can detect the genus of $\Sigma$ as the maximal number of pairwise disjoint elements of $C_{1HT}$ we can find in $\Sigma$.

To detect the number of holes, we set $m(\Sigma)$ to be the maximal $m$ such that there is a chain of slices, $X_1 < X_2 < \cdots < X_m$, of length $m$.

Lemma 6.4. Assuming that $\xi(S_{g,p}) \geq 4$, we have: $m(S_{0,p}) = p-5$, $m(S_{1,p}) = p-2$ and $m(S_{g,p}) = 2g + p - 3$ for all $g \geq 2$.

Proof. To begin, recall that $2g + p - 2$ is the number of pants in any pants decomposition of $\Sigma = S_{g,p}$. It is therefore also the maximal number of essential $S_{0,3}$’s we can embed disjointly in $\Sigma$.

Suppose first that $g \geq 2$. In this case, we can find a pants decomposition, $F_1, F_2, \ldots, F_{2g+p-2}$, such that if $i < j < k$ then $F_j$ separates $F_i$ from $F_k$ in $\Sigma$, and moreover such that $F_1$ and $F_{2g+p-2}$ each have two of their boundary curves identified, and so give rise to $S_{1,1}$’s in $\Sigma$. Now let $X_i^- = \bigcup_{j=1}^i F_j$ and $X_i^+ = \bigcup_{j=i+1}^{2g+p-2} F_j$, for $i = 1, \ldots, 2g+p-3$. This gives a chain $X_1 < X_2 < \cdots < X_{2g+p-3}$. Conversely, given any chain $X_1 < X_2 < \cdots < X_m$, each of the surfaces $X_i^-$, $X_m^+$ and $X_i^+ \cap X_{i+1}^-$ for $1 \leq i \leq m - 1$ must contain an $S_{0,3}$, showing that $m + 1 \leq 2g + p - 2$. Therefore $m(\Sigma) = 2g + p - 3$.

The case when $g = 1$ is essentially the same, except in this case we lose 1, since one of the extreme surfaces (i.e. $X_1^-$ or $X_m^+$) must be an $S_{0,4}$, and this must accommodate two $S_{0,3}$’s. Similarly, if $g = 0$, we lose 2, since then both extreme surfaces will be $S_{0,4}$’s. $\square$
Let us summarise what we have so far detected in terms of $G_1(\Sigma)$. Note first that if $\xi(\Sigma) \leq 2$ then $G_1(\Sigma) = \emptyset$, and if $\xi(\Sigma) = 3$, then $G_1(\Sigma)$ is just an infinite set of vertices. If $\xi(\Sigma) = 4$, then $m(\Sigma) = 2$. If $\xi(\Sigma) = 5$, then $m(\Sigma) = 3$. If $\xi(\Sigma) \geq 6$, then $m(\Sigma) \geq 4$ unless $\Sigma = S_{3,0}$, in which case, $m(S_{3,0}) = 3$. However, we can distinguish $S_{3,0}$ from the complexity-5 surfaces (namely $S_{2,2}, S_{1,5}, S_{0,8}$) by the fact that in $S_{3,0}$ we can find three disjoint curves in $C_1$. Moreover, if $\xi(\Sigma) \geq 6$, we can determine $g$, via Lemma 6.3, as we have already noted. Since we also know $m(\Sigma)$, we can also determine $p$.

We have shown:

**Lemma 6.5.** Suppose that $\Sigma, \Sigma'$ are compact orientable surfaces with $G_1(\Sigma)$ isomorphic to $G_1(\Sigma')$. Then either $\xi(\Sigma), \xi(\Sigma') \leq 2$ or $\xi(\Sigma) = \xi(\Sigma') \geq 3$. Moreover, if $\xi(\Sigma) = \xi(\Sigma') \geq 6$, then $\Sigma$ and $\Sigma'$ are homeomorphic.

Note that this leaves open the question of distinguishing the different complexity-4 surfaces and the different complexity-5 surfaces (namely the classes $\{S_{2,1}, S_{1,4}, S_{0,7}\}$ and $\{S_{2,2}, S_{1,5}, S_{0,8}\}$).

