THE COARSE GEOMETRY OF THE TEICHMÜLLER METRIC

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ABSTRACT. We study the coarse geometry of the Teichmüller space of a compact surface in the Teichmüller metric. We show that this admits a ternary operation, natural up to bounded distance, which endows the space with the structure of a coarse median space whose rank is equal to the complexity of the surface. We deduce that Teichmüller space satisfies a coarse quadratic isoperimetric inequality. We describe when it admits a quasi-isometric embedding of a euclidean space, or a euclidean half-space. We give a weak form of quasi-isometric rigidity for Teichmüller space, and deduce that, apart from some well known coincidences, the Teichmüller spaces are quasi-isometrically distinct. We show that any asymptotic cone is bilipschitz equivalent to a CAT(0) space, and so in particular, is contractible.

1. INTRODUCTION

In this paper we explore various properties of the large scale geometry of the Teichmüller space in the Teichmüller metric of a compact surface. Our starting point is a combinatorial model of the Teichmüller space $\mathcal{R}$, which we use to show that Teichmüller space admits a coarse median structure in the sense of [Bo1]. From this, a number of facts follow immediately, though others require additional work. As we will note, some of these results have been obtained in some form before, while others seem to be new. The paper makes use of constructions in [Bo4], which studies the geometry of the mapping class group and the Weil-Petersson metric from a similar perspective. The idea of using medians (or “centroids”) in the mapping class group originates in [BeM].

Let $\Sigma$ be a compact orientable surface of genus $g$ with $p$ boundary components. Let $\xi = \xi(\Sigma) = 3g + p - 3$ be the complexity of $\Sigma$. Unless otherwise stated, we will assume in this introduction that $\xi(\Sigma) \geq 2$. We will sometimes use $S_{g,p}$ to denote a generic surface of this type.
We write $T(\Sigma)$ for the Teichmüller space of $\Sigma$, that is, the space of marked finite-area hyperbolic structures on the interior of $\Sigma$. We give $T(\Sigma)$ the Teichmüller metric. This endows it with the structure of a complete Finsler manifold, diffeomorphic to $\mathbb{R}^{\mathfrak{r}(\Sigma)}$. Note that the mapping class group, $\text{Map}(\Sigma)$, acts properly discontinuously on $T(\Sigma)$. The properties we are mainly interested in here are quasi-isometry invariant, so for most of the paper we will be referring instead to the “decorated marking graph”, $\mathcal{R}(\Sigma)$, which is a slight variation on the “augmented marking graph” of [D]. Both these spaces are equivariantly quasi-isometric to $T(\Sigma)$.

The central observation of this paper is that Teichmüller space admits an equivariant median, with similar properties to that of the mapping class group [BeM, Bo1]. More specifically, we show:

**Theorem 1.1.** There is a ternary operation, $\mu : T(\Sigma)^3 \rightarrow T(\Sigma)$, which is canonical up to bounded distance, and which endows $T(\Sigma)$ with the structure of a coarse median space of rank $\mathfrak{r}(\Sigma)$.

A more precise formulation of this (for $\mathcal{R}(\Sigma)$) is given as Theorem 4.1. Roughly speaking, this means that $\mathcal{R}(\Sigma)$ behaves like a median algebra of rank $\mathfrak{r}(\Sigma)$ up to bounded distance. In fact, any finite subset of $T(\Sigma)$ lies inside a larger finite subset of $T(\Sigma)$ which can be identified with the vertex set of a finite CAT(0) cube complex of dimension at most $\mathfrak{r}(\Sigma)$, in such a way that the median operation in $T(\Sigma)$ agrees up to bounded distance with the usual median operation in a cube complex (see Section 2). As with the mapping class group, or the Weil-Petersson metric, the median can be characterised in terms of subsurface projection maps. Moreover, it is $\text{Map}(\Sigma)$-equivariant up to bounded distance.

Theorem 1.1 is used here to prove various facts about the quasi-isometry type of $T(\Sigma)$.

For example:

**Theorem 1.2.** For any compact surface $\Sigma$, $T(\Sigma)$ satisfies a coarse quadratic isoperimetric inequality.

Here the term “coarse quadratic isoperimetric inequality” refers to the standard quasi-isometric invariant formulation of the quadratic isoperimetric inequality, and will be made more precise in Section 5.

We also have the following consequence of Theorem 1.1, which can be found in [EMR1] (modulo the comment below).

**Theorem 1.3.** There is a quasi-isometric embedding of a euclidean $n$-dimensional half-space into $T(\Sigma)$ if and only if $n \leq \mathfrak{r}(\Sigma)$. 
Theorem 1.3 was proven by different methods in [EMR1]. (There, the result was erroneously stated for the full euclidean plane, though their argument gives a correct proof if one substitutes this for a half-space. There is a mistake in the very last sentence of the paper: the proposed construction of a quasiflat can only be applied in the cases described below.)

Regarding quasi-isometric embeddings of $\mathbb{R}^\xi(\Sigma)$, we will show:

**Theorem 1.4.** There is a quasi-isometric embedding of the euclidean space, $\mathbb{R}^{\xi(\Sigma)}$, into $\mathcal{T}(\Sigma)$ if and only if $\Sigma$ has genus at most 1, or is a closed surface of genus 2.

Theorem 1.4 turns out to be equivalent to finding an embedded $(\xi(\Sigma) - 1)$-sphere in the curve complex of $\Sigma$, which we will see happens precisely in the above cases (see Proposition 5.6).

Note that Theorem 1.3 immediately implies that if $\Sigma$ and $\Sigma'$ are both compact surfaces and $\mathcal{T}(\Sigma)$ is quasi-isometric to $\mathcal{T}(\Sigma')$, then $\xi(\Sigma) = \xi(\Sigma')$. (One can distinguish some further cases by bringing Theorem 1.4 into play.) In fact we have:

**Theorem 1.5.** If $\mathcal{T}(\Sigma)$ is quasi-isometric to $\mathcal{T}(\Sigma')$ then $\xi(\Sigma) = \xi(\Sigma')$, and if $\xi(\Sigma) = \xi(\Sigma') \geq 4$, then $\Sigma$ and $\Sigma'$ are homeomorphic. Moreover, if one of $\Sigma$ or $\Sigma'$ is homeomorphic to $S_{1,3}$, then they both are.

This statement can be paraphrased by saying that if two Teichmüller spaces are quasi-isometric then they are isometric, since it is well known each of the pairs $\{S_{2,0}, S_{0,6}\}$, $\{S_{1,2}, S_{0,5}\}$ and $\{S_{1,1}, S_{0,4}\}$ have isometric Teichmüller spaces. They are all different apart from the above coincidences. Theorem 1.5 is a simple consequence of Theorem 1.6 below, together with the corresponding statement for the mapping class group.

Recall that we have a thick part, $\mathcal{T}_T(\Sigma)$, of $\mathcal{T}(\Sigma)$. Up to bounded Hausdorff distance, it can be defined in a number of equivalent ways. For example, given $\epsilon > 0$, it can be defined as the set those finite area hyperbolic structures with injectivity radius at least $\epsilon$. For this, we need to choose $\epsilon$ sufficiently small so that $\mathcal{T}_T(\Sigma)$ is connected. Note that $\text{Map}(\Sigma)$ acts cocompactly on $\mathcal{T}_T(\Sigma)$. So, for example, $\mathcal{T}_T(\Sigma)$ is a bounded Hausdorff distance from any $\text{Map}(\Sigma)$-orbit.

**Theorem 1.6.** If $\phi : \mathcal{T}(\Sigma) \rightarrow \mathcal{T}(\Sigma)$ is any quasi-isometry, then the Hausdorff distance from $\phi(\mathcal{T}_T(\Sigma))$ and $\mathcal{T}_T(\Sigma)$ is finite, and bounded above in terms of $\xi(\Sigma)$ and the quasi-isometric parameters of $\phi$.

Our proof will use asymptotic cones. However, it has been pointed out to me by Kasra Rafi that it is also a consequence of the results of
Mosher [Mo] and Minsky [Mi1, Mi2] which imply that a quasigeodesic in Teichmüller space is stable if and only if it lies a bounded distance from the thick part. Thus, up to bounded distance, the thick part can be characterised as the union of all stable quasigeodesics.

Now, up to bounded Hausdorff distance, $T_T(\Sigma)$, can be viewed as a uniformly embedded copy of (any Cayley graph of) $\text{Map}(\Sigma)$. It follows that $\phi$ gives rise to a quasi-isometry of $\text{Map}(\Sigma)$ to $T_T(\Sigma)$. By quasi-isometric rigidity of $\text{Map}(\Sigma)$, [BeKMM, Ham] (see also [Bo4]), it follows that $\phi|_{T_T(\Sigma)}$ agrees up to bounded distance with the map induced by an element of $\text{Map}(\Sigma)$.

With some more work, one can get:

**Theorem 1.7.** There is a function $l : [0, \infty) \rightarrow [0, \infty)$ depending on $\xi(\Sigma)$ and quasi-isometry parameters, with $l(t)/t \rightarrow 0$ as $t \rightarrow \infty$, such that if $\phi : T(\Sigma) \rightarrow T(\Sigma)$ is a quasi-isometry, then $\rho(x, \phi x) \leq l(\phi(x, T_T(\Sigma)))$ for all $x \in T(\Sigma)$.

This is a weak form of quasi-isometric rigidity. In fact, I am informed that full quasi-isometric rigidity (replacing the sublinear bound above with a constant one) has been obtained independently by other means, by Eskin, Masur and Rafi [EMR2]. From this, Theorems 1.5 and 1.6 follow.

The proofs of Theorems 1.3 to 1.7 involve studying an asymptotic cone, $T^\infty(\Sigma)$, of $T(\Sigma)$ (see [G, VaW]). As a consequence of Theorem 1.1, we know that $T^\infty(\Sigma)$ is a topological median algebra of rank at most $\xi(\Sigma)$, and can be biLipschitz embedded in a finite product of $\mathbb{R}$-trees [Bo1, Bo2]. In particular, it is biLipschitz equivalent to a median metric space. We also derive the following facts:

**Theorem 1.8.** Any asymptotic cone, $T^\infty(\Sigma)$, of $T(\Sigma)$ has locally compact dimension $\xi(\Sigma)$. It is biLipschitz equivalent to a CAT(0) space (and so, in particular, is contractible).

Here the *locally compact dimension* of a topological space is the maximal dimension of any locally compact subset.

In fact, analysing the structure of $T^\infty(\Sigma)$ will be a significant part of the work of this paper.

We remark that there are various other spaces naturally associated to a compact surface. Of particular note are the Weil-Petersson metric on Teichmüller space, the (Cayley graph of the) mapping class group, and the curve complex. Some discussion of the quasi-isometry types of these and other related spaces can be found in [Y]. Various results regarding rank and rigidity of such spaces can be found, for example, in [BeKMM, Ham, EMR1, Bo1, Bo4], and references therein.
Theorem 1.1 of this paper is proven in Section 4 (modulo the lower bound on rank, which will be a consequence of Theorem 1.3). Theorems 1.2, 1.3, 1.8 and half of 1.4 are proven in Section 5. Theorems 1.5, 1.6 and 1.7 are proven in Section 7. The remaining half of Theorem 1.4 is proven in Section 9.

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2. Background

In this section, we review a few items of background material.

2.1. Conventions and terminology. If \( x, y \in \mathbb{R} \), we often the notation \( x \sim y \) to mean that \( |x - y| \) is bounded above by some additive constant. The factors determining the relevant constant at any given moment, if not specified, should be clear from context. Ultimately, it should only depend on the parameters of the hypotheses, namely the complexity of the surface, or quasi-isometry constants etc. If \( a, b \) are points in a metric space, we write \( a \sim b \) to mean that they are bounded distance apart.

If \( x, y > 0 \), we will similarly write \( x \asymp y \) to mean that \( y \) is bounded above and below by increasing linear functions of \( x \). Again, the factors determining these functions should be clear from context.

We will generally behave as though the relations \( \sim \) and \( \asymp \) were transitive, though clearly each application of the transitive law will implicitly entail a change in the defining constants.

Except when we are working in the asymptotic cone, maps between our various metric spaces are generally defined up to bounded distance. It will generally be assumed that maps between graphs send vertices to vertices.

In the context of asymptotic cones, we will be using ultraproducts of various sets or graphs associated to a surface, for example curves, multicurves and markings. In this case, we will understand a “curve” to mean an element of the ultraproduct of the set of curves, and a “standard curve” to mean an element of the original set, that is, a curve in the traditional sense. We apply similar terminology to multicurves and subsurfaces etc. We will elaborate on this later.

Let \( \phi : (M, \rho) \to (M', \rho') \) is a map (not necessarily continuous) between metric spaces. We say that \( \phi \) is coarsely lipschitz if for all \( x, y \in M \), \( \rho'(\phi x, \phi y) \) is bounded above by a linear function of \( \rho(x, y) \). We say it is a quasi-isometric embedding if, in addition, \( \rho(x, y) \) is
bounded above by a linear function of $\rho'(\phi x, \phi y)$. A quasi-isometry is a quasi-isometric map with cobounded image (i.e., every point of $M'$ is a bounded distance from $\phi(M)$).

2.2. Marking graphs. Let $\Sigma$ be a compact orientable surface of genus $g$ with $p$ boundary components. Let $\xi = \xi(\Sigma) = 3g + p - 3$ be the complexity of $\Sigma$. We will write $S_{g,p}$ to denote the topological type of $\Sigma$.

As usual, a curve in $\Sigma$ will mean a homotopy class of essential non-peripheral simple closed curves (except when it refers to an element of the ultraproduct of such, as mentioned above). We write $\iota(\alpha, \beta)$ for the geometric intersection number of two curves, $\alpha, \beta$. We write $\text{Map}(\Sigma)$ for the mapping class group of $\Sigma$. A multicurve in $\Sigma$ is a set of pairwise disjoint curves. Here we will generally allow empty multicurves. We write $S = S(\Sigma)$ for the set of multicurves on $\Sigma$. A multicurve is complete if it cuts $\Sigma$ into $S_{0,3}$'s. A complete multicurve as exactly $\xi(\Sigma)$ components.

Central to our discussion will be the notion of a “marking graph”. One version of this was given in [MaM2], though there are many variations. We first describe the essential properties we require of it.

A marking graph is a connected graph, $M$, whose vertices are “markings” of $\Sigma$. By a “marking”, we mean a finite of curves in $\Sigma$ which fill $\Sigma$. We require that the set, $M^0$, of vertices, is $\text{Map}(\Sigma)$-invariant (under the natural action on markings induced by the action on curves). We also require that this action extends to an action on $M$, and that the action is cofinite. Note, in particular, that if $a, b \in M^0$ are equal or adjacent, and $\alpha \in a$, $\beta \in b$, then $\iota(\alpha, \beta)$ is bounded. (We should therefore think of a marking has having bounded self-intersection.) For many purposes, the above would be sufficient. Examples of such are the marking graph described in [MaM2]. Alternatively, we could choose any sufficiently large integers, $q \geq p > 0$ and set $M^0$ to be the set of markings, $a$, with $\iota(\alpha, \beta) \leq p$ for all $\alpha, \beta \in a$, and deem markings $a, b$ to be adjacent of $\iota(a, b) \leq q$ for all $\alpha \in a$ and $\beta \in b$. (In fact, we could take $p = q = 4$.) This was the definition used in [Bo1].

Here we will require a bit more, namely that some marking should contain a complete multicurve. Note that it follows that if $a \in M^0$ contains a multicurve, $\tau$, then $a$ is a bounded distance from some $b \in M^0$, where $b$ contains a complete multicurve containing $\tau$. We also require the following. Moreover, if $a, b \in M^0$ both contain a multicurve $\tau$, then they can be a connected by a path in $M$, whose vertices all contain $\tau$, and whose length is bounded above in terms of the distance
between $a$ and $b$ in $\mathcal{M}$. This is easy to arrange. For example, it is true of the marking graph described in [MaM2].

(We note that, in [MaM2] and its application in [D], the complete multicurve is viewed as part of the structure of the marking, though that is not essential here.)

In any case, the notion is quite robust. For some applications, it is convenient to suppose that the self-intersection bound on markings is sufficiently large.

We will fix one marking graph, and denote it by $\mathcal{M}(\Sigma)$.

2.3. Subsurfaces. By a subsurface in $\Sigma$, we mean a subsurface $X$ of $\Sigma$, defined up to homotopy, such that the intrinsic boundary, $\partial X$, of $X$ is essential in $\Sigma$, and such that $X$ is not a three-holed sphere. We write $\mathcal{X}$ for the set of subsurfaces. We can partition $\mathcal{X}$ as $\mathcal{X}_A \sqcup \mathcal{X}_N$ into annular and non-annular subsurfaces. Given a curve, $\gamma$, in $\Sigma$, we write $X(\gamma) \in \mathcal{X}_A$, for the regular neighbourhood of $\gamma$. Given $X \in \mathcal{X}$, write $\partial_{\Sigma} X$ for the relative boundary of $X$ in $\Sigma$, thought of as a multicurve in $\Sigma$.

Given $X, Y \in \mathcal{X}$, we have the following pentachotomy:

- $X = Y$.
- $X \prec Y$: $X \neq Y$, and $X$ can be homotoped into $Y$ but not into $\partial Y$.
- $Y \prec X$: $Y \neq X$, and $Y$ can be homotoped into $X$ but not into $\partial X$.
- $X \land Y$: $X \neq Y$ and $X, Y$ can be homotoped to be disjoint.
- $X \pitchfork Y$: none of the above.

We will be using subsurface projections to curve graphs and marking complexes.

We can associate to each $X \in \mathcal{X}$ the curve graph, $\mathcal{G}(X)$, in the usual way. Thus, if $X \in \mathcal{X}_N$, then the vertex set, $\mathcal{G}^0(X)$, is the set of curves in $X$, where two curves are adjacent if they have minimal possible intersection number. If $\xi(\Sigma) \geq 2$, this is 0. We write $\sigma^0_X$ for the combinatorial metric on $\mathcal{G}(X)$. (The caret subscript will be explained later.)

We will need to deal with annular surfaces differently. We begin with a general discussion.

Suppose that $A$ is a compact topological annulus. Let $\mathcal{G}(A)$ be the graph whose vertex, $\mathcal{G}^0(A)$, consists of arcs connecting the two boundary components of $A$, defined up to homotopy fixing their endpoints, and where two such arcs are deemed adjacent in $\mathcal{G}(A)$ if they can be realised so that the meet at most at their endpoints. We write $\sigma^*_A$ for the combinatorial metric on $\mathcal{G}(A)$. It is easily seen that $\mathcal{G}(A)$ is quasi-isometric to the real line. In fact, we will choose points $x, y$ in the
different boundary components, and let $\mathcal{G}_0(A)$ be the full subgraph of $\mathcal{G}(A)$, whose vertex set, $\mathcal{G}_0^0(A)$, consists of those arcs with endpoints at $x$ and $y$. Now $\mathcal{G}_0(A)$ can be identified with real line, $\mathbb{R}$, with vertex set $\mathbb{Z}$. In fact, if $t$ is the Dehn twist in $A$, and $\delta \in \mathcal{G}_0^0(A)$ is any fixed arc, then the map $[r \mapsto t^r \delta]$ gives an identification of $\mathbb{Z}$ with the vertex set. One can also check that the inclusion of $\mathcal{G}_0(A)$ into $\mathcal{G}(A)$ is an isometric embedding. Moreover, $\mathcal{G}(A)$ is the 1-neighbourhood of its image. It follows that $|\sigma_\delta^\Lambda(\delta, t^r \delta) - |r|| \leq 1$, for all $r \in \mathbb{Z}$. In fact, since $x$, $y$ and $\delta$ can be chosen arbitrarily, this holds for all $\delta \in \mathcal{G}_0^0(A)$. (The fact that we have only an additive error here is important for later discussion.)

Now given $X \in \mathcal{X}$, we have a well defined “subsurface projection” map: $\theta^\Lambda_X : \mathcal{M}(\Sigma) \to \mathcal{G}(X)$, well defined up to bounded distance (see [MaM2]). (Here we are using the notation “$\theta^\Lambda_X$” to remind us that we are dealing with marking graphs and curve graphs. In [Bo1], this was simply denoted $\theta_X$. However, we use that notation here for projection between “decorated” graphs, which will play an equivalent role in this paper. This also explains the notation $\sigma^\Lambda_X$.)

We also have a projection map $\psi^\Lambda : \mathcal{M}(\Sigma) \to \mathcal{M}(X)$. In fact, these maps can be defined intrinsically to subsurfaces. In this way, if $Y \preceq X$, then $\theta^\Lambda_Y \circ \psi^\Lambda_Y = \theta^\Lambda_X : \mathcal{M}(\Sigma) \to \mathcal{G}(X)$ and $\psi^\Lambda \circ \psi^\Lambda_Y = \psi^\Lambda_X : \mathcal{M}(\Sigma) \to \mathcal{M}(X)$. Moreover, if $\gamma \in \mathcal{M}(\Sigma)$ with $\gamma \prec X$, we may always assume that $\gamma \in \psi^\Lambda_Xa$.

If $Y \preceq X$ or $Y \cap X$, then we also have a projection, $\theta^\Lambda_XY \in \mathcal{G}(X)$.

The distance formula of [MaM2] relates distances in $\mathcal{M}(\Sigma)$ to subsurface projection distances. In particular, they show:

**Lemma 2.1.** There is some $r_0 \geq 0$ depending only on $\xi(\Sigma)$, such that if $r \geq r_0$ a, $b \in \mathcal{M}(\Sigma)$ then the set $\mathcal{A}(a, b; r) = \{X \in \mathcal{X} : \sigma^\Lambda_X(\theta^\Lambda_Xa, \theta^\Lambda_Xb) \geq r\}$ is finite. Moreover, $\rho(a, b) \asymp \sum_{X \in \mathcal{A}(a, b)} \sigma^\Lambda_X(\theta^\Lambda_Xa, \theta^\Lambda_Xb)$.

Here the linear bounds implicit in $\asymp$ depend only on $\xi(\Sigma)$ and $r$. (A similar formula for Teichmüller space was given in [R]. It is given as Proposition 4.9 here.)

One immediate consequence is the following:

**Lemma 2.2.** Given $r \geq 0$, there is some $r' \geq 0$, depending only on $\xi(\Sigma)$ and $r$ such that if $a, b \in \mathcal{M}(\Sigma)$ and $\sigma^\Lambda_X(\theta^\Lambda_Xa, \theta^\Lambda_Xb) \leq r$ for all $X \in \mathcal{X}$, then $\rho(a, b) \leq r$.

Another important ingredient is the following lemma of Behrstock [Be] (given as Lemma 11.3 of [Bo1]).

**Lemma 2.3.** There is a constant, $l$, depending only on $\xi(\Sigma)$ with the following property. Suppose that $X, Y \in \mathcal{X}$ with $X \cap Y$, and that $a \in \mathcal{M}^0$. Then $\min\{\sigma^\Lambda_X(\theta^\Lambda_Xa, \theta^\Lambda_XY), \sigma^\Lambda_Y(\theta^\Lambda_Ya, \theta^\Lambda_YX)\} \leq l$. 

2.4. Median algebras. Let $(M, \mu)$ be a median algebra; that is, a set, $M$, equipped with a ternary operation, $\mu : M^3 \rightarrow M$, such that $\mu(a, b, c) = \mu(b, c, a) = \mu(b, a, c)$, $\mu(a, a, b) = a$ and $\mu(a, b, \mu(c, d, e)) = \mu(\mu(a, b, c), \mu(a, b, d), e)$, for all $a, b, c, d, e \in M$. (For an exposition, see for example [BaH].) Given $a, b \in M$, write $[a, b] = \{x \in M \mid \mu(a, b, x) = x\}$, for the median interval from $a$ to $b$. A subset, $A$, of $M$ is a subalgebra if $\mu(a, b, c) \in A$ for all $a, b, c \in A$. It is convex if $[a, b] \subseteq A$ for all $a, b \in A$. One defines homomorphisms and isomorphisms between median algebras in the obvious way. If $a, b \in M$, then $[a, b]$ admits a partial order, $\leq$, defined by $x \leq y$ if $x \in [a, y]$ (or equivalently $y \in [b, x]$). If $[a, b]$ has intrinsic rank 1, then this is a total order.

An $n$-cube in $M$ is a subalgebra isomorphic to the direct product on $n$ two-point median algebras: $\{-1, 1\}^n$. The rank of $M$ is the maximal $n$ such that $M$ contains an $n$-cube (deemed infinite, if there is no such bound). In [Bo1], we also defined “$n$-colourability” for a median algebra, though that will only be referred to indirectly here, so we will not repeat the definition.

We will refer to a 2-cube as a square. We will generally denote a square by cyclically listing its points as $a_1, a_2, a_3, a_4$, so that $\{a_i, a_{i+1}\}$ is a side for all $i$. Note that $a_i \in [a_{i-1}, a_{i+1}]$ for all $i$ (which one can check is, in fact, an equivalent way of characterising a square, provided we assume the $a_i$ to be pairwise distinct).

Two ordered pairs, $x, y$ and $x', y'$ of elements of $M$ are said to be parallel if $(x = y$ and $x' = y')$ or $(x = x'$ and $y = y')$ or $x, y, y', x'$ form a square.

Suppose $C \subseteq M$ is any subset. A map $\omega : M \rightarrow C$ is a gate map or quasiprojection if $\omega(x) \in [c, x]$ for all $x \in M$ and $c \in C$. If such a map exists, then it is unique, and $\omega|C$ is the identity. Moreover, $C$ must be convex (since if $a, b \in C$ and $d \in [a, b]$, then $\omega(d) \in [a, d] \cap [d, b] = \{d\}$, so $d \in C$).

We will also use the notion of a topological median algebra. This consists of a hausdorff topological space, $M$, and a continuous ternary operation $\mu : M^3 \rightarrow M$ such that $(M, \mu)$ is a median algebra. We say that $M$ is locally convex if every point has a base of convex neighbourhoods. We say that $M$ is weakly locally convex if, given any open set $U \subseteq M$ and any $x \in U$, there is another open set, $V \subseteq U$ containing $x$ such that if $y \in V$, then $[x, y] \subseteq U$. (In fact, finite rank together with weakly locally convex implies locally convex, see Lemma 7.1 of [Bo1] for an exposition.) All the topological median algebras that arise here will be locally convex. Examples of topological median algebras arise as median metric spaces. Again, we only refer to this latter notion in passing here. More detailed discussion can be found in [Ve, Bo3].
2.5. **Coarse median spaces.** Let \((\Lambda, \rho)\) be a geodesic metric space. The following definition was given in [Bo1]:

**Definition.** We say that \((\Lambda, \rho, \mu)\) is a coarse median space if it satisfies:

(C1): There are constants, \(k, h(0)\), such that for all \(a, b, c, a', b', c' \in \Lambda\),
\[
\rho(\mu(a, b, c), \mu(a', b', c')) \leq k(\rho(a, a') + \rho(b, b') + \rho(c, c')) + h(0).
\]

(C2): There is a function \(h : \mathbb{N} \rightarrow [0, \infty)\) such that \(1 \leq |A| \leq p < \infty\), then there is a finite median algebra \((\Pi, \mu_{\Pi})\) and a map \(\lambda : \Pi \rightarrow \Lambda\) such that
\[
\rho(\lambda \mu_{\Pi}(x, y, z), \mu(\lambda x, \lambda y, \lambda z)) \leq h(p)
\]
for all \(x, y, z \in \Pi\) and such that \(\rho(a, \lambda \pi a) \leq h(p)\) for all \(a \in A\).

We say that \((\Lambda, \rho, \mu)\) has rank at most \(n\) if \(\Pi\) can always be chosen to have rank at most \(n\).

We say that \((\Lambda, \rho, \mu)\) is \(n\)-colourable if we can always choose \(\Pi\) to be \(n\)-colourable.

Given \(a, b \in \Lambda\), write \([a, b] = \{\mu(a, b, x) \mid x \in \Lambda\}\), for the coarse interval from \(a\) to \(b\). If \(c \in [a, b]\), then one can check that \(\mu(a, b, c) \sim c\).

**Definition.** We say that \(C \subseteq \Lambda\) is \(r\)-convex if \([a, b] \subseteq N(C; r)\) for all \(a, b \in C\). We say that \(C\) is coarsely convex if it is \(r\)-convex for some \(r \geq 0\).

Note that a coarsely convex set, \(C\), is always quasi-isometrically embedded, or more precisely, there is some \(s\) depending only on \(r\) and the parameters of \(\Lambda\), such that the inclusion of \(N(C; s)\) into \(\Lambda\) is a quasi-isometric embedding.

We can define a coarse gate map to be a map \(\omega : \Lambda \rightarrow C\) such that \(\mu(x, \omega(x), c) \sim \omega(x)\) for all \(x \in \Lambda\) and \(c \in C\). Similarly as with gate maps in a median algebra, the existence of such a map implies that \(C\) is coarsely convex.

We note the following:

**Lemma 2.4.** Suppose that \(a, b, c \in \Lambda\) and \(r \geq 0\), with \(\rho(\mu(a, b, c), c) \leq r\). Then \(\rho(a, c) + \rho(c, b) \leq k_1 \rho(a, b) + k_2\), where \(k_1, k_2\) are constants depending only on \(r\) and the parameters of \(\Lambda\).

**Proof.** This is equivalent to saying that \(\rho(a, c)\) is bounded above in terms of \(\rho(a, b)\). This, in turn follows from the fact that the map \([x \mapsto \mu(a, b, x)]\) is coarsely lipschitz, which is an immediate consequence of hypothesis (C1) of the definition of coarse median. \(\square\)
In other words, if \( c \) lies “between” \( a \) and \( b \) in the coarse median sense, then \( \rho(a, c) + \rho(c, b) \) agrees with \( \rho(a, b) \) up to linear bounds.

A map, \( \phi : (\Lambda, \rho, \mu) \rightarrow (\Lambda', \rho', \mu') \) between two coarse median spaces is said to be a \( h \)-quasimorphism if \( \rho'(\phi \mu(x, y, z), \mu(\phi x, \phi y, \phi z)) \) for all \( x, y, z \in \Lambda \). We abbreviate this to quasimorphism if the constant is understood from context.

A particular example of a coarse median space is a Gromov hyperbolic space. In this case, the median is the usual centroid of three points (a bounded distance from geodesic triangle connecting any two of these points). Such a coarse median space has rank at most 1. (In fact, any rank-1 coarse median space arises in this way.)

2.6. Asymptotic cones. Let \( Z \) be a countable set equipped with a non-principal ultrafilter. Given a \( Z \)-sequence of sets, \( A = (A_\zeta) \), we write \( UA \) for its ultraproduct, that is, \( UA = (\prod_{\zeta \in Z} A_\zeta) / \approx \), where \( (a_\zeta) \approx (b_\zeta) \) if \( a_\zeta = b_\zeta \) almost always. If \( A_\zeta = A \) is constant, we write \( UA = UA \). We can identify \( A \) as a subset of \( UA \) via constant sequences. We refer to an element of \( A \) in \( UA \) as being standard.

Note that the ultraproduct of the reals \( UR \) is an ordered field. We say that \( x \in UR \) is infinitesimal if \( |x| < y \) for all \( y \in \mathbb{R} \) with \( y > 0 \). We say that \( x \in UR \) is limited if \( |x| < y \) for some \( y \in \mathbb{R} \). If we quotient \( UR \) by the infinitesimals (i.e. two numbers are equivalent if they differ by an infinitesimal), we get the “extended reals”, \( R^* \), which is also an ordered field, containing \( \mathbb{R} \) as the subfield of limited extended reals.

Suppose that \( ((\Lambda_\zeta, \rho_\zeta))_\zeta \) is a \( Z \)-sequence of metric spaces. Then \( (UA, U\rho) \) is a \( (UR) \)-metric space. After identifying points an infinitesimal distance apart, we get a quotient \( (\Lambda^*, \rho^*) \), which is an \( R^* \)-metric space. We say that two points of \( \Lambda^* \) lie in the same component if they are a limited distance apart. Thus, each component of \( \Lambda \) is a metric space in the usual sense (that is, the metric takes real values). In fact, one can show that such a component is a complete metric space.

In particular, suppose that \( (\Lambda, \rho) \) is a fixed metric space, and that \( t \in UR \) is a positive infinitesimal. Let \( (\Lambda_\zeta, \rho_\zeta) = (\Lambda, t_\zeta \rho) \). In this case, we write \( (\Lambda^*, \rho^*) \) for the resulting \( R^* \)-metric space (where the scaling factors, \( t_\zeta \), are implicitly understood). We will use \( \Lambda^\infty \) to denote a general component of \( \Lambda^* \), and \( \rho^\infty \) for the restriction of \( \rho^* \). Thus, \( (\Lambda^\infty, \rho^\infty) \) is a complete metric space in the usual sense. This is called an asymptotic cone of \( \Lambda \) and we refer to \( \Lambda^* \) as the extended asymptotic cone. Given a \( Z \)-sequence of points \( x_\zeta \in \Lambda_\zeta \), and \( x \in \Lambda^* \), write \( x_\zeta \to x \) to mean that \( x \) is the class corresponding to \( (x_\zeta)_\zeta \). If \( \Lambda \) is a geodesic space, so is \( \Lambda^\infty \). We put the metric topology on each component of
Λ*, and topologise Λ* as the disjoint union of its components. Note that Λ* has a preferred basepoint, namely that corresponding to any constant sequence in Λ. The component of Λ* containing this basepoint is sometimes referred to as the asymptotic cone of Λ. (Again, the choice of scaling factors is implicitly assumed.)

If a group Γ acts by isometry on Λ, we get an action of its ultraproduct UΓ on Λ*.

If (Λ, ρ) and (Λ′, ρ′) are metric spaces, and \( f : Λ \to Λ' \) is a coarsely lipschitz map, we get an induced map \( f^* : Λ^* \to (Λ')^* \), which is lipschitz (in the sense that the multiplicative bound is real). This restricts to a lipschitz map \( f^\infty : Λ^\infty \to (Λ')^\infty \). If \( f \) is a quasi-isometric embedding, then \( f^\infty \) is bilipschitz onto its range. In particular, if \( f \) is a quasi-isometry then, \( f^\infty \) is a bilipschitz homeomorphism.

A similar discussion applies if we have a \( \mathbb{Z} \)-sequence of uniformly coarsely lipschitz maps, \( f^ζ : Λ \to Λ' \).

If \( (Λ, ρ, µ) \) is a coarse median space, then we get a ternary operation \( µ^* : (Λ^*)^3 \to Λ^* \) which restricts to \( µ^\infty : (Λ^\infty)^3 \to Λ^\infty \). One can check that \( (Λ^*, µ^*) \) is a median algebra, with \( (Λ^\infty, µ^\infty) \) as a subalgebra. Note that \( (Λ^\infty, µ^\infty) \) is a topological median algebra, in the sense that the median is continuous with respect to the topology induced by \( ρ^\infty \).

A standard example is that of a Gromov hyperbolic space, Λ, in which case, Λ* is an \( \mathbb{R}^* \)-tree. Each component, Λ^\infty, is an \( \mathbb{R} \)-tree. For example, if Λ is the hyperbolic plane, then Λ^\infty is the unique complete \( 2^{ℵ_0} \)-regular tree. If Λ is a horodisc, then Λ^\infty is a closed subtree thereof. In both cases, Λ^\infty is “almost furry” which means that it is non-trivial, and no point has valence 2.

### 3. A combinatorial model

In this section, we describe a combinatorial model, \( \mathcal{R}(Σ) \), for \( \mathcal{T}(Σ) \). It is a slight variation on the “augmented marking complex” described in [R] and in [D]. In order to distinguish it, we will refer to the model described here as the “decorated marking complex”. The model \( \mathcal{R}(Σ) \) contains \( \mathcal{M}(Σ) \) as a subgraph. We define \( \mathcal{R}(Σ) \) as follows.

A vertex, \( a \), of \( \mathcal{R} \) consists of a marking, \( \bar{a} \in \mathcal{M}^0 \), together with a map \( η = η_a : \bar{a} \to \mathbb{N} \) such that if \( α, β \in \bar{a} \), with \( η(α) > 0 \) and \( η(β) > 0 \), then \( i(α, β) = 0 \). Thus, \( \bar{a} = \{ α ∈ \bar{a} | η(α) > 0 \} \) is a (possibly empty) multicurve in Σ. We refer to such an \( a \) as a decorated marking, and to \( η_a(α) \) as the decoration on \( α \). Two decorated markings, \( a, b \in \mathcal{R}^0 \) are deemed adjacent in \( \mathcal{R} \) if one of the following three conditions hold:

(E1): \( \bar{a} = \bar{b} \) and \( |η(α) - η(β)| ≤ 1 \) for all \( α ∈ \bar{a} \).
(E2a): \( \hat{a} = \hat{b}, \eta_a|\hat{a} = \eta_b|\hat{b} \) and \( \bar{a}, \bar{b} \) are adjacent in \( \mathcal{M} \).

(E2b): \( \hat{a} = \hat{b}, \eta_a|\hat{a} = \eta_b|\hat{b} \) and \( \hat{b} = t_\alpha^*\hat{a}, \) where \( \alpha \in \bar{a}, t_\alpha \) is the Dehn twist about \( \alpha \), and \( |r| \leq 2n_a(\alpha) \).

We refer to condition (E1) as “vertical adjacency” and to (E2) (that is (E2a) or (E2b)) as “horizontal adjacency”. Given that \( \mathcal{M} \) is connected, it is easily seen that \( \mathcal{R} \) is connected also. We write \( \rho \) for the combinatorial metric on \( \mathcal{R} \) (assigning each edge unit length).

We say that an element \( a \in \mathcal{R}^0 \) is thick if \( \hat{a} = \emptyset \). The thick part, \( \mathcal{R}_T \), of \( \mathcal{R} \) is the full subgraph of \( \mathcal{R} \) whose vertex set consists of thick decorated markings. Note that there is a natural embedding, \( \nu: \mathcal{M} \rightarrow \mathcal{R}_T \subseteq \mathcal{R} \), extending this inclusion. It is easily seen that this is a quasi-isometry, with respect to the intrinsic path metric induced on \( \mathcal{R}_T \). Note that this is again robust — if we take a different marking complex satisfying the conditions laid out in Section 2, we get a quasi-isometric space.

In Section 2 above we described subsurface projections, \( \theta_X^\Sigma: \mathcal{M}(\Sigma) \rightarrow \mathcal{G}(X) \). Here we need to modify that in the case where \( X \) is an annulus.

Recall, first, that we have associated to any annulus, graphs \( \mathcal{G}(A) \) etc., as described in Section 2. From this, we can define the decorated arc graph, \( \mathcal{H}(A) \). Its vertex set, \( \mathcal{H}^0(A) \) is \( \mathcal{G}^0(A) \times \mathbb{N} \), where \( (\delta, i) \) and \( (\epsilon, j) \) are deemed adjacent if either \( \delta = \epsilon \) and \( |i - j| = 1 \), or if \( i = j \) and \( \sigma_\delta(\delta, \epsilon) \leq 2^i \). We write \( \sigma_A \) for the induced combinatorial metric. One can see that \( \mathcal{H}(A) \) is quasi-isometric to a horodisc in the hyperbolic plane. In fact, if \( \mathcal{H}_0(A) \) is the full subgraph of \( \mathcal{H}(A) \) with vertex set \( \mathcal{G}_0(A) \times \mathbb{N} \), then the inclusion of \( \mathcal{H}_0(A) \) in \( \mathcal{H}(A) \) is an isometric embedding with 1-cobounded image. Moreover the map sending \( (t^r\delta, i) \) to the point \( (r, 1 + i \log 2) \) in the upper-half-plane model gives a quasi-isometry of \( \mathcal{H}_0(A) \) to the horodisc \( \mathbb{R} \times [1, \infty) \).

Now suppose that \( X \in \mathcal{X}_A \). The open annular cover of \( \Sigma \) corresponding to \( X \) has a natural compactification to a compact annulus, \( A(X) \) (cf. [MaM2]). We write \( (\mathcal{G}(X), \sigma_X^\Sigma) = (\mathcal{G}(A(X)), \sigma_A^\Sigma(X)), \) and \( (\mathcal{H}(X), \sigma_X) = (\mathcal{H}(A(X)), \sigma_A(X)) \). Write \( \mathcal{H}_T(X) = \mathcal{G}(X) \subseteq \mathcal{H}(X) \).

Let \( \theta_X^\Sigma: \mathcal{M} \rightarrow \mathcal{G}_0(X) \) be the usual subsurface projection map (which we assume sends vertices to vertices). This commutes with the Dehn twist, \( t \), about the core curve of \( X \). In particular, it follows from the above discussion that \( |\sigma_\delta(\theta_X^\Sigma m, \theta_X^\Sigma t^r m) - |r|| \leq 1 \) for all \( r \in \mathbb{Z} \) and \( m \in \mathcal{M} \). If \( a \in \mathcal{R}^0 \), set \( \theta_X a = (\theta_X^\Sigma \hat{a}, i) \), where \( i = \eta_a(\alpha) \) if \( \alpha \in \bar{a} \), and \( i = 0 \) if \( \alpha \notin \bar{a} \).

If \( X \in \mathcal{X}_\Sigma \), we simply set \( (\mathcal{H}(X), \sigma_X) = (\mathcal{G}(X), \sigma_X^\Sigma) \). We define \( \theta_X: \mathcal{R}^0 \rightarrow \mathcal{H}^0(X) \) just by setting \( \theta_X a = \theta_X^\Sigma (\hat{a}) \).
Lemma 3.1. If $X \in \mathcal{X}$, then the map $\theta_X : \mathcal{R}^0 \rightarrow \mathcal{H}^0(X)$ extends to a coarsely lipschitz map $\theta_X : \mathcal{R} \rightarrow \mathcal{H}(X)$.

Proof. In other words, we claim that if $a, b \in (\mathcal{R}^0)$ are adjacent in $\mathcal{R}$, then $\sigma_X(\theta_X a, \theta_X b)$ is bounded above (in terms of $\xi(\Sigma)$). We deal with the three types of edges in turn.

(E1): We have $\tilde{a} = \tilde{b}$. If $X \in \mathcal{X}_N$ then $\theta_X a = \theta_X b$. If $X \in \mathcal{X}_A$, then the first coordinates of $\theta_X a$ and $\theta_X b$ are equal, and their second coordinates are either differ by 1 or both equal 0 (depending on whether or not the core curve of $X$ lies in $\tilde{a}$). In all cases $\sigma_X(\theta_X a, \theta_X b) \leq 1$.

(E2a): We have $\hat{a} = \hat{b}$ and $\eta_a|\hat{a} = \eta_b|\hat{b}$, and that $\sigma_X^\xi(\theta_X^\xi a, \theta_X^\xi b)$ is bounded. $X \in \mathcal{X}_N$, we are done. If $X \in \mathcal{X}_A$, the first coordinates are a bounded distance apart in $\mathcal{G}(X)$, and the second coordinates are equal (to $\eta_a\alpha = \eta_b\alpha$ or to 0, depending on whether or not the core curve of $X$ lies in $\tilde{a}$).

(E2b): We have $b = t^r a$ (so that $\hat{a} = \hat{b}$). Suppose first that $X$ is not a regular neighbourhood of $\alpha$. In this case, we have $\sigma_X^\xi(\theta_X^\xi \hat{a}, \theta_X^\xi \hat{b})$ bounded. (To see this, let $\gamma$ be any curve not homotopic into $\Sigma \setminus X$, with $\iota(\gamma, \alpha) = 0$, and with $\iota(\gamma, \delta)$ bounded for all $\delta \in \tilde{a}$. Note that $t^\alpha \gamma = \gamma$, and we see that $\iota(\gamma, \varepsilon) = \iota(\gamma, t^\alpha \varepsilon)$ is bounded for all $\varepsilon \in \hat{b}$. It follows that $\theta_X^\xi \hat{a}$ and $\theta_X^\xi \hat{b}$ are both a bounded distance from the projection of the curve $\gamma$ to $X$ in $\mathcal{G}(X)$.) We see that in this case $\sigma_X(\theta_X a, \theta_X b)$ is bounded. We are therefore reduced to considering the case where $X$ is an annulus with core curve $\alpha$. From the earlier discussion, we know that $\sigma_X^\xi(\theta_X^\xi \hat{a}, \theta_X^\xi \hat{b}) = \sigma_X^\xi(\theta_X^\xi \hat{a}, \theta_X^\xi t^r \hat{a})$ differs by at most 1 from $|r|$. Moreover, $|r| \leq 2^k$, where $i = \eta_a\alpha = \eta_b\alpha$. Note that $i$ is also the second coordinate of $\theta_X a$ and $\theta_X b$. By construction of $\mathcal{H}(X)$, we see that $\sigma_X(\theta_X a, \theta_X b) \leq 2$ in this case. \hfill \square

Given $a \in \mathcal{R}$, $\alpha \in \hat{a}$ and $i \in \mathbb{N}$, write $a_i = a_i(\alpha) \in \mathcal{R}$ for the decorated marking obtained by changing the decoration on $\alpha$ to $i$. (That is, $\tilde{a}_i = \tilde{a}$, $\eta_a(\beta) = \eta_a(\beta)$ for all $\beta \in \tilde{a} \setminus \{\alpha\}$, and $\eta_a(\alpha) = i$.) Write $t = t^\alpha$ for Dehn twist about $\alpha$. Write $H_a(\alpha) = \{t^r a_i \mid r \in \mathbb{Z}, i \in \mathbb{N}\}$.

Lemma 3.2. Suppose $a \in \mathcal{R}$, $\alpha \in \hat{a}$ and $X = X(\alpha)$. There is a quasi-isometric embedding $\psi : \mathcal{H}(X) \rightarrow \mathcal{R}$ with image a bounded Hausdorff distance from $H_a(\alpha)$, and with $\theta_X \circ \psi$ a bounded distance from the identity on $\mathcal{H}(X)$.

Proof. Let $\delta = \theta_X^\xi \hat{a}$. We can assume that $\delta \in \mathcal{H}_0^0(X) \subseteq \mathcal{H}^0(X)$. Thus, $\mathcal{H}_0^0(X) = \{(t^r \delta, i) \mid r \in \mathbb{Z}, i \in \mathbb{N}\}$. Define $\psi|\mathcal{H}_0^0(X)$ by $\psi((t^r \delta, i)) = t^r a_i$. Thus, by construction, $\psi|\mathcal{H}_0^0(X) = H_a(\alpha)$ and $\theta_X \circ \psi|\mathcal{G}_0^0(X)$ is the identity. If $b, c \in (\mathcal{H}_0^0(X))$, the $\psi b, \psi c$ are connected by an edge of $\mathcal{R}$.
(of type (E1) or (E2b)). Thus, we can extend this to an embedding of \( \mathcal{H}_0(X) \) in \( \mathcal{R} \), and hence to a coarsely lipschitz map, \( \psi: \mathcal{H}(X) \to \mathcal{R} \). Given that it has a left quasi-inverse, this must be a quasi-isometric embedding.

Note that, in fact, we can see that the multiplicative constant of the quasi-isometry is 1 in this case, i.e. distances agree up to an additive constant. In other words, we see that if \( b, c \in \mathcal{H}_a(\alpha) \), then \( |\rho(b, c) - \theta_X(b, c)| \) is bounded.

We can extend this to a statement about twists on multicurves. Given \( a, b \in \mathcal{R} \), suppose that there is some \( \tau \subseteq \hat{a} \cap \hat{b} \) such that \( b \) is obtained from \( a \) by applying powers of Dehn twists about elements of \( \tau \), and changing the decorations on these curves. In this case, we get:

**Lemma 3.3.** If \( a, b \in \mathbb{R} \) are as above, then \( |\rho(a, b) - \sum_{\alpha \in \tau} \sigma_{X(\alpha)}(a, b)| \) is bounded.

(We will only really need that \( \rho(a, b) \approx \sum_{\alpha \in \tau} \sigma_{X(\alpha)}(a, b) \).)

Note that for all \( X \in \mathcal{X} \), \( \mathcal{H}(X) \) is uniformly hyperbolic (in the sense of Gromov). In particular, they each admit a median operation \( \mu_X: \mathcal{H}(X)^3 \to \mathcal{H}(X) \), well defined up to bounded distance, and such that \( (\mathcal{H}(X), \mu_X) \) is a coarse median space of rank 1.

We will need the following observation:

**Lemma 3.4.** \( \mathcal{R}_T(\Sigma) \) is uniformly embedded in \( \mathcal{R}(\Sigma) \).

*Proof.* In fact, \( \mathcal{R}_T(\Sigma) \) is exponentially distorted in \( \mathcal{R}(\Sigma) \). Given \( a, b \in \mathcal{R}_T(\Sigma) \), let \( n = \rho(a, b) \). Let \( a = a_0, \ldots, a_n = b \) be vertex path from \( a \) to \( b \) in \( \mathcal{R}(\Sigma) \). Certainly, \( \rho(a_i, \hat{a}_i) = \rho(a_i, \mathcal{R}_T(\Sigma) \leq n \) for all \( i \). So by construction of \( \mathcal{R}(\Sigma) \), we have \( \rho(a_i, a_{i+1}) \leq 2^n \), so \( \rho(a, b) \leq 2^n n \). □

There are many variations on the construction of \( \mathcal{R} \) which would give rise to quasi-isometrically equivalent graphs. Suppose that \( \mathcal{M}_0 \) is any \( \text{Map}(\Sigma) \)-invariant collection of markings of \( \Sigma \), where a “marking” here can be any collection of curves which fill \( \Sigma \). We suppose that the action of \( \text{Map}(\Sigma) \) is cofinite (which is equivalent to bounding the self-intersection of each element of \( \mathcal{M}_0 \)), and that every complete multicurve is a subset of some element of \( \text{Map}(\Sigma) \). Now suppose that \( \mathcal{M} \) is any connected locally finite graph with vertex set \( \mathcal{M} \) and that the action of \( \text{Map}(\Sigma) \) extends to a cofinite action on \( \mathcal{M}_0 \). We can define “decorated markings” similarly as before, and construct a graph \( \mathcal{R} \) of decorated markings. Again there are a number of variations. For example, in place of the exponent base 2 in (E2b) we could use any
number bigger than 1. Note that $\mathcal{M}$ embeds in $\mathcal{M}(p,q)$ for all sufficiently large $p,q$ and that the inclusion is a quasi-isometry. From this one can construct a quasi-isometry from the new $\mathbb{R}$ to the decorated marking graph as we have defined it. (This is where we use the fact that two markings a bounded distance apart in $\mathcal{M}$, both containing a given multicurve, $\tau$, can be connected by a bounded-length path of markings in $\mathcal{M}$ all containing $\tau$.)

The construction of the "augmented marking graph" in [D] fits (more or less) into this picture. There, the marking graph, $\mathcal{M}$, is taken to be the marking graph as defined in [MaM2]. There is a restriction that $\hat{a}$ is required to be a subset of the "base" of a marking $a$ (but any multicurve in $a$ will be the base of some nearby marking, so this requirement does not change things on a large scale). Our edges of type (E1), (E2a) and (E2b) correspond to "vertical moves", "flip moves" and "twist moves" in the terminology there. Note that the exponent $e$ is used instead of 2 for the twist moves. In any case, it is easily seen from the above, that the augmented marking complex of [D] is equivariantly quasi-isometric to the decorated marking complex as we have defined it.

It was shown in [D] that the augmented marking graph is equivariantly quasi-isometric to $T_{\Sigma}$. We deduce:

**Proposition 3.5.** There is a $\text{Map}(\Sigma)$-equivariant quasi-isometry of $T_{\Sigma}$ to $\mathbb{R}_{\Sigma}$.

Note that this quasi-isometry necessarily maps $T_{\Sigma}$ to within a bounded Hausdorff distance of $\mathbb{R}_{\Sigma}$. (This can be seen explicitly from the various constructions, but also follows from the fact that the constructions are equivariant and that $\text{Map}(\Sigma)$ acts coboundedly on both $T_{\Sigma}$ and $\mathbb{R}_{\Sigma}$.) In fact, one can see from Lemma 7.8 that any quasi-isometry from $T_{\Sigma}$ to $\mathbb{R}_{\Sigma}$ must have this property.

As a special case of the above construction, we consider the situation where $\Sigma$ is a $S_{1,1}$ or $S_{0,4}$. In this case, we can take $G^0(\Sigma)$, as usual, to be the set of curves in $\Sigma$. We deem $\alpha, \beta$ to be adjacent in $G^0$ if they have minimal intersection (that is, $\iota(\alpha, \beta) = 1$ for the $S_{1,1}$ and $\iota(\alpha, \beta) = 2$ for the $S_{0,4}$). Thus, $G(\Sigma)$ is a Farey graph, which we can identify with the 1-skeleton of a regular ideal tessellation of the hyperbolic plane, $\mathbb{H}^2$.

We can take $\mathcal{M}(\Sigma)$ to be the dual 3-regular tree. Its vertices are at the centres of the ideal triangles, and its edges are geodesic segments. In this way, an element of $\mathbb{R}(\Sigma)$ consists of a triple of curves $\{\alpha, \beta, \gamma\}$ corresponding to a triangle in $G(\Sigma)$, with decorations assigned to these curves, at most one of which is non-zero. We define a map $f : \mathcal{R}^0 \to \mathbb{H}^2$ as follows. If all the decorations of $\{\alpha, \beta, \gamma\}$ are 0, then we map it...
to the centre, \( m \), of the corresponding triangle. If the decoration on \( \alpha \), say, is \( i > 0 \), then we map it to \( \lambda(i \log 2) \) where \( \lambda : [0, \infty) \rightarrow [0, \infty) \) is the geodesic ray with \( \gamma(0) = m \), and tending the ideal point of \( \mathbb{H}^2 \) corresponding to \( \alpha \). Mapping edges to geodesic segments, we get a map \( f : \mathcal{R} \rightarrow \mathbb{H}^2 \), which extends the inclusion of \( \mathcal{M} \) into \( \mathbb{H}^2 \). It is easily checked that \( f \) is a quasi-isometry.

Note that each curve \( \alpha \in \mathcal{G}^0 \) corresponds to a component, \( C(\alpha) \), of the complement of \( \mathcal{M} \) in \( \mathbb{H}^2 \). Now \( f|\mathcal{H}(\alpha) \) is a quasi-isometry of \( \mathcal{H}(\alpha) \) to \( C(\alpha) \), which is in turn a bounded Hausdorff distance from a horodisc in \( \mathbb{H}^2 \).

Finally note that there is a natural \( \text{Map}(\Sigma) \)-equivariant identification of \( \mathbb{H}^2 \) with the Teichmüller space of \( \Sigma \) with the Teichmüller metric.

### 4. The Median Construction

The main aim of this section will be to prove the following:

**Theorem 4.1.** There is a coarsely lipschitz ternary operation \( \mu : \mathcal{R}(\Sigma)^3 \rightarrow \mathcal{R}(\Sigma) \), unique up to bounded distance, such that for all \( a, b, c \in \mathcal{R}(\Sigma) \) and all \( X \in \mathcal{X} \), \( \theta_X \mu(a, b, c) \) agrees up to bounded distance with \( \mu_X(\theta_X a, \theta_X b, \theta_X c) \). Moreover, \((\mathcal{R}(\Sigma), \mu)\) is a finitely colourable coarse median metric space of rank \( \xi(\Sigma) \). All constants involved in the conclusion depend only on \( \xi(\Sigma) \) and the constants of the hypotheses.

As we will see, the median is characterised up to bounded distance by the fact that all the subsurface projection maps, \( \theta_X : \mathcal{R}(\Sigma) \rightarrow \mathcal{H}(X) \) are uniform quasimorphisms for all \( X \in \mathcal{X} \). Since this condition is \( \text{Map}(\Sigma) \) equivariant the median is necessarily \( \text{Map}(\Sigma) \)-equivariant up to bounded distance.

In view of Proposition 3.5, we see that this will then imply Theorem 1.1.

It seems likely that Theorem 4.1 could be proven directly by following through similar arguments to those of [BeKMM, BeM] or [Bo4], replacing markings with decorated markings. However, here we shall simply make use of the fact that we already know that there are medians in the marking graph.

To summarise so far, we have a graph, \((\mathcal{R}, \rho)\), and a collection of uniformly hyperbolic spaces, \((\mathcal{H}(X), \sigma_X)\), indexed by the set, \( \mathcal{X} \), of subsurfaces of \( \Sigma \), together with uniformly coarsely lipschitz maps \( \theta_X : \mathcal{R}(\Sigma) \rightarrow \mathcal{H}(X) \). The maps \( \theta_X \) were constructed out of the uniformly lipschitz maps \( \theta_X^\mathcal{M} : \mathcal{M}(\Sigma) \rightarrow \mathcal{G}(X) \), for the family of graphs, \((\mathcal{G}(X), \sigma_X^\mathcal{G})\). Note that these constructions are all \( \text{Map}(\Sigma) \)-equivariant up to bounded distance.
Lemma 4.2. For all $l \geq 0$, there is some $l' \geq 0$, depending only on $l$ and $\xi(\Sigma)$ such that of $a, b \in R$ satisfy $\sigma_X(a, b) \leq l$ for all $X \in \sigma_X$, then $\rho(a, b) \leq l'$.

Proof. First note that we can assume that $\hat{a} = \hat{b}$. For suppose that $\alpha \in \hat{a} \setminus \hat{b}$. Let $X(\alpha)$ be the regular neighbourhood of $\alpha$. Then $\eta_\alpha(\alpha) \leq \sigma_X(\alpha, b) \leq l$. Let $a_0 = a_0(\alpha) \in R^0$ (i.e. we reset the decoration on $\alpha$ to 0). Thus, $\rho(a, a_0) = \eta_\alpha(\alpha) \leq l$, and $\hat{a}_0 = \hat{\alpha} \setminus \{\alpha\}$. Note that by construction, $\theta_X(a_0) = \theta_X(a)$ for all $X \in \mathcal{X} \setminus X(\alpha)$. We now replace $\alpha$ with $a_0$. We continue with this process for all curves in $\hat{a} \setminus \hat{b}$ and in $\hat{b} \setminus \hat{a}$, until $\hat{a}$ and $\hat{b}$ are both equal to some (possibly empty) multicurve, $\tau$, say. Note that this process does not change $\hat{a}$ or $\hat{b}$, and moves $a$ and $b$ each a bounded amount.

If $X$ is not a regular neighbourhood of any element of $\tau$, then we see that $\sigma_X(\alpha, \beta)$ is bounded in terms of $l$. Moreover, for each $\beta \in \tau$, we can find some $r(\beta) \in \mathbb{Z}$ such that $\sigma_X(\beta, t^{r(\beta)}_\beta a)$ is bounded. Let $g$ be the composition of the twists $t^{r(\beta)}_\beta$ as $\beta$ ranges over $\tau$, and set $e = G(A) \in R^0$. We now get that $\sigma_X(\hat{b}, e)$ is bounded for all $X \in \mathcal{X}$. Therefore by Corollary 2.2, we get $\rho(e, b)$ is bounded in terms of $l$.

After moving $b$ a bounded distance (across edges of type (E2a)) in $\mathcal{M}$, we assume that $\hat{b} = \hat{e}$. In other words, $a, b$ now satisfy the hypotheses of Corollary 2.2, and so $\rho(a, b)$ is bounded above as a function of $\sum_{\alpha \in \tau} \sigma_X(a, b)$, and hence bounded.

The following is an analogue of Behrstock’s Lemma [Be], given as Lemma 2.3 here.

Lemma 4.3. There is a constant, $l_1$, depending only on $\xi(\Sigma)$ with the following property. Suppose that $X, Y \in \mathcal{X}$ with $X \cap Y$, and that $a \in R^0$. Then $\min\{\sigma_X(a, Y), \sigma_Y(a, X)\} \leq l_1$.

Proof. Suppose, for contradiction, that $\sigma_X(a, Y)$ and $\sigma_Y(a, X)$ are both large. Now by Lemma 2.3, we can assume, without loss of generality that $\sigma_X(a, X)$ is bounded. This can only happen if $X = X(\gamma)$ for some curve $\gamma \in \hat{a}$. By assumption, $\gamma$ crosses $Y$, so we see that $\sigma_X(a, X)$ is also bounded. We must therefore have $Y = X(\beta)$ for some $\beta \in \hat{a}$. But $\beta \cap \alpha$, giving a contradiction, since $\hat{a}$ is assumed to be a multicurve.
We will write

\[(x, y : z, w)_{\rho} = \]

\[\frac{1}{2} \left( \max\{\sigma(x, z) + \sigma(y, w), \sigma(x, w) + \sigma(y, z)\} - (\sigma(x, y) + \sigma(z, w)) \right).\]

We will abbreviate \((x, y : z, w) = (x, y : z, w)_{\rho}\). (Note that \((x, y : z, z)\) is the Gromov product of \(x\) and \(y\) relative the the basepoint, \(z\).) We abbreviate \((a, b : c, d)_X = (\theta_X a, \theta_X b : \theta_X c, \theta_X d)_{\sigma_X}\), and \((a, b : c, d)_Y = (\theta_Y a, \theta_Y b : \theta_Y c, \theta_Y d)_{\sigma_Y}\).

The following is an analogue of Lemma 11.4 of [Bo1]:

**Lemma 4.4.** There are some \(l_1, l_2\), depending only on \(\xi(\Sigma)\), with the following property. Suppose that \(X, Y \in \mathcal{X}\) with \(Y \prec X\), and suppose that \(a, b \in \mathcal{R}\) with \((a, b : Y, Y)_X \geq l_1\). Then \(\sigma_Y(a, b) \leq l_2\).

**Proof.** Note that \(X \in \mathcal{X}_\mathcal{N}\), so we have that \((a, b : Y, Y)_X = (a, b : Y, Y)_\mathcal{X}\) is large. From the bounded geodesic image theorem of [MaM2] (see Lemma 11.4 of [Bo1]), it follows that \(\sigma_Y(a, b)\) is bounded. If \(Y \in \mathcal{X}_\mathcal{N}\), \(\sigma_Y(a, b) = \sigma_Y(a, b)\) and we are done. So suppose that \(Y = X(\gamma)\) for some curve \(\gamma\). If \(\sigma_\gamma(a, b)\) is large, then we must have \(\gamma \in \hat{a} \cup \hat{b}\), and so we suppose \(\gamma \in \hat{a} \subseteq \hat{a}\). But then \((a, b : Y, Y)_X \leq \sigma_X(a, Y)\) is bounded, giving a contradiction. \(\square\)

Suppose that to each \(X \in \mathcal{X}\), we have associated a point, \(p_X \in \mathcal{H}(X)\). Let \(l \geq 0\).

We say that \((p_X)_X\) is **transversely l-consistent** if for all \(X, Y \in \mathcal{X}\) with \(X \pitchfork Y\), we have \(\min\{\sigma_X(p_X, \theta_X Y), \sigma_Y(p_Y, \theta_Y X)\} \leq l\). We will abbreviate this to “transversely consistent” when the constant \(l\) is clear from context.

Note that, in this terminology, Lemma 4.3 tells us that for any \(a \in \mathcal{R}\), the family \((\theta_X a)_X\) is transversely consistent.

**Lemma 4.5.** For all \(l \geq 0\), there is some \(l' \geq 0\) depending only on \(l\) and \(\xi(\Sigma)\) such that if \((p_X)_X\), \((q_X)_X\) and \((r_X)_X\) are all transversely \(l\)-consistent, then \((\mu_X(p_X, q_X, r_X))_X\) is transversely \(l'\)-consistent.

**Proof.** Suppose \(X, Y \in \mathcal{X}\) with \(X \pitchfork Y\). Now \(\min\{\sigma_X(p_X, \theta_X Y), \sigma_Y(p_Y, \theta_Y X)\}, \min\{\sigma_X(q_X, \theta_X Y), \sigma_Y(q_Y, \theta_Y X)\}\) and \(\min\{\sigma_X(r_X, \theta_X Y), \sigma_Y(r_Y, \theta_Y X)\}\) are each at most \(l\). Thus, after permuting \(\{X, Y\}\) and \(\{p, q, r\}\), we can assume that \(\theta_X(p_X, \theta_X Y) \leq l\) and \(\theta_X(q_X, \theta_X Y) \leq l\). But this bounds \(\sigma_X(\mu_X(p_X, q_X, r_X, \theta_X Y))\) as required. \(\square\)

Note that, if \(\alpha\) and \(\beta\) are curves, then \(\sigma_\alpha(p_\alpha, \theta_\alpha \beta) \geq \eta_\alpha(p_\alpha)\) and \(\sigma_\beta(p_\beta, \theta_\beta \alpha) \geq \eta_\beta(p_\beta)\). Therefore, if \((p_X)_X\) is transversely \(l\)-consistent, and \(\alpha \pitchfork \beta\), then \(\min\{\eta_\alpha(p_\alpha), \eta_\beta(p_\beta)\} \leq l\). Therefore, the set of curves,
γ, for which \( \eta_\gamma(p_\gamma) > l \) is a (possibly empty) multicurve, \( \tau \), which we refer to as the \textit{short multicurve} associated to \((p_X)\). We will refer to the “short multicurve” of a transversely consistent family, when an appropriate constant \( l \) is assumed.

Given \( X \in \mathcal{X} \) and a multicurve, \( \tau \), we write \( X \cap \tau \) to mean that \( \gamma \cap X \) or \( \gamma \prec X \) for some \( \gamma \in \tau \). In this case, we can define \( \theta_X(\tau) \in \mathcal{H}_T(X) \subseteq \mathcal{H}(X) \) by setting \( \theta_X(\tau) = \theta_X(\gamma) \) for some such \( \gamma \). This is well defined up to bounded distance. Note that it follows from [Bo2] (Lemma 7.8 thereof) that if \( a \in \mathcal{R}(\Sigma) \) and \( \sigma_X(\theta_X a, \theta_X \tau) \) is bounded for all \( X \) with \( X \cap \tau \), then \( \tau \) has bounded intersection with \( \bar{a} \). (For further discussion of this, see Section 6.)

**Lemma 4.6.** Suppose that \( a \in \mathcal{R}(\Sigma) \) and that \((p_X)\) is a transversely consistent family with short multicurve, \( \tau \). Suppose that \( \sigma_X(p_X, \theta_X a) \) is bounded for all \( X \in \mathcal{X} \setminus \{X(\gamma) \mid \gamma \in \tau \} \). Then there is some \( b \in \mathcal{R}(\Sigma) \) such that \( \sigma_X(p_X, \theta_X b) \) is bounded for all \( X \in \mathcal{X} \).

**Proof.** Suppose \( X \in \mathcal{X} \) with \( X \cap \tau \). Now \( \sigma_\gamma(p_\gamma, \theta_\gamma X) > \eta_\gamma(p_\gamma) > l \), so by transverse consistency, we have that \( \sigma_X(p_X, \theta_X \tau) = \sigma_X(p_X, \theta_X \gamma) \) is bounded. By hypothesis, we have \( \sigma_X(p_X, \theta_X a) \) bounded, so we get that \( \sigma_X(\theta_X a, \theta_X \tau) \) is bounded for all such \( X \). It now follows from the observation above, that \( \tau \) has bounded intersection with \( \bar{a} \). Therefore, after moving \( a \) a bounded distance in \( \mathcal{R}(\Sigma) \), we can assume that \( \tau \subseteq \bar{a} \).

But now, after applying a Dehn twist, and redecorating the elements of \( \tau \), we can find another \( b \in \mathcal{R} \), with \( \sigma_\gamma(p_\gamma, \theta_\gamma b) \) bounded for all \( \gamma \in \tau \). Note that this changes projection to other subsurfaces a bounded amount. It follows that \( \sigma_X(p_X, \theta_X b) \) is bounded for all \( X \in \mathcal{X} \) as required. \( \square \)

**Lemma 4.7.** Suppose that \( a, b, c \in \mathcal{R}(\Sigma) \). Then there is some \( d \in \mathcal{R}(\Sigma) \) such that for all \( X \in \mathcal{X} \), we have that \( \sigma_X(\theta_X d, \mu_X(\theta_X a, \theta_X b, \theta_X c)) \) is bounded.

**Proof.** Set \( d_X = \mu_X(\theta_X a, \theta_X b, \theta_X c) \). By Lemma 4.3, the families \((\theta_X a)_X\), \((\theta_X b)_X\) and \((\theta_X c)_X\) are all transversely consistent. Therefore, by Lemma 4.5, so is the family \((d_X)_X\). Let \( \tau \) be the short multicurve associated to \((d_X)_X\). Let \( e \in \mathcal{R}(\Sigma) \) be such that \( \bar{e} \) is a median of \( \bar{a}, \bar{b}, \bar{c} \) in \( \mathcal{M}(\Sigma) \). In other words, \( \sigma_X(\theta_X \bar{e}, \mu_X(\theta_X \bar{a}, \theta_X \bar{b}, \theta_X \bar{c})) \) is bounded for all \( X \in \mathcal{X} \). Note that, by the definition of \( \tau \), this implies that \( \sigma_X(\theta_X e, d_X) \) is bounded for all \( X \in \mathcal{X} \setminus \{X(\gamma) \mid \gamma \in \tau \} \). By Lemma 4.6, we therefore have some \( d \in \mathcal{R}(\Sigma) \), such that \( \sigma_X(\theta_X d, d_X) \) is bounded for all \( X \in \mathcal{X} \) as required. \( \square \)

By Lemma 3.1, \( d \) is unique up to bounded distance. We denote it by \( \mu(a, b, c) \).
By Proposition 10.1 of [Bo1], it now follows that $\mu$ is coarse median on $\mathcal{R}(\Sigma)$. To bound the rank, we need the following analogue of Lemma 1.7 of [Bo1]:

**Lemma 4.8.** There is some $l_0 \geq 0$, depending only $\xi(\Sigma)$ such that if $X, Y \in \mathcal{X}$ and there exist $a, b, c, d \in \mathcal{R}(\Sigma)$ with $(a, b : c, d)_X \geq l_0$ and $(a, b : c, d)_Y \geq l_0$, then $X \land Y$.

**Proof.** This is a simple consequence of Lemmas 4.4 and 4.5 above, exactly as with the proof of Lemma 11.7 of [Bo1]. □

As noted in [Bo1], $\xi(\Sigma)$ is the maximal cardinality of a set of pairwise homotopically disjoint surfaces which can be embedded in $\Sigma$.

We have now verified properties (P1)–(P4) as given in Section 10 of [Bo1], and so we deduce that $(\mathcal{R}(\Sigma), \rho, \mu)$ is a coarse median space of rank at most $\xi(\Sigma)$. The fact that it has rank at least $\xi(\Sigma)$ will follow from the “if” part of Theorem 1.3 which we prove in Section 5. The fact that it is finitely colourable follows exactly as in Section 12 of [Bo1].

This proves Theorem 1.1 modulo the lower bound on rank. We move on to some further discussion used later.

If $X \in \mathcal{X}_N$, we can define a map, $\psi_X : \mathcal{R}(\Sigma) \to \mathcal{R}(X)$ as follows. Suppose that $a \in \mathcal{R}^0(\Sigma)$. We can suppose that if $\gamma \in \hat{a}$ with $\gamma \prec X$, then $\gamma \in \psi^X_\Lambda a$. Let $\tau = \{ \gamma \in \hat{a} | \gamma \prec X \} \subseteq \psi^X_\Lambda a$. Now let $\psi_X a$ be the decorated marking with base marking $\psi^X_\Lambda$ with with decorations determined by $\eta_a|\tau$. It is easily seen that $\psi_X$ is uniformly lipschitz. Moreover, if $Y \preceq X$, then $\theta_Y \circ \psi_X \simeq \theta_Y$. It now follows from the characterisation of medians that the projection is a uniform quasimorphism. In the case where $X \in \mathcal{X}_A$, we will interpret $\mathcal{R}(X) = \mathcal{H}(X)$ and $\psi_X = \theta_X$.

For future reference, we note the following variation, due to Rafi, the distance formula of Masur and Minsky, mentioned in Section 2.

Given $a, b \in \mathcal{R}(\Sigma)$ and $r \geq 0$, let $\mathcal{A}(a, b; r) = \{ X \in \mathcal{X} | \sigma_X(a, b) > r \}$. We have noted that this is finite. We have:

**Proposition 4.9.** There is some $r_0 \geq 0$ depending only on $\xi(\Sigma)$ such that for all $r \geq r_0$, then for all $a, b \in \mathcal{R}(\Sigma)$, $\rho(a, b) \simeq \sum_{X \in \mathcal{A}(a, b; r)} \sigma_X(a, b)$.

The corresponding statement for Teichmüller space is proven in [R]. Its reinterpretation for the augmented marking complex is used in [D]. Given that this is (equivariantly) quasi-isometric to $\mathcal{R}(\Sigma)$, the statement can be easily deduced.

We remark that this is the only input that is not derived directly from the defining properties of $\mathcal{R}(\Sigma)$. (Though it can probably be deduced with some work.)
We can apply these constructions to the extended asymptotic cone. We write $R^\infty(\Sigma) \subseteq R^*(\Sigma)$ for the asymptotic cone and extended asymptotic cone of $R(\Sigma)$, and $R^\infty(\Sigma)$ for some component thereof. We similarly have spaces $G^\infty(\Sigma) \subseteq G^*(\Sigma)$ etc.

For the remainder of this section, we will refer to an element of the ultrapoduct, $UG^0(\Sigma)$, as a curve and to an element of $G(\Sigma)$ as a standard curve. We will similarly refer to elements of $UX$ and $X$ as subsurfaces and standard subsurfaces respectively. We apply similar terminology to multicurves etc.

Note that $U\text{Map}(\Sigma)$ acts by isometry of $R^*(\Sigma)$, and $U^0\text{Map}(\Sigma)$ acts on $R^\infty(\Sigma)$. Unlike the case of the marking graph, however, these spaces are not homogeneous (as we will see in Section 6).

In fact, we have a $U\text{Map}(\Sigma)$-invariant thick part, $R_T^\infty(\Sigma)$, of $R^*$. We say that a component, $R^\infty(\Sigma)$, of $R^*(\Sigma)$ is thick if $R^\infty(\Sigma) \cap R_T^\infty(\Sigma) \neq \emptyset$, in which case, we denote $R^\infty(\Sigma) \cap R_T^\infty(\Sigma)$ by $R_T(\Sigma)$. In fact, $U\text{Map}(\Sigma)$ acts transitively on $R_T^\infty(\Sigma)$. Note that $R_T(\Sigma)$ contains the standard basepoint of $R^*(\Sigma)$. It follows that every thick component of $R^*(\Sigma)$ is a $U\text{Map}(\Sigma)$-image of the standard component.

The various coarsely lipschitz quasimorphisms we described in Section 4 now give rise to lipschitz median homomorphisms. Specifically, we have maps: $\theta^*_X : R^*(\Sigma) \rightarrow G^*(X)$, and $\psi^*_X : R^*(\Sigma) \rightarrow R^*(X)$ for all $X \in UX$.

5. Some applications of the coarse median property

In this section, we describe a few immediate consequences of what we have shown. In particular, we will give proofs of Theorems 1.2, 1.3, and 1.8, and the “if” part of Theorem 1.4. Theorems 1.2, 1.8 and the “only if” part of 1.3 are simple consequences of the preceding discussion, so we discuss these first.

The first result is a coarse quadratic isoperimetric inequality. This is a quasi-isometrically invariant property which one could equivalently formulate in a number of different ways. For definiteness we will say that a geodesic space, $\Lambda$, satisfies a coarse quadratic isoperimetric inequality if the following holds. There is some $r \geq 0$ and some $k \geq 0$ such that if $\gamma$ is any closed path in $\Lambda$ of length at most $nr$, where $n \geq 1$ is some natural number, then we can find a triangulation of the disc with at most $kn^2$ 2-simplexes and a map of its 1-skeleton into $\Lambda$ such that the image of every 1-cell has length at most $r$ and such that the map restricted to the boundary agrees with $\gamma$ (thought of as a map of the boundary of the disc into $\Lambda$). One can easily check that this property is quasi-isometry invariant.
It is shown in [Bo1] (Proposition 8.2) than any coarse median space has this property. We immediately deduce from Theorem 1.1 that:

**Proposition 5.1.** \( \mathcal{R}(\Sigma) \) has a coarse quadratic isoperimetric inequality.

This is, of course, equivalent to Theorem 1.2.

We next prove Theorem 1.8. Recall that it is shown in [Bo2] that any asymptotic cone of a finitely colourable coarse median space embeds as a closed subset of a finite product of \( \mathbb{R} \)-trees. The embedding can be taken to be a bilipschitz median homomorphism. The image of the embedding is a median metric space, in which all intervals are compact. One can go on to show that this is in turn bilipschitz equivalent to a CAT(0) metric [Bo3]. Note that, in view of Theorem 4.1, this applies in particular to any asymptotic cone of \( \mathcal{R}(\Sigma) \), hence also any asymptotic cone of \( T(\Sigma) \). This proves Theorem 1.8.

Some other consequences follow on from the fact that \( \mathcal{R}_T^*(\Sigma) \) is a locally convex topological median algebra of finite rank. For example, the topological dimension of any locally compact subset of any asymptotic cone of \( \mathcal{R}_T^*(\Sigma) \) is at most \( \xi(\Sigma) \) (see Theorem 2.2 and Lemma 7.6 of [Bo1]). In particular, it does not admit any continuous injective map of \( \mathbb{R}^{\xi+1} \). From this we get the following. (A similar statement can be found in [EMR1].) Write \( B^n_r \) for the ball of radius \( r \) in the euclidean space \( \mathbb{R}^n \).

**Proposition 5.2.** Given parameters of quasi-isometry, there is some constant \( r \geq 0 \), such that there is no quasi-isometric embedding of \( B^{\xi+1}_r \) into \( \mathcal{R}(\Sigma) \) with these parameters.

*Proof.* This is a standard argument involving asymptotic cones. Suppose that, for each \( i \in \mathbb{N} \), the ball, \( B_i \), of radius \( i \) admits a uniformly quasi-isometric embedding, \( \phi_i : B_i \rightarrow \mathcal{R}(\Sigma) \). Now pass to the asymptotic cone with scaling factors, \( i \). We end up with a bilipschitz map, \( \phi^\infty : B_1 \rightarrow \mathcal{R}^{\infty}(\Sigma) \), giving contradicting the dimension bound. \( \Box \)

(Indeed, the above holds in any coarse median space of rank at most \( \xi \).)

An immediate consequence is that \( \mathcal{R}(\Sigma) \) does not admit any quasi-isometric embedding of a euclidean \( (\xi(\Sigma) + 1) \)-dimension half-space. This proves the “only if” part of Theorem 1.3.

For the “if” part, we need to construct such an embedding in dimension \( \xi(\Sigma) \). We use the same construction as in [EMR1], though base the proof on the arguments here. This will show, in addition, that the image can be assumed coarsely convex.
Given any \( a \in \mathcal{M}^0(\Sigma) \), and any \( t \subseteq a \), let \( O(t) = O_\Delta(t) = \{ b \in R^0(\Sigma) \mid b = a, \hat{a} \subseteq t \} \subseteq R(\Sigma) \). In other words, we take all possible decorations on \( a \) subject to the constraint that all the decorated curves must lie in \( t \).

**Lemma 5.3.** \( O(t) \) is coarsely convex in \( R(\Sigma) \).

**Proof.** We define a map \( \omega : R(\Sigma) \to O(t) \) as follows. Given \( x \in R^0(\Sigma) \), let \( \tau = \hat{x} \cap t \subseteq a \). Let \( b \in R^0 \) be such that \( b = a, \eta_k|\tau = \eta_k|\tau \) and \( \eta_k|\hat{b} \setminus \tau \equiv 0 \), and set \( \omega(x) = b \). In other words, we take the base marking \( a \), and decorate curves in \( a \) if and only if they also happen to be decorated curves of \( x \). We claim that \( \omega \) is a coarse gate map.

To this end, let \( c \in O(t) \), so \( \hat{c} \subseteq a \cap \hat{x} \). If \( \gamma \in \tau \), then \( \theta_\gamma \omega(x) \sim \theta_\gamma x \), so \( \theta_\gamma \mu(x, \omega(x), c) \sim \theta_\gamma \omega(x) \). If \( \gamma \notin \hat{c} \), then \( \gamma \notin \hat{x} \), so \( \theta_\gamma \omega(x) \sim \theta_\gamma a \sim \theta_\gamma c \) (since \( \hat{c} = a \)), and \( \theta_\gamma \mu(x, \omega(x), c) \sim \theta_\gamma \omega(x) \). Finally, if \( X \in X \) does not have the form \( X(\gamma) \) for such \( \gamma \), then \( \theta_X c \sim \theta_X a \sim \theta_X \omega(x) \), so \( \theta_X \mu(x, \omega(x), c) \sim \theta_X \omega(x) \). By Lemma 4.2, \( \mu(x, \omega(x), c) \sim \omega(x) \) as claimed.

Given a multicurve \( \tau \in S \), write \( O(\tau) = (0, \infty)^\tau \) and \( O^0(\tau) = \mathbb{N}^\tau \subseteq O(\tau) \). Note that \( O(\tau) \) is a median algebra with the product structure, and that \( O^0(\tau) \) is a subalgebra.

Suppose \( a \in \mathcal{M}^0 \equiv R^0_\tau \), with \( \tau \subseteq a \). Define a map \( \lambda = \lambda^\tau : O^0(\tau) \to R(\Sigma) \) by setting \( \lambda^\tau(v) = b \) where \( b = \hat{a}, \hat{\tau} = \tau, \eta_\tau|\tau = v \), and \( \eta_\tau|\hat{a} \setminus \tau \equiv 0 \). (In other words, we take the base marking \( a \), with decorations determined by \( v \).) The map \( \lambda \) is easily seen to be a quasimorphism. Note that \( O(\tau) = O_\Delta(\tau) = \lambda(O^0(\tau)) \). Also, if \( x \in R(\Sigma) \), then \( \omega(x) = \lambda(u) \), where \( u \in O(\tau) \) is defined by \( u|\tau \cap \hat{x} = \eta_\tau|\tau \cap \hat{x} \) and \( u|\tau \setminus \hat{x} \equiv 0 \).

By Lemma 5.3, \( O(\tau) \) is coarsely convex in \( R(\tau) \). Moreover, it follows that \( \lambda \) extends to a quasi-isometric embedding of \( O(\tau) \) into \( R(\Sigma) \).

We have shown that:

**Lemma 5.4.** The map \( \lambda^\tau : O(\tau) \to R(\Sigma) \) is a quasi-isometric embedding with coarsely convex image.

If we take \( \tau \) to be any complete multicurve and \( a \) to be any marking containing it, then we get a quasi-isometric embedding of a \( \xi \)-orthant (or \( \xi \)-dimensional half-space).

This proves the “if” part of Theorem 1.3.

Note that this shows that \( R(\Sigma) \) has coarse median rank exactly \( \xi(\Sigma) \) (given the observation after Proposition 5.2).

For the “if” part of Theorem 1.4, we want to construct a quasi-isometric embedding of \( \mathbb{R}^\xi \), in the cases described. (We will deal with the “only if” part at the end of Section 8.)
Suppose that $\Upsilon$ is a finite simplicial complex. We can construct a singular euclidean space, $O(\Upsilon)$, by taking an orthant for every simplex of $\Upsilon$ and gluing them together in the pattern determined by $\Upsilon$. This has vertex $o$ and we can identify $\Upsilon$ as the spherical link of $o$ in $O(\Upsilon)$. (For example, the cross polytope gives a copy of euclidean space.) Note that if $\Upsilon$ is bilipschitz equivalent to the standard $(n - 1)$-sphere, then $O(\Upsilon)$ is bilipschitz equivalent to $\mathbb{R}^n$. We write $O^0(\Upsilon)$ for the set of integer points in $O(\Upsilon)$.

Now let $C(\Sigma)$ be the curve complex associated to $\Sigma$. This is the flag complex with 1-skeleton $G(\Sigma)$. In particular, $C^0(\Sigma) = G^0(\Sigma)$. We can identify the set of simplices of $C(\Sigma)$ with the set, $\mathcal{S}\setminus\{\emptyset\}$, of non-empty multicurves in $\Sigma$.

Given a finite subcomplex, $\Upsilon$, of $C(\Sigma)$, write $\mathcal{S}(\Upsilon) \subseteq \mathcal{S}$ for the set of multicurves corresponding to the simplices of $\Upsilon$. Write $\Upsilon^0 = \bigcup \mathcal{S}(\Upsilon)$ for the set of vertices. If $\tau \in \mathcal{S}$, we can identify $O(\tau)$ as a subset of $O(\Upsilon)$, and $O^0(\tau)$ as a subset of $O^0(\Upsilon)$.

Suppose $a \in M^0$ is marking of $\Sigma$ with $\Upsilon^0 \subseteq a \subseteq M^0$. We can define a map $\lambda = \lambda_a : O^0(\Upsilon) \longrightarrow R(\Sigma)$ by combining the maps $\lambda_\gamma : O^0(\tau) \longrightarrow R(\Sigma)$ for $\gamma \in \mathcal{S}$. Write $O(\Upsilon) = \lambda(O(\Upsilon)) = \bigcup_{\tau \in \mathcal{S}(\Upsilon)} O(\tau)$.

Suppose now that $\Upsilon$ is a full subcomplex of $C(\Sigma)$ (in other words, if the vertices of a simplex in $C(\Sigma)$ are contained in $\Upsilon^0$, then the whole simplex is contained in $\Upsilon$). In this case, we have $O(\Upsilon) = O(\Upsilon^0)$ as previously defined. In particular, Lemma 5.3 tells us that $O(\Upsilon)$ is coarsely convex.

As in the case of a single orthant, we now see:

**Lemma 5.5.** If $\Upsilon$ is a full subcomplex of $C(\Sigma)$, with $\Upsilon^0 \subseteq a \subseteq M^0$, then $O(\Upsilon)$ is coarsely convex in $R(\Sigma)$.

We see that $\lambda_a$ extends to a quasi-isometric embedding of $O(\Upsilon)$ into $R(\Sigma)$. (In fact, one can show that $\lambda_a$ is a quasi-isometric embedding even if we do not assume that $\Upsilon$ is full, though of course, its image need not be coarsely convex in this case.)

Note that if $\Upsilon$ is PL homeomorphic to the standard $(n - 1)$-sphere, we get a bilipschitz embedding of $\mathbb{R}^n$ into $R(\Sigma)$.

We will show:

**Proposition 5.6.** $C(\Sigma)$ contains a full subcomplex PL homeomorphic to the standard $(\xi(\Sigma) - 1)$-sphere if and only if $\Sigma$ has genus at most 1, or is the closed surface of genus 2.

This is based on results and constructions in [Har]. (For further related discussion, see [Br].) If $\Sigma \cong S_{g,p}$, write $\xi'(\Sigma) = 2g + p - 2$ if
26 BRIAN H. BOWDITCH

\[ g, p > 0, \xi'(\Sigma) = 2g - 1 \text{ if } p = 0 \text{ and } \xi'(\Sigma) = p - 3 \text{ if } g = 0. \] (Here, we are assuming that \( \xi(\Sigma) \geq 2 \).)

Note that \( \xi'(\Sigma) \leq \xi(\Sigma) \) with equality precisely in the cases described by Proposition 5.6.

It is shown in [Har] (Theorem 3.5) that \( C(\Sigma) \) is homotopy equivalent to a wedge of spheres of dimension \( \xi'(\Sigma) - 1 \). In particular, the homology is trivial in dimension \( \xi'(\Sigma) \). (It does not matter which homology theory we use here.) Now \( C(\Sigma) \) has dimension \( \xi(\Sigma) \), so if \( \xi'(\Sigma) < \xi(\Sigma) \), it follows that \( C(\Sigma) \) cannot contain any \((\xi(\Sigma) - 1)\)-dimensional homology cycle, and so in particular, no topologically embedded \((\xi(\Sigma) - 1)\)-manifold. This proves the “only if” part of Proposition 5.6.

For the “if” part, we need to construct such a sphere. In the planar (genus-0) case there is a simple explicit construction described in [AL], which involves doubling the arc complex of a disc. The latter is known to be a sphere by [HuM]. (It is easily checked that this gives a PL sphere.) However, it is unclear how to adapt this to the genus-1 case. Below, we give an argument which deals with all cases. It is based on an idea in Harer’s proof of the result mentioned above. We first show:

**Lemma 5.7.** Suppose \( g, p, n \in \mathbb{N} \) and \( p \geq 2 \). Suppose that \( C(S_{g,p}) \) contains a full subcomplex PL homeomorphic to the standard \( n \)-sphere. Then \( C(S_{g,p+1}) \) contains a full subcomplex PL homeomorphic to the standard \((n + 1)\)-sphere.

**Proof.** For this discussion, it will be convenient to view \( S_{g,p} \) as a closed surface, \( S \), of genus \( g \), with a set \( A \subseteq S \) of \( p \) preferred points. Let \( \Upsilon \subseteq C(S_{g,p}) \) be a full subcomplex PL homeomorphic to the standard \( n \)-sphere. Now realise the elements of \( \Upsilon^0 \) as closed curves in \( S \setminus A \), so that no three curves intersect in a point, and such that the total number of intersections is minimal, subject to this constraint. (It is well known that this necessarily minimises pairwise intersection numbers.)

Now let \( I \subseteq S \) be an embedded arc meeting \( A \) precisely at its endpoints, \( a, b \), say. We may assume that no point of \( I \) lies in two curves of \( \Upsilon^0 \) and that (subject to this constraint) the total number, \( m \), of intersections, \( I \cap \bigcup \Upsilon^0 \subseteq \Sigma \), is minimal, in the homotopy class of \( I \) in \( S \setminus A \) relative to its endpoints. Let \( I_0, I_1, \ldots, I_m \), be the components of \( I \setminus (\{a, b\} \cup \bigcup \Upsilon^0) \), consecutively ordered, so that \( a \) and \( b \) lie respectively in the closures of \( I_0 \) and \( I_m \).

Choose any point \( c_i \in I_i \) and an arbitrary point \( d \in I \setminus \{a, b\} \). Let \( B = A \cup \{d\} \), and think of \( S_{g,p+1} \) as \( S \) with the points of \( B \) removed. The following can be thought of intuitively as sliding the point \( d \) from \( a \) to \( b \) along \( I \). However, formally it is better expressed as keeping \( d \) fixed and applying an isotopy to the curves, as we now describe.
Given any \( i \in \{0, \ldots, m\} \), we obtain a map \( f_i : \Upsilon \to C(S_{g,p+1}) \) as follows. Take an isotopy of \( S \) supported on a small neighbourhood of the interval \([d, c_i]\), fixing \( I \) setwise, and carrying \( c_i \) to \( d \). At the end of the isotopy we get a map sending each curve in \( \Upsilon^0 \) to a curve in \( S \setminus B \), and so gives rise to a map \( f_i : \Upsilon^0 \to C(S_{g,p+1}) \). Note that postcomposing with the map which forgets the point \( d \), we get the inclusion of \( \Upsilon^0 \) into \( C(S_{g,p}) \). Now it is easily seen that \( f_i \) preserves disjointness of curves, and so extends to a map \( f_i : \Upsilon \to C(S_{g,p+1}) \), which maps \( \Upsilon \) isomorphically to a full subcomplex \( \Upsilon_i = f_i(\Upsilon) \subseteq C(S_{g,p+1}) \). Now let \( \alpha \) and \( \beta \) be, respectively, the boundary curves of small regular neighbourhoods of \([a, d]\) and \([d, b]\) in \( I \). Let \( \Omega^0 = \{\alpha, \beta\} \cup \bigcup_{i=0}^{m} \Upsilon^0_i \), and let \( \Omega \) be the full subcomplex of \( C(\Sigma) \) with vertex set \( \Omega_0 \). We claim that \( \Omega \) is PL homeomorphic to the standard \((n+1)\)-sphere.

Note first that \( \Upsilon_0^0 \) and \( \Upsilon_m^0 \) are respectively the sets of points adjacent to \( \alpha \) and \( \beta \).

Now, given \( i \in \{0, \ldots, m\} \), let \( \Omega^0_i = \{\alpha\} \cup \bigcup_{j=0}^{i} \Upsilon^0_j \), and let \( \Omega_i \) be the full subcomplex with vertex set \( \Omega^0_i \). Now \( \Omega_0 \) is the cone on \( \Upsilon_0 \) with vertex \( \alpha \), and so PL homeomorphic to a ball with boundary \( \Upsilon_0 \). We claim that, for all \( i \), there is a PL homeomorphism of \( \Omega_0 \) to \( \Omega_i \) whose restriction to \( \Upsilon_0 \) is the map \( f_i \circ f_0^{-1} : \Upsilon_0 \to \Upsilon_m \).

Suppose, inductively, that this holds for \( i \). Moving from \( \Upsilon_i \) to \( \Upsilon_{i+1} \) corresponds to pushing one of the curves of \( \Upsilon_i^0 \) across the hole, \( d \), of \( S_{g,p+1} \). In other words, there is some \( \gamma \in \Upsilon_i^0 \) such that \( f_i(\Upsilon_i^0 \setminus \{\gamma\}) = f_{i+1}(\Upsilon_i^0 \setminus \{\gamma\}) \), but with \( \delta = f_i(\gamma) \neq \epsilon = f_{i+1}(\gamma) \). Thus, \( \Omega_i^{0+1} = \Omega_i^0 \cup \{\epsilon\} \). Now \( \delta \) and \( \epsilon \) are clearly adjacent. In fact, it is easily checked that the set of curves in \( \Omega_i \) adjacent to \( \epsilon \) are precisely those of the form \( f_i(\zeta) \), where \( \zeta \in \Upsilon \) is equal to or adacent to \( \gamma \). Thus, \( \Omega_{i+1} \) is obtained from \( \Omega_i \) by attaching a cone with vertex \( \epsilon \) to the star of \( \delta \) in \( \Upsilon_i \). Given that \( \Omega_i \) is a PL \((n+1)\)-ball with boundary \( \Upsilon_i \), we can find a PL homeomorphism of \( \Omega_i \) to \( \Omega_{i+1} \) which restricts to \( f_{i+1} \circ f_i^{-1} \) on \( \Upsilon_i \). Precomposing this with the homeomorphism from \( \Omega_0 \) to \( \Omega_i \) proves the inductive step of the statement.

In particular, we have a PL homeomorphism from \( \Omega_0 \) to \( \Omega_m \) whose restriction to \( \Upsilon_m \) is \( f_m \circ f_0^{-1} \). Now \( \Omega \) is obtained from \( \Omega_m \) by coning the boundary, \( \Upsilon_m \), with vertex \( \beta \). Thus, \( \Omega \) is PL homeomorphic to a suspension of \( \Upsilon \), hence a PL \((n+1)\)-sphere. \( \square \)

Now it is easy to find a pentagon which is a full subcomplex of \( C(S_{0,5}) \) or equivalently in \( C(S_{1,2}) \). Note that these cases correspond to \( \xi = \xi' = 2 \). Therefore, applying Lemma 5.7 inductively, we find a \((\xi - 1)\)-sphere for all surfaces of genus at most 1, where \( \xi \geq 2 \). Note
that $C(S_{2,0}) \cong C(S_{0,6})$ so this deals with that case also. (An explicit description of a sphere in the case of $S_{2,0}$ can be found in [Br].)

This completes the proof Proposition 5.6.

To construct our quasi-isometric embedding of $\mathbb{R}^\xi$ into $R(\Sigma)$ in these cases, we could assume that, in defining the marking complex, we have taken the intersection bounds large enough so that some marking $a$ contains all the vertices of a $(\xi - 1)$-sphere, $\Upsilon$, in $C(\Sigma)$. Now construct $\lambda_a : \mathcal{O}(\Upsilon) \to R(\Sigma)$ as above and apply Lemma 5.5. (Alternatively, we could take different marking containing each multicurve of $S(\Upsilon)$.) Note that these markings are all a bounded distance apart. Thus, if $\tau' \subseteq \tau \in S(\Upsilon)$, then the map of $\mathcal{O}(\tau')$ into $R(\Sigma)$ agrees up to bounded distance with the restriction of the map of $\mathcal{O}(\Upsilon)$ into $R(\Sigma)$. Therefore, these maps again combine to give a quasi-isometric embedding of $\mathcal{O}(\Upsilon)$ into $R(\Sigma)$.)

This proves the “if” part of Theorem 1.4.

6. Product structure and stratification

In this section, we show how the extended asymptotic cone, $R^*(\Sigma)$ can be partitioned into “strata” indexed by (ultralimits of) multicurves. The strata have a local product structure, where the factors correspond to extended asymptotic cones of subsurfaces. This structure arises from a kind of coarse stratification of $R(\Sigma)$. We begin by describing a coarse product structure based on multicurves.

Recall that $\mathcal{S}$ is the set of all multicurves in $\Sigma$. We allow $\emptyset \in \mathcal{S}$. If $\tau \in \mathcal{S}$, we will use the following notation (as in [Bo4]). We write $\mathcal{X}_A(\tau) = \{X(\gamma) \mid \gamma \in \tau\}$. We write $\mathcal{X}_N(\tau)$ for the set of components of $\Sigma \setminus \tau$ which are not $S_{0,3}$’s, and set $\mathcal{X}(\tau) = \mathcal{X}_N(\tau) \cup \mathcal{X}_A(\tau)$. We write $\mathcal{X}_F(\tau) = \{Y \in \mathcal{X}(\tau) \mid \gamma \cap Y \neq \emptyset \}$; that is, there is some $\gamma \in \tau$ with $\gamma \cap Y$ or $\gamma \prec Y$.

Let $\tau$ be a (possibly empty) multicurve. Let $L(\tau)$ be the set of $a \in R(\Sigma)$ such that $\tau \subseteq a$ and $\tau$ does not cross $\hat{a}$ (so that $\tau \cup \hat{a}$ is a multicurve). Thus $a \in L(\hat{a})$ for all $a \in R(\tau)$. Given $r \geq 0$, let $L(\tau; r) = \{a \in R(\Sigma) \mid \varepsilon(\hat{a}, \tau) \leq r\}$. One can check that for all sufficiently large $r$ (in relation to $\xi(\Sigma)$) we have $\text{hd}(L(\tau), L(\tau; r))$ is bounded. As in Lemma 7.8 of [Bo2], we see that $a \in R(\Sigma)$ is a bounded distance from $L(\tau)$ if and only if $\sigma_Y(\theta_Y a, \theta_Y \tau)$ is bounded for all $Y \in \mathcal{X}_F(\tau)$.

Let $\mathcal{L}(\tau) = \prod_{X \in \mathcal{X}(\tau)} R(X)$. We give this the $l^1$ metric (though any quasi-isometrically equivalent geodesic metric would serve for our purposes). Note that this has a product coarse median structure. Combining the maps $\psi_X : R(\Sigma) \to R(X)$, we get a coarsely lipschitz quasimorphism $\psi_\tau : R(\Sigma) \to \mathcal{L}(\tau)$.
There is also a quasimorphism \( v_\tau : \mathcal{L}(\tau) \rightarrow \mathcal{R}(\Sigma) \) with image \( L(\tau) \) and with \( \psi_\tau \circ v_\tau \) the identity up to bounded distance. Such a map for markings was described in [Bo1]: essentially the same construction having featured in the proof of the consistency theorem in [BeKMM]. This is obtained by taking the given markings on the non-\( S_{0,3} \) components of \( \Sigma \setminus \tau \) and combining them, together with \( \tau \) and a suitable “marking curve” for each element of \( \tau \), to give a marking of \( \Sigma \). The marking curves are chosen (after applying suitable powers of Dehn twists) so that the projections to the annuli corresponding to \( \tau \) are as prescribed by the element of \( \mathcal{L}(\tau) \). Note the union of the original markings are a subset of the combined marking, and so if we started with decorated markings, we can combine the decorations to give a decoration of the combined marking. Any new marking curves introduced (that is those which cross an element of \( \tau \)) are assigned a decoration of \( 0 \). It is easily checked that the map \( v_\tau \) thus defined has the stated properties.

We now set \( \omega_\tau : v_\tau \circ \psi_\tau : \mathcal{R}(\Sigma) \rightarrow L(\tau) \). Exactly as in the case of markings (Lemma 7.9 of [Bo2]), we see that this is a coarse gate map the set \( L(\tau) \), that is, \( \rho(\omega_\tau x, \mu(x, \omega_\tau(x), c)) \) is bounded for all \( x \in \mathcal{R}(\Sigma) \) and all \( c \in L(\tau) \).

We now apply this to the extended asymptotic cone, \( \mathcal{R}^*(\Sigma) \). Recall that we have maps: \( \theta_X^* : \mathcal{R}^*(\Sigma) \rightarrow \mathcal{G}^*(X) \) and \( \psi_X^* : \mathcal{R}^*(\Sigma) \rightarrow \mathcal{R}^*(X) \) for all \( X \in \mathcal{U} \).

If \( \tau \) is a multicurve, we have \( \psi_X^* \circ \theta_X^* : \mathcal{R}^*(\Sigma) \rightarrow \mathcal{G}^*(X) \) and \( \theta_X^* \circ \psi_X^* : \mathcal{G}^*(X) \rightarrow \mathcal{R}^*(\Sigma) \). We write \( \omega_X^* = \theta_X^* \circ \psi_X^* : \mathcal{G}^*(X) \rightarrow \mathcal{R}^*(\Sigma) \). Note that \( \mathcal{G}(\tau) \) is a direct product of the spaces \( \mathcal{G}^*(X) \) as \( X \) varies in \( \mathcal{U}(\tau) \).

Write \( L^*(\tau) = \omega_X^*(\mathcal{G}^*(\tau)) = \omega_\tau(\mathcal{R}^*(\Sigma)) \). This is the limit of the sets \( (\mathcal{L}(\tau_\xi))_\xi \). Note that \( L^*(\tau) \) is median convex in \( \mathcal{R}^*(\Sigma) \), and that \( \omega_X : \mathcal{R}^*(\Sigma) \rightarrow L^*(\tau) \) is the gate map (that is \( \omega_X(x) \in [x, c] \) for all \( x \in \mathcal{R}^*(\Sigma) \) and all \( c \in L^*(\tau) \)). In particular, \( \omega_X^* L^*(\tau) \) is the identity.

Note that if \( \tau \) is big, then \( L^\infty(\tau) \) is the direct product of \( \xi \) almost furry \( \mathbb{R} \)-trees. (Since in this case each factor of \( L(\tau_\xi) \) is quasi-isometric to either a hyperbolic plane or a horodisc.)

We now move on to describe the stratification.

Let \( \mathcal{S} \) be the set of standard multicurves in \( \Sigma \). We allow \( \emptyset \in \mathcal{S} \). Given \( \tau \in \mathcal{S} \), let \( \Theta(\tau) = \{ a \in \mathcal{R}(\Sigma) \mid \hat{a} = \tau \} \). Thus \( \Theta(\tau) \subseteq L(\tau) \), and \( \Theta(\emptyset) = \mathcal{R}(\Sigma) \).

Lemma 6.1. Given any \( \tau, \tau' \in \mathcal{S} \) and any \( a \in \Theta(\tau) \) and \( b \in \Theta(\tau') \), there is some \( c \in \Theta(\tau \cap \tau') \) with \( \mu(a, b, c) \sim c \).

Proof. Let \( d \in \mathcal{R}^0(\Sigma) \) be obtained from \( a \) by setting \( \bar{d} = \bar{a} \), rescaling the decorations on \( \tau \setminus \tau' \subseteq \hat{a} \) equal to \( 0 \), and leaving all other decorations on
a unchanged. Thus \( d \in \Theta(\tau \cap \tau') \). Now apply Dehn twists to \( d \) about the curves of \( \tau \setminus \tau' \) to give \( c \in \mathcal{R}^0(\Sigma) \), so that \( \theta_{\gamma} c \sim \theta_{\gamma} b \) for all \( \gamma \in \tau \setminus \tau' \). We also have \( c \in \Theta(\tau \cap \tau') \). Suppose \( X \in \mathcal{X} \). If \( X = X(\gamma) \) for some \( \gamma \in \tau \setminus \tau' \), then \( \theta_X c \sim \theta_{\tau} b \). If \( X \) is not of this form, we get \( \theta_X c \sim \theta_{\tau} a \). In all cases, we have \( \theta_X \mu(a, b, c) \sim \mu(\theta_X a, \theta_X b, \theta_X c) \sim \theta_X c \). It follows by Lemma 4.2 that \( \mu(a, b, c) \sim c \). \[ \square \]

We now pass to the asymptotic cone. Write \( \mathcal{US} \) for the ultraproduct of \( \mathcal{S} \). Recall that we have an intersection operation defined on \( \mathcal{US} \).

Given any \( \tau \in \mathcal{US} \), let \( \Theta(\tau) \) be the limit of the sets \( \Theta^*(\tau) \) be the limit of \( \left( \Theta(\tau_i) \right)_i \). Note that \( \Theta^*(\tau) \) is closed, and \( \Theta^*(\tau) \subseteq L^*(\tau) \). Also, clearly \( \mathcal{R}^*(\Sigma) = \bigcup_{\tau \in \mathcal{US}} \Theta^*(\tau) \).

**Lemma 6.2.** Given any \( \tau, \tau' \in \mathcal{US} \), and any \( a \in \Theta^*(\tau) \), \( b \in \Theta^*(\tau') \), then \( [a, b] \cap \Theta^*(\tau \cap \tau') \neq \emptyset \).

**Proof.** Choose \( a_\zeta \in \Theta(\tau) \) and \( b_\zeta \in \Theta(\tau') \) with \( a_\zeta \to a \) and \( b_\zeta \to b \). Let \( c_\zeta \in \Theta(\tau \cap \tau') \) be as given by Lemma 6.1. Then \( c_\zeta \to c \in \Theta^*(\tau \cap \tau') \) and \( \mu(a, b, c) = c \), that is, \( c \in [a, b] \). \[ \square \]

In particular, it follows that \( \Theta^*(\tau) \cap \Theta^*(\tau') \subseteq \Theta^*(\tau \cap \tau') \). Therefore, given any \( a \in \mathcal{R}^* \), there is a unique minimal \( \tau \in \mathcal{US} \) with \( a \in \Theta^*(\tau) \). We write \( \tau(a) = \tau \). Since the sets \( \Theta^*(\tau) \) are all closed, we see that the map \( \tau : \mathcal{R}^*(\Sigma) \to \mathcal{US} \) is lower semicontinuous. Moreover, given any \( a, b \in \mathcal{R}^*(\Sigma) \), there is some \( c \in \mathcal{R}^*(\Sigma) \) with \( \tau(c) \subseteq \tau(a) \cap \tau(b) \).

More generally, suppose that \( C \subseteq \mathcal{R}^* \) is convex. Choose \( a \in C \) with \( \tau(a) \) minimal. If \( b \in C \), then there is some \( c \in [a, b] \subseteq C \) with \( \tau(c) \subseteq \tau(a) \cap \tau(b) \), so \( \tau(a) = \tau(c) \subseteq \tau(b) \). We write \( \tau(C) = \tau(a) \).

Thus, \( \tau(C) \) is uniquely determined by the property that \( \tau(C) \subseteq \tau(b) \) for all \( b \in C \) and \( \tau(C) = \tau(a) \) for some \( a \in C \). Note that this applies in particular, if \( C \) is a component of \( \mathcal{R}^* \). Note that a component, \( C \), of \( \mathcal{R}^* \) is thick (i.e. \( C \cap \mathcal{R}^*_T(\Sigma) \neq \emptyset \)) if and only if \( \tau(C) = \emptyset \).

Now, given \( \tau \in \mathcal{US} \), let \( \Xi(\tau) = \{ a \in \mathcal{R}^*(\Sigma) \mid \tau(a) = \tau \} \). Clearly, \( \Xi(\tau) \subseteq \Theta^*(\tau) \) and \( \Xi(\emptyset) = \Theta^*(\emptyset) = \mathcal{R}^*_T \). Since \( \tau : \mathcal{R}^* \to \mathcal{US} \) is lower semicontinuous, we have that \( \Xi(\tau) \) is open in \( \Theta^*(\tau) \). Also:

**Lemma 6.3.** For all \( \tau \in \mathcal{US} \), \( \Xi(\tau) \) is dense in \( \Theta^*(\tau) \).

**Proof.** Let \( a \in \Theta^*(\tau) \), and choose \( a_\zeta \in \Theta^*(\tau_i) \), with \( a_\zeta \to a \). Thus \( \tau(a_\zeta) \subseteq \tau_\zeta \). Given any \( i \in \mathbb{N} \), let \( a_{i, \zeta} \in \mathcal{R} \) be the decorated multicurve with \( \tilde{a}_{i, \zeta} = \tilde{a}_\zeta \), and resetting the decoration, \( \eta_{a_{i, \zeta}}(\gamma) \), on each \( \gamma \in \tau_\zeta \) equal to \( \eta_{a_\zeta}(\gamma) + i \). Now \( (a_{i, \zeta})_i \) is a (quasi)geodesic sequence in \( \mathcal{R} \) with \( a_0 = a \) and with \( N(a_{i, \zeta}, i) \subseteq \Theta(\tau_\zeta) \) for all \( i \). In fact, \( \rho(a_i, \Theta(\tau_i')) \geq i \) for all \( \tau' \neq \tau \).
Passing to the asymptotic the cone, we see that from any \( a \in \Theta^*(\tau) \), there is a bilipschitz embedded ray, \( \lambda \), emanating for which \( \lambda(t) \in \Xi(\tau) \) for all \( t > 0 \).

In fact, the argument shows that \( \Xi(\tau) \cap C \) is dense in \( \Theta^*(\tau) \cap C \) for any component, \( C \) of \( \mathcal{R}^* \).

Write \( \mathcal{S}_C \subseteq \mathcal{S} \) for the set of standard complete multicurves, and write \( \mathcal{US}_C \subseteq \mathcal{US} \) for the set of complete multicurves.

Note that, if \( a \in \mathcal{R}(\Sigma) \), then \( a \) is a bounded distance from a marking, \( b \), which contains a complete multicurve, \( \tau \supseteq \hat{a} \). Thus \( \bigcup_{\tau \in \mathcal{S}_C} \Theta(\tau) \) is cobounded in \( \mathcal{R}(\Sigma) \). We deduce:

**Lemma 6.4.** \( \mathcal{R}^*(\Sigma) = \bigcup_{\tau \in \mathcal{US}_C} \Theta^*(\tau) \).

From this, we immediately get:

**Lemma 6.5.** If \( \tau \in \mathcal{US}_C \), then \( \Xi(\tau) \) is open in \( \mathcal{R}^* \). Also \( \bigcup_{\tau \in \mathcal{US}_C} \Xi(\tau) \) is dense in any component of \( \mathcal{R}^* \).

In this way, we see that \( (\Xi(\tau))_{\tau \in \mathcal{US}} \) defines a stratification of \( \mathcal{R}^* \).

We also note:

**Lemma 6.6.** For all \( \tau \in \mathcal{US} \), \( \Xi(\tau) \) lies in the interior of \( L^*(\Sigma) \) in \( \mathcal{R}^*(\Sigma) \).

*Proof.* For all \( a, b \in \Xi(\tau) \), then by lower semicontinuity, there is some open \( U \subseteq \mathcal{R}^* \) with \( \tau(b) \supseteq \tau \) for all \( b \in U \). So \( b \in L^*(\tau(b)) \subseteq L^*(\tau) \).

This shows that \( U \subseteq L^*(\tau) \). \( \square \)

**Lemma 6.7.** Suppose \( \tau \in \mathcal{S} \), \( d \in \Theta(\tau) \) and \( \tau' \subseteq \tau \). Then there exist \( a, b, c \in \Theta(\tau') \cap L(\tau) \) with \( d \sim \mu(a, b, c) \). Moreover, we can choose \( a, b, c \) so that \( \rho(d, a), \rho(d, b) \) and \( \rho(d, c) \) are all bounded above by a fixed linear function of \( \rho(d, \Theta(\tau')) \).

*Proof.* Let \( a \in \mathcal{R} \) be obtained from \( d \) by resetting all decorations on \( \tau \setminus \tau' \) equal to 0. Thus \( a \in \Theta(\tau') \cap L(\tau) \). By applying suitable powers of Dehn twists about each curve \( \gamma \in \tau \setminus \tau' \), we obtain \( b, c \in \Theta(\tau') \cap L(\tau) \), so that \( \theta_\gamma d \sim \mu(\theta_\gamma a, \theta_\gamma b, \theta_\gamma c) \) in \( H(\gamma) \), and such that \( \sigma_\gamma(d, a) \sim \sigma_\gamma(d, b) \sim \sigma_\gamma(d, c) \). Note that \( \theta_\gamma^X a \sim \theta_\gamma^X b \sim \theta_\gamma^X c \sim \theta_\gamma^X d \) for all \( X \in \mathcal{X} \) not of the form \( X(\gamma) \) for \( \gamma \in \tau \setminus \tau' \). It now follows that \( \mu(a, b, c) \sim d \), so that \( \rho(d, a) \asymp \rho(d, b) \asymp \rho(d, c) \asymp \sum_{\gamma \in \tau \setminus \tau'} \sigma(d, a) \asymp d(a, \Theta(\tau')). \) \( \square \)

**Lemma 6.8.** Suppose \( \tau \in \mathcal{US} \), \( d \in \Theta^*(\tau) \) and \( \tau' \subseteq \tau \). Then there exist \( a, b, c \in \Theta^*(\tau') \cap L^*(\tau) \) with \( d = \mu(a, b, c) \). Moreover, \( \rho(d, a), \rho(d, b) \) and \( \rho(d, c) \) are all bounded above by a fixed multiple of \( \rho(d, \Theta^*(\tau)) \).
Proof. Take \( d, \zeta \in \Theta(\tau) \) and apply Lemma 6.7 to give \( a, b, c \in \Theta(\tau) \cap L(\tau) \). Let \( a \rightarrow a, b \rightarrow b \) and \( c \rightarrow c \). The additive constants go away, and we get \( d = \mu(a, b, c) \). \( \square \)

Note in particular, setting \( \tau' = \emptyset \), we get that if \( d \in \Theta^\infty(\tau) \subseteq \mathcal{R}^\infty \) lies in a thick component, \( \mathcal{R}^\infty \), of \( \mathcal{R}^* \), then we can find \( a, b, c \in L^\infty(\tau) \cap \mathcal{R}^\infty_T \) with \( d = \mu(a, b, c) \).

In what follows, let \( \mathcal{R}^\infty(\Sigma) \) be any component of any extended asymptotic cone, \( \mathcal{R}^*(\Sigma) \) of \( \mathcal{R}(\Sigma) \).

If \( \tau \in \mathcal{S}_B \), the \( L^\infty(\tau) \) is a closed convex subset which is a direct product of \( \xi \) almost furry \( \mathbb{R} \)-trees. (Recall that “almost furry” means non-trivial and with no 2-valent element.) Now it was shown in [Bo2], that any continuous injective map of a direct product of \( \xi \) almost furry trees with closed image is necessarily a median isomorphism onto its range, which must be convex. In particular, we see that the median structure on \( L^\infty(\tau) \) is determined by its intrinsic topology. We deduce:

**Lemma 6.9.** Suppose that \( f : \mathcal{R}^\infty(\Sigma) \rightarrow \mathcal{R}^\infty(\Sigma) \) is a homeomorphism and that \( \tau \in \mathcal{US}_B \). Then \( f|L^\infty(\tau) \) is a median isomorphism onto its range, \( L^\infty(\tau) \), which is convex.

We also get:

**Lemma 6.10.** Suppose that \( \mathcal{R}^\infty \) is a thick component of \( \mathcal{R}^* \) and that \( f : \mathcal{R}^\infty \rightarrow \mathcal{R}^\infty \) is a homeomorphism with \( f|\mathcal{R}^\infty_T \) the identity. Then \( f \) is the identity on \( \mathcal{R}^\infty \).

**Proof.** Let \( x \in \mathcal{R}^\infty(\Sigma) \). By Lemma 6.4, \( x \in \Theta^\infty(\tau) \) for some \( \tau \in \mathcal{US}_C \). By Lemma 6.8, we can find \( a, b, c \in \mathcal{R}^\infty_T \cap \Theta^\infty(\tau) \) with \( x = \mu(a, b, c) \). By Lemma 6.9, \( f|L^\infty(\tau) \) is a median homomorphism, and so \( fx = x \). \( \square \)

### 7. Quasi-isometric maps

In this section, we apply the results of Section 6 to quasi-isometries between Teichmüller spaces. In particular, we give proofs of Theorems 1.5, 1.6 and 1.7.

Let \( M \) be a topological median algebra. We say that a subset \( P \subseteq M \) is square-free if there is no square in \( M \) with a side contained in \( P \).

If \( P \) is closed and convex, a projection map to \( P \) is a (necessarily continuous) map \( \omega : M \rightarrow P \) such that for all \( x \in M \) and \( y \in P \) we have \( \omega(x) \in [x, y] \). In particular, \( \omega|P \) is the identity. (Such a map is unique if it exists.)

Note that if \( P \) is square-free and \( x, y \in M \) with \( [x, y] \cap P = \emptyset \), then \( \omega(x) = \omega(y) \) (otherwise, we would have a square \( \omega(x), \omega(y), \mu(x, y, \omega(y)), \mu(x, y, \omega(x)) \))
with side $\{\omega(x), \omega(y)\}$ in $P$. If $M$ is weakly locally convex, it then follows that $\omega : M \to P$ is locally constant on $M \setminus P$.

Now suppose $P \subseteq M$ with $\omega : M \to P$ a locally constant retraction. If $p \in P$ separates $x, y \in P$, then $p$ also separates $x, y$ in $M$. (Note that $\omega^{-1}(p) \setminus \{p\}$ is open $M$. Thus, if $P \setminus \{p\} = U \cup V$ is an open partition of $P \setminus \{p\}$, then $M \setminus \{p\} = (\omega^{-1}(U \cup \{p\}) \setminus \{p\}) \cup \omega^{-1}V$ is an open partition of $M \setminus \{p\}$. Taking $U, V$ so that $x \in U$ and $y \in V$, the claim follows.)

Putting the above observations together, we have shown:

**Lemma 7.1.** Suppose that $M$ is a weakly locally convex topological median algebra, and $P \subseteq M$ is closed convex and square-free and admits a gate map. If $p, x, y \in P$, and $p$ separates $x$ from $y$ in $P$, then it also separates $x$ from $y$ in $M$.

In particular, any cut point of $P$ will also be a cut point of $M$.

In our situation, $R^\infty$ is certainly (weakly) locally convex, see [Bo1]).

We now consider some constructions of such $P$.

**Definition.** We say that a set $P \subseteq R(\Sigma)$ is coarsely square-free if given any square, $Q$, and any quasimorphism $\phi : Q \to R(\Sigma)$ which maps some side of $Q$ into $P$, there is a (possibly different) side, $c, d$, of $P$ with $\rho(\phi c, \phi d)$ bounded.

(Evidently, this entails implicit constants of quasimorphism and side-length.)

**Definition.** Given $k \geq 0$ and $P \subseteq R(\Sigma)$, we say that $P$ is $k$-straight if $\mathrm{diam} \, \theta_X(P) \leq k$ for $X \in X \setminus \{\Sigma\}$. We say $P$ is coarsely straight if it is $k$-straight for some $k \geq 0$.

Note that by the distance formula of Rafi (Proposition 4.9 here), we see that if $a, b \in P$, then $\rho(a, b) \asymp \sigma_\Sigma(a, b)$. (Here $\asymp$ denotes agreement to within linear bounds depending on $k$.) In other words, the map $\theta_\Sigma : P \to \mathcal{H}(\Sigma) = \mathcal{G}(\Sigma)$ is a quasi-isometric embedding.

The following makes use of a more general statement about quasicubes, whose formal statement we postpone until Section 8, see Lemma 8.3. In fact, it is a simple consequence of Lemma 4.8.

**Lemma 7.2.** A coarsely straight set is coarsely square-free.

**Proof.** Let $Q = \{a, b, d, c\}$ be a square, with sides, $\{a, b\}$ and $\{a, c\}$, and let $\phi : Q \to R$ be a quasimorphism, with $\phi a, \phi b \in P$. Let $k_0$ be the constant of Lemma 8.3. Now $\rho(a, b) \asymp \sigma_\Sigma(a, b)$, so $\rho(a, b)$ is sufficiently large (in relation to $\xi$ and the straightness constant) we will have $\sigma_\Sigma(a, b) \geq k_0$. Similarly, from Lemma 4.2, if $\rho(a, c)$ is sufficiently
large, we will have $\sigma_X(a,c) \geq k_0$ for some $X \in \mathcal{X}$. But Lemma 8.3 now gives the contradiction that $X \land \Sigma$.

\begin{lemma}
Suppose that $P$ is coarsely straight. Then $P$ is coarsely convex in $\mathcal{R}(\Sigma)$ if and only if $\theta_{\Sigma}P$ is coarsely median convex in $\mathcal{G}(\Sigma)$.
\end{lemma}

\begin{proof}
The “only if” part is an immediate consequence of the fact that $\theta_{\Sigma} : \mathcal{R}(\Sigma) \to \mathcal{G}(X)$ is a quasimorphism. For the converse, suppose $a, b \in P$ and $c \in \mathcal{R}$. By hypothesis, there is some $d \in P$ with $\theta_{\Sigma}a \sim \mu(\theta_{\Sigma}a, \theta_{\Sigma}b, \theta_{\Sigma}c)$, and so $\theta_{\Sigma}d \sim \theta_{\Sigma}\mu(a, b, c)$. If $X \in \mathcal{X} \setminus \{\Sigma\}$, then $\theta_Xa \sim \theta_Xb \sim \theta_Xd \sim \theta_X\mu(a, b, c)$, and so $\thetaXd \sim \mu(\theta_Xa, \theta_Xb, \theta_Xc) \sim \theta_X\mu(a, b, c)$. It follows by Lemma 4.2, that $d \sim \mu(a, b, c)$. In other words, this shows that the coarse interval $[a, b]$ lies in a bounded neighbourhood of $P$ as required.

Clearly, if $P$ is coarsely convex and coarsely square free, then $P^\infty$ is closed convex and square-free in $\mathcal{R}$.

We can construct examples of such $P$ from coarsely straight sequences.

\begin{definition}
We say that a bi-infinite sequence, $(a_i)_{i \in \mathbb{Z}}$ in $\mathcal{R}(\Sigma)$ is coarsely straight if $\rho(a_i, a_{i+1})$ is bounded above for all $i$, and if $\sigma_{\Sigma}(a_i, a_j)$ is bounded below by an increasing linear function of $|i - j|$.\end{definition}

Note that, the since the first condition implies also that $\sigma_X(a_i, a_{i+1})$ is bounded above, the second condition is equivalent to saying that the sequence $(\theta_{\Sigma}a_i)_i$ is quasigeodesic in $\mathcal{G}(\Sigma)$.

\begin{lemma}
If $(a_i)_i$ is a coarsely straight sequence, then the set $\{a_i \mid i \in \mathbb{Z}\}$ is coarsely straight in $\mathcal{R}(\Sigma)$.
\end{lemma}

\begin{proof}
Since $(\theta_{\Sigma}a_i)_i$ is quasigeodesic in $\mathcal{G}(\Sigma)$, we can find $m, n \in \mathbb{Z}$, with $0 \leq n - m$ bounded, and with $\sigma_{\Sigma}(\theta_{\Sigma}a_i, \theta_{\Sigma}(\partial X)) \geq t$ and $\sigma_{\Sigma}(\theta_{\Sigma}a_j, \theta_{\Sigma}(\partial X)) \geq t$ for all $i \leq m$ and all $j \geq n$, where $t$ is an upper bound on $\sigma_X(a_p, a_{p+1})$. (Of course, this might hold for all $i, j \in \mathbb{Z}$.) By the Bounded Geodesic Image Theorem of [MaM2] (Theorem 3.1 thereof), it follows that $\sigma_X(a_i, a_m)$ and $\sigma_X(a_j, a_n)$ are bounded for all $i \leq m$ and all $j \geq n$. Also, if $m \leq i \leq j \leq n$, then $\rho(a_i, a_j)$ is bounded, so $\sigma_X(a_i, a_j)$ is bounded. It follows that $\sigma_X(a_i, a_j)$ is bounded for all $i, j \in \mathbb{Z}$ as required.

Note that it now follows that $(a_i)_i$ is coarsely convex, and it is also quasigeodesic in $\mathcal{R}(\Sigma)$.

We can also define a coarse gate map to $P = \{a_i \mid i \in \mathbb{Z}\}$. Given $c \in \mathcal{R}$, we can find $n \in \mathbb{Z}$ such that $\mu(\theta_{\Sigma}a_n, \theta_{\Sigma}c, \theta_{\Sigma}a_i) \sim \theta_{\Sigma}a_n$ for all $i \in \mathbb{Z}$. Then $\theta_{\Sigma}\mu(a_n, c, a_i) \sim \theta_{\Sigma}a_n$. Also, for all $X \in \mathcal{X} \setminus \{\Sigma\}$, we
have $\theta_X a_i \sim \theta_X a_n$, so $\theta_X \mu(a_n, c, a_i) \sim \theta_X a_n$. It follows that by setting $\omega(c) = a_n$ we obtain a coarse gate map $\omega : \mathcal{R}(\Sigma) \rightarrow P$.

Now $P^*$ is a convex subset of $\mathcal{R}^*(\Sigma)$ median isomorphic to $\mathbb{R}^*$. Restricting to the standard component, $\mathcal{R}^\infty(\Sigma)$, of $\mathcal{R}^*(\Sigma)$, we see that $P^\infty$ is closed and convex in $\mathcal{R}^\infty(\Sigma)$ and median isomorphic to $\mathbb{R}$. In particular, it is a bi-lipschitz embedding of $\mathbb{R}$. Moreover, $P^\infty$ is square-free and admits a gate map, $\omega^\infty : \mathcal{R}^\infty(\Sigma) \rightarrow P^\infty$. It follows that any point of $P^\infty$ is a cut point of $\mathcal{R}^\infty(\Sigma)$.

Finally, note that if $g \in \operatorname{Map}(\Sigma)$ is pseudoanosov, then the $\langle g \rangle$-orbit of any point of $\mathcal{G}(X)$ is quasigeodesic (see [MaM1]). We see that if $a \in \mathcal{R}(\Sigma)$, then $(g' a)_i$ is a coarsely straight sequence, hence coarsely convex by the above. This gives rise to a line in $\mathcal{R}^\infty(\Sigma)$ all of whose points are cut points of $\mathcal{R}^\infty_T$.

Since $\mathcal{U} \operatorname{Map}(\Sigma)$ acts transitively on $\mathcal{R}^*_T$, we deduce:

**Lemma 7.5.** Each point of $\mathcal{R}^*_T$ is a cut point of the component of $\mathcal{R}^*_T$ in which it lies.

In fact, we have a converse:

**Lemma 7.6.** If $x \in \mathcal{R}^* \setminus \mathcal{R}^*_T$, then $x$ is not a cut point of the component in which it lies.

**Proof.** Let $\tau = \tau(x) \in \mathcal{S}$. By assumption, $\tau \neq \emptyset$. By Lemma 6.6, $x$ lies in the interior of $L^\infty(\tau)$ in $\mathcal{R}^\infty_T$. But $L^\infty(\tau)$ is a median, hence topological, direct product of at least two non-trivial path-connected spaces, and so any two points lie in an embedded disc. \(\square\)

We see that $\mathcal{R}^*_T$ is determined by the topology of $\mathcal{R}^*$ as the set of cut points. We deduce:

**Lemma 7.7.** Suppose that $\Sigma, \Sigma'$ are compact surfaces, and $f : \mathcal{R}^*(\Sigma) \rightarrow \mathcal{R}^*(\Sigma')$ is a homeomorphism, then $f(\mathcal{R}^*_T(\Sigma)) = \mathcal{R}^*_T(\Sigma')$.

(Given that $\xi$ is the locally compact dimension of $\mathcal{R}^\infty(\Sigma)$, if such a homeomorphism exists, then $\xi(\Sigma) = \xi(\Sigma')$.)

Note that the above holds for any extended aysmptotic cones, for any choice of scaling factors.

Now suppose that $\Sigma$ and $\Sigma'$ are compact surfaces of complexity at least 2 and that $\phi : \mathcal{R}(\Sigma) \rightarrow \mathcal{R}(\Sigma')$ is a quasi-isometry. This induces a (bilipschitz) homoeomorphism, $f = \phi^* : \mathcal{R}^*(\Sigma) \rightarrow \mathcal{R}^*(\Sigma')$.

**Lemma 7.8.** There is some $k \geq 0$ such that $\operatorname{hd}(f(\mathcal{R}_T(\Sigma)), \mathcal{R}_T(\Sigma')) \leq k$.

**Proof.** By symmetry, it’s enough to show that $f(\mathcal{R}_T(\Sigma))$ lies in a bounded neighbourhood of $\mathcal{R}_T(\Sigma')$. Suppose to the contrary that we
can find an \( N \)-sequence, \((x_i)_i\), in \( \mathcal{R}(\Sigma) \) with \( r_i = \rho(f(x_i), \mathcal{R}_T(\Sigma')) \to \infty \). Let \( \mathcal{R}^\infty \) and \( \mathcal{R}^\infty(\Sigma') \) be asymptotic cones with \( Z = \mathbb{N} \), scaling factors \((r_i)_i\) and basepoints \((x_i)_i\) and \((\phi x_i)_i\). We get a homeomorphism, \( f = \phi^\infty : \mathcal{R}^\infty(\Sigma) \to \mathcal{R}^\infty(\Sigma') \) with \( \rho(f(x), \mathcal{R}^\infty(\Sigma')) = 1 \). But \( f(x) \in f(\mathcal{R}^\infty(\Sigma)) = \mathcal{R}^\infty(\Sigma') \), giving a contradiction. \( \square \)

This proves Theorem 1.6.

It now follows that \( \phi \) is a bounded distance from a map from \( \mathcal{R}_T(\Sigma) \) to \( \mathcal{R}_T(\Sigma') \). Recall (Lemma 3.4) that \( \mathcal{R}_T(\Sigma) \) is a uniformly embedded copy of the marking complex \( \mathcal{M}(\Sigma) \) of \( \Sigma \). Thus \( \phi \) gives rise to a quasi-isometry from \( \text{Map}(\Sigma) \) to \( \text{Map}(\Sigma') \).

Now, if \( \mathcal{M}(\Sigma) \) and \( \mathcal{M}(\Sigma') \) are quasi-isometric, then \( \Sigma \) and \( \Sigma' \) are homeomorphic, under the conditions described by Theorem 1.5. This follows using the result of [BeKMM, Ham], and is shown directly in [Bo4].

This proves Theorem 1.5.

Henceforth, we assume that \( \Sigma = \Sigma' \), so \( \phi : \mathcal{R}(\Sigma) \to \mathcal{R}(\Sigma) \) is a quasi-isometry. By [BeKMM, Ham], there is some \( g \in \text{Map}(\Sigma) \) such that \( \rho(gx, \phi x) \) is bounded for all \( x \in \mathcal{R}_T(\Sigma) \). Postcomposing with \( g^{-1} \), we may as well assume that \( g \) is the identity, so \( \rho(x, \phi x) \) is bounded. Thus, up to bounded distance, we can assume that \( \phi|\mathcal{R}_T(\Sigma) \) is the identity.

If we pass to any asymptotic cone, we will now have that \( f = \phi^*|\mathcal{R}^*_T \) is the identity. By Lemma 7.7, it follows that \( f \) is the identity on any thick component of \( \mathcal{R}^*_T(\Sigma) \). From this we get:

**Lemma 7.9.** There is a function, \( l : [0, \infty) \to [0, \infty) \) with \( l(t)/t \to 0 \) as \( t \to \infty \), such that for all \( x \in \mathcal{R}(\Sigma) \), we have \( \rho(x, \phi x) \leq l(\rho(x, \mathcal{R}_T(\Sigma))) \).

**Proof.** We claim that, for all \( \epsilon > 0 \), there is some \( u = u(\epsilon) \geq 0 \), such that if \( \rho(x, \mathcal{R}_T(\Sigma)) \geq u \), then \( \rho(x, \phi x) \leq \epsilon \rho(x, \mathcal{R}_T) \). For suppose not.

We can find a \( \mathbb{N} \)-sequence \((r_i)_i\), with \( r_i \to \infty \), \( x_i \in \mathcal{R} \), \( y_i \in \mathcal{R}_T \) with \( \rho(x_i, y_i) \leq r_i \) and \( \rho(x_i, \phi x_i) \geq \epsilon r_i \). Passing to an asymptotic cone, with scaling factors \((r_i)_i\) and basepoints \( x_i \) and \( \phi x_i \), we get \( f : \mathcal{R}^\infty \to \mathcal{R}^\infty \), and \( x \in \mathcal{R}^\infty \), \( y \in \mathcal{R}_T^\infty \) with \( \rho^\infty(x, y) \leq 1 \) and \( \rho^\infty(f, f) \geq \epsilon \). But, by the above, \( f \) is the identity on \( \mathcal{R}^\infty \) giving a contradiction.

We take \( u_n = u(1/n) \) for \( n \in \mathbb{N}, n \geq 1 \). We an suppose that \( u_{n+1} \leq u_n \). We take \( l \) equal to \( 2u_1 \) on \([0, u_1]\), and equal to \([t \to t/n]\) on \([u_n, u_{n+1}]\) for \( n \geq 1 \). \( \square \)

This now proves Theorem 1.7.
8. Quasicubes

A “quasicube” in $R(\Sigma)$ can be thought of as a subset median isomorphic to a cube up to bounded distance. We give a description of “nondegenerate” quasicubes of maximal dimension, and use this to describe cubes in $R^*(\Sigma)$. This will be used in Section 9 to conclude the proof of Theorem 1.4.

Given $a, b \in R$ and $r \geq 0$, let $A(a, b; r) = \{ X \in \sigma_X | \sigma_X(a, b) > r \}$. (As in the case of markings, there is some $r_0 \geq 0$, depending only on such $\xi(\Sigma)$ that $A(a, b; r_0)$ is always finite. In what follows, we can always take $r \geq r_0$, though we won’t be explicitly needing this.)

Given $a, b \in X$ and $r \geq 0$, we say that $a, b$ are $(X, r)$-related if $\rho(a, L(\partial X)) \leq r$, $\rho(b, L(\partial X)) \leq r$ and for all $Y \in \mathcal{X}$ we have either $Y \preceq X$ or $Y \not\in \mathcal{A}(a, b; r)$. We will often suppress mention of $r$ where a choice (ultimately depending on $\xi(\Sigma)$) is clear from context, and simply refer to $a, b$ as being “$X$-related”.

Note that this property is “median convex” in the sense that if $c \in R$, with $\rho(\mu(a, b, c), c) \leq l$, then $c$ is $(X, r')$-related to $a$ and to $b$, where $r'$ depends only on $r$, $l$ and $\xi(\Sigma)$.

Note also that if $X = X(\gamma) \in \mathcal{X}_A$, and $a, b$ are $X(\gamma)$-related, then $a, b$ differ, up to bounded distance, by some power of a Dehn twist about $\gamma$, possibly together with a redecoration of $\gamma$.

**Lemma 8.1.** Suppose $a, b$ are $(X, r)$-related. Then $\rho(a, b)$ agrees with $\rho_X(a, b)$ up to linear bounds depending only on $r$ and $\xi(\Sigma)$.

**Proof.** The fact that $\psi_X : R(\Sigma) \rightarrow R(X)$ is uniformly coarsely lipschitz immediately gives a linear upper bound for $\rho_X(a, b)$. For the other direction, set $\tau = \partial X$. Then $\psi_\tau a$ and $\psi_\tau b$ differ only in the $X$-coordinate. Since $\psi_\tau$ is uniformly coarsely lipschitz, we have that $\rho(\omega_\tau a, \omega_\tau b) = \rho(\psi_\tau a, \psi_\tau b)$ is linearly bounded above by $\rho_X(\psi_X a, \psi_X b) = \rho_X(a, b)$. By assumption $\rho(a, L(\tau))$ and $\rho(b, L(\tau))$ are bounded, and since $\omega_\tau$ is a coarse gate map to $L(\tau)$, it follows that $\rho(a, \omega_\tau a)$ and $\rho(b, \omega_\tau b)$ are bounded. This bounds $\rho(a, b)$, as required. \qed

Here is a criterion which implies that two decorated markings are $X$-related:

**Lemma 8.2.** Then there is a constant $r_1 \geq 0$ depending only on $\xi(\Sigma)$ with the following property. Suppose that $a, b \in R(\Sigma)$, $X \in \mathcal{X}$ and $r_2 \geq r_1$ such that for all $Y \in \mathcal{X}$, either $Y \not\in \mathcal{A}(a, b; r_2)$ or $Y \preceq X$, and if $\gamma$ is a curve with $\gamma \pitchfork X$, then there is some $Y \in \mathcal{A}(a, b; r_1)$ with $\gamma \pitchfork Y$. Then $a, b$ are $(X, r)$-related for some $r$ depending only on $r_2$ and $\xi(\Sigma)$.
We just need to check that $\rho(a, L(\tau))$ and $\rho(b, L(\tau))$ are bounded, where $\tau = \partial c(X)$. If $X \in \mathcal{X}_T(\tau)$, then there is some $Y \in \mathcal{A}(a, b; r_1)$ with $Y \cap X$ or $Y \preceq X$. Exactly as in Lemma 7.12 of [Bo4] we see that $\sigma_X(\theta_X a, \theta_X \tau)$ and $\rho_X(\psi_X b, \psi_X \tau)$ are bounded. Since this holds for all $X \in \mathcal{X}_T(\tau)$, the statement follows.

We now move on to consider quasicubes.

Suppose that $Q = \{-1, 1\}^n$ is an $n$-cube. By an $i$th side of $Q$, we mean an unordered pair, $c, d \in Q$, which differ precisely in their $i$th coordinates. Note that any two $i$th sides are parallel in the median sense. If $a, b \in Q$, we can speak of the $i$th sides crossed by $Q$, that is those which (up to parallelism) correspond to the coordinates for which $a, b$ differ.

Suppose that $\phi : Q \rightarrow \mathcal{R}(\Sigma)$ is an $h$-quasimorphism. If $c, d$ and $c', d'$ are both $i$th sides of $Q$, then $\rho(\phi c, \phi d) \asymp \rho(\phi c', \phi d')$. (Since $\mu(\phi c, \phi d, \phi c') \sim \phi c$ and $\mu(\phi c, \phi d, \phi d') \sim \phi d$, we get a linear upper bound for $\rho(\phi c, \phi d)$, and the lower bound follows symmetrically.) We will write $s_i = \min \rho(\phi c, \phi d)$ as $c, d$ ranges over all $i$th sides. Thus $\rho(\phi c, \phi d') \asymp s_i$ for any other $i$th sides, $c', d'$. We also note that for all $X \in \mathcal{X}$, we have $\sigma_X(\phi c, \phi d) \asymp \sigma_X(\phi c', \phi d')$ and $\rho_X(\phi c, \phi d) \asymp \rho_X(\phi c', \phi d')$ (similarly, since $\theta_X \circ \phi$ and $\psi_X \circ \phi$ are quasimorphisms to $\mathcal{G}(X)$ and $\mathcal{R}(X)$ respectively). Here, the linear bounds depend implicitly on $h$.

If $a, b \in Q$, then a repeated application of Lemma 2.4 shows that $\rho(\phi a, \phi b) \asymp \sum_i s_i$, where the sum is taken over all sides of $Q$ crossed by $a, b$.

**Lemma 8.3.** There is some $k_0 \geq 0$, depending only on $X, Y \in \mathcal{X}$ such that if $X, Y \in \mathcal{X}$ with $\sigma_X(\phi c, \phi d) \geq k_0$ and $\sigma_Y(\phi c', \phi d') \geq k_0$, where $c, d$ and $c', d'$ are respectively $i$th and $j$th sides of $Q$, then either $i = j$ or $X \land Y$.

**Proof.** This is an immediate consequence of Lemma 4.8.

It follows from Lemma 8.3 that for each $i$, there is a (possibly empty, possibly disconnected) “subsurface”, $Y_i$, of $\Sigma$, which contains all $X \in \mathcal{X}$, for which $\sigma_X(\phi c, \phi d) \geq k_0$ for any $i$th side, $c, d$, of $Q$. We can also take $Y_i$ to be minimal with this property. To ensure that each $Y_i$ must be non-empty, we will assume that $\phi$ is non-degenerate, that is, for each side, $c, d$, $\mathcal{A}(c, d; r) \neq \emptyset$, for a fixed $r \geq k_0$. Note that, by Lemma 4.2, this is implied by placing a suitable lower bound on $\min\{s_i \mid 1 \leq i \leq n\}$. We will also take $r$ to be at least the constant $r_1$ featuring in Lemma 8.2.
Recall that $\xi(\Sigma)$ is the maximal number of disjoint and distinct (non-$S_{0,3}$) subsurfaces we can embed in $\Sigma$. Thus, if $n = \xi(\Sigma)$, we see that if $\phi$ is non-degenerate, then each $Y_i$ is connected and is either an annulus, or has complexity-1 (that is a $S_{0,4}$ or $S_{1,1}$).

**Definition.** We say that a multicurve $\tau$ is big if each component of $\Sigma \setminus \tau$ is a $S_{0,3}$, $S_{0,4}$ or $S_{1,1}$.

Note that, in this case, $L(\tau)$ is quasi-isometric to a direct product of uniformly hyperbolic spaces. In fact, each factor is quasi-isometric to either a hyperbolic plane or a horodisc.

**Lemma 8.4.** Suppose that $Q$ is a $\xi$-cube, and that $\phi : Q \to \mathcal{R}(\Sigma)$ is a non-degenerate quasimorphism. Then there is a big multicurve, $\tau \subseteq \Sigma$, such that we can write $X(\tau) = \{Y_1, \ldots, Y_\xi\}$, so that if $c, d$ is any $i$th side of $Q$, then $\phi c, \phi d$ are $Y_i$-related. Moreover, $\phi(Q)$ lies in a bounded neighbourhood of $L(\tau)$.

**Proof.** We construct disjoint surfaces $Y_i$ as above, and as already observed, the set of these $Y_i$ is precisely $\mathcal{X}(\tau)$ for a big multicurve, $\tau$. Recall that for all $X \in \mathcal{X}$, if $c, d$ and $c', d'$ are $i$th sides of $Q$, the $\sigma_X(\phi c, \phi d) \approx \sigma_X(\phi c', \phi d')$. Let $r_1$ be the constant of Lemma 8.2. If the nondegeneracy constant is sufficiently large, the $A(\phi c, \phi d, r) \neq 0$. So if $r \geq \max\{r_1, k_0\}$, the subsurfaces of $A(\phi c, \phi d, r)$ fill $Y_i$, so $\phi c, \phi d, Y_i$ satisfy the hypotheses of Lemma 8.2. We see that $\phi c, \phi d$ are $Y_i$-related as claimed.

Note that a big multicurve, $\tau$, satisfying the conclusion of Lemma 8.4 might not be unique. For example, if $\gamma \in \tau$ bound a $S_{0,3}$ component of $\Sigma \setminus \tau$ on both sides (perhaps the same $S_{0,3}$), then we can remove it, and the conclusion will still hold. However, this is essentially the only ambiguity that can arise.

We will also need:

**Lemma 8.5.** Suppose that $Q$ is a $\xi$-cube, and that $\phi : Q \to \mathcal{R}(\Sigma)$ is a non-degenerate quasimorphism, and that $c, d$ is an $i$th side of $Q$. Let $Y_i$ be as given by Lemma 8.4. Suppose that $x, y \in \mathcal{R}(\Sigma)$ with $\mu(\phi c, \phi d, x) \sim x$ and $\mu(\phi c, \phi d, y) \sim y$. Then $\rho(x, y) \approx \rho_{Y_i}(x, y)$.

**Proof.** By Lemma 8.3, and $\phi c, \phi d$ are $Y_i$-related, and so therefore are $x, y$ (with suitable constants). The statement then follows by Lemma 8.1.

We now consider cubes in $\mathcal{R}^*(\Sigma)$.

Let $Q \subseteq \mathcal{R}(\Sigma)$ be an $n$-cube. If $c, d$ and $c', d'$ are both $i$th sides of $Q$, then the intervals $[c, d]$ and $[c', d']$ are parallel. That is, the maps...
Suppose now that $Q \subseteq \mathcal{R}^*(\Sigma)$ is a $\xi$-cube. Applying Lemma 8.6 to the faces of $Q$, we see that if $\alpha, \beta \in D_0^i(Q)$ for any $i$, then either $\alpha = \beta$ or $\alpha \wedge \beta$.

We also have:

**Lemma 8.7.** If $\alpha \in D_0^0(Q)$ and $\beta \in D_0^j(Q)$, then either $i = j$ or $\alpha \wedge \beta$.

**Proof.** Let $\phi_\zeta : Q \to \mathcal{R}^*(\Sigma)$ be $Z$-sequence of uniform quasimorphisms, with $\phi_\zeta x \to x$ for all $x \in Q$. Let $c, d$ and $c', d'$ be $i$th and $j$th sides of $Q$, respectively. Then $\sigma_\alpha(\phi_\zeta c, \phi_\zeta d) \to \infty$ and $\sigma_\beta(\phi_\zeta c', \phi_\zeta d') \to \infty$, so for almost all $\zeta$, $\sigma_\alpha(\phi_\zeta c, \phi_\zeta d) \geq k_0$ and $\sigma_\beta(\phi_\zeta c', \phi_\zeta d') \geq k_0$, where $k_0$ is the constant of Lemma 8.3. If $i \neq j$, then by Lemma 8.3, $X_\zeta \wedge Y_\zeta$, so it follows that $X \wedge Y$.

**Lemma 8.8.** Suppose that $\gamma \in D_0^0(Q)$ and that $X \in D_j(Q)$ is a complexity-1 surface (i.e. a $S_{0,1}$ or $S_{1,1}$). If $\gamma \prec X$, the $i = j$.

**Proof.** Let $\phi_\zeta c, d, c', d'$ be as in the proof