We now move on to consider rigidity. We say that a slice, $X$, is *simple* if $\Sigma \setminus (X^- \cup X^+)$ is connected (i.e. a single annulus). In other words, a simple slice is essentially the same thing as a $(1+)$-separating curve together with a transverse orientation.

We claim that we can detect simple slices. First note that if $\xi(\Sigma) \leq 5$, then all slices are simple (since at least one of $X^-$ or $X^+$ is complexity-1, hence terminal, but the definition of a slice). We therefore suppose that $\xi(\Sigma) \geq 6$. Now a slice, $X$ is simple if and only if $\text{genus}(X^-) + \text{genus}(X^+) = \text{genus}(\Sigma)$. We can assume that $X^-$ and $X^+$ are big, otherwise $X$ is certainly simple (again from the definition). But now we can detect $\text{genus}(X^\pm)$ as the maximal number of disjoint $C_{1HT}$ curves contained in $X^\pm$.

We therefore have a means of describing $(1+)$-separating curves in $\Sigma$ in terms of $G_1(\Sigma)$. Such a curve, $\gamma$, corresponds to an unoriented simple slice. That is, either it is already an element of $C_1$, or else it corresponds to the unordered pair, $\{X^-(\gamma), X^+(\gamma)\}$, of subsets of $C_1$, and we have seen that we can recognise the set of unordered pairs which arise in this way. Moreover, we can also detect the disjointness of $(1+)$-separating curves from this information. (They are disjoint if and only if we can choose the transverse orientation so that that slices are nested.) In other words, we can reconstruct the whole of $G_{1+}(\Sigma)$ from $G_1(\Sigma)$.

We have shown:

**Lemma 6.6.** Suppose that $\Sigma, \Sigma'$ are compact orientable surfaces, and that $\xi(\Sigma) = \xi(\Sigma') \geq 4$. Then any isomorphism from $G_1(\Sigma)$ to $G_1(\Sigma')$ extends to an isomorphism from $G_{1+}(\Sigma)$ to $G_{1+}(\Sigma')$.

(We have not shown that the extension is unique, but in certain cases at least, it must be, as will follow from the discussion below.)
The above shows that if $G_{1+}(\Sigma)$ is rigid, then so is $G_1(\Sigma)$. (Indeed, the converse also holds, since it is not hard to recognise 1-separating curves in $G_{1+}(\Sigma)$.)

**Definition.** We say that $\Sigma$ is of rigid type if $G_{1+}(\Sigma)$ is rigid.

This is taken to imply that $\xi(\Sigma) \geq 4$. (Otherwise $G_{1+}(\Sigma)$ is either empty or a discrete set of points.) Note that if $p \leq 1$, then $G_{1+}(\Sigma) = G_s(\Sigma)$, and so applying the result of [BreM, Ki], given as Theorem 6.1 here, we see that $S_{g,0}$ and $S_{g,1}$ are of rigid type if $g \geq 3$. Also in [Bo6], it is shown that $\Sigma$ is of rigid type if $g(\Sigma) + p(\Sigma) \geq 7$.

We now proceed to applications to the pants graph, $P(\Sigma)$, of $\Sigma$. We immediately get:

**Theorem 6.7.** Suppose that $\Sigma, \Sigma'$ are compact orientable surfaces with $P(\Sigma)$ quasi-isometric to $P(\Sigma')$. Then $\xi(\Sigma) = \xi(\Sigma')$. Moreover, if $\xi(\Sigma) = \xi(\Sigma') \geq 6$, then $\Sigma$ is homeomorphic to $\Sigma'$.

**Proof.** Note that if $\xi(\Sigma) = 1$ if and only if $P(\Sigma)$ is a quasitree; $\xi(\Sigma) = 2$ if and only if $P(\Sigma)$ is hyperbolic and not a quasitree; and $\xi(\Sigma) \geq 3$ if and only of $P(\Sigma)$ is not hyperbolic. We can therefore assume that $\xi(\Sigma) \geq 3$. As observed at the end of Section 5, we then have that $G_1(\Sigma)$ and $G_1(\Sigma')$ are isomorphic, and so the statement then follows by Lemma 6.5. □

It is well known that $P(S_{1,2})$ is quasi-isometric to $P(S_{0,5})$ and that $P(S_{2,0})$ is quasi-isometric to $P(S_{0,6})$. (Indeed the corresponding Weil-Petersson spaces are isometric.) It remains unclear whether or not $P(S_{1,3})$ is quasi-isometric to $P(S_{0,6})$. Also the classes $\{S_{2,1}, S_{1,4}, S_{0,7}\}$ and $\{S_{2,2}, S_{1,5}, S_{0,8}\}$ remain unresolved by the above.

Regarding quasi-isometric rigidity, we can now show:

**Theorem 6.8.** Suppose that $\Sigma$ is a compact orientable surface of rigid type, and that $\phi : P(\Sigma) \rightarrow P(\Sigma)$ is a quasi-isometry. Then there is some $h \in \text{Map}(\Sigma)$ such that if $a \in P(\Sigma)$, then $\rho(\phi a, ha) \leq k$, where $k$ depends only on $\xi(\Sigma)$ and the parameters of the quasi-isometry, $\phi$.

As observed above, this applies, in particular, if $\Sigma = S_{g,0}$ or $\Sigma = S_{g,1}$ for $g \geq 3$, or if $g(\Sigma) + p(\Sigma) \geq 7$. We therefore see that Theorem 6.8 implies Theorem 1.4.

**Proof.** The map $\pi$ given by Lemma 5.24 determines an automorphism of $G_1(\Sigma)$. Therefore, by Lemma 6.6, after applying some element of $\text{Map}(\Sigma)$, we can assume this to be the identity on $G_1(\Sigma)$. In other words, if $X$ is any terminal subsurface of $\Sigma$, we have $\pi X = X$. But now if $X$ is any admissible subsurface of $\Sigma$, each component of $\Sigma \setminus X$ is filled by terminal subsurfaces of $\Sigma$, in the sense that any curve in $X$ must cross some terminal subsurface. (Possibly it is itself a terminal subsurface.) Now these subsurfaces determine $X$ uniquely, and so it follows that $X$ must be fixed by $\pi$. In other words, we have $\phi B(X) = B(X)$ for every admissible subsurface, $X$, of $\Sigma$. 

Now suppose that \( \tau \) is a good multicurve in \( \Sigma \). Lemma 5.10 gives us a good multicurve \( \tau' \) in \( \Sigma \) such that \( \text{hd}(T(\tau'), \phi(T(\tau))) \) is finite and bounded above in terms of \( \xi(\Sigma) \) and the parameters of \( \phi \). Also (as observed after Lemma 5.24) the map, \( \pi \), given by Lemma 5.24, when restricted to \( \mathcal{X}_N(\tau) \), agrees with the map \( \pi \) given by Proposition 5.11. Since this is the identity here, it implies that \( \mathcal{X}_N(\tau') = \mathcal{X}_N(\tau) \), and so \( \tau' = \tau \). In other words, we have shown that \( \text{hd}(T(\tau), \phi(T(\tau))) \) is uniformly bounded above for all good multicurves \( \tau \).

The remainder of the proof follows as with that of Theorem 5.12 of [Bo8]. Note that \( W(\Sigma) \) is undefined (or trivial) for \( S_{0,0}, S_{0,1}, S_{0,2}, S_{0,3} \). For \( S_{0,4}, S_{1,0}, S_{1,1} \) it is quasi-isometric to an infinite-valence tree (or the Farey graph), and is certainly not quasi-isometrically rigid in these cases. The cases left unresolved by the present paper are therefore:

\[ S_{0,5}, S_{0,6}, S_{1,2}, S_{1,3}, S_{1,4}, S_{1,5}, S_{2,0}, S_{2,1}, S_{2,2}, S_{2,3}, S_{2,4}, S_{3,2}, S_{3,3}, S_{4,2}. \]

The pairs \( \{S_{0,5}, S_{1,2}\} \) and \( \{S_{0,6}, S_{2,0}\} \) are essentially equivalent, since they have isometric Weil-Petersson geometries and commensurable mapping class groups. There are therefore essentially twelve cases left open. The case of \( W(S_{0,5}) \equiv W(S_{1,2}) \) is different in nature, since this space hyperbolic, as we have already noted. One could probably deal with a few more cases by elaborating on the arguments of [Bo6], though a complete answer is likely to require some new ideas.

\section*{References}


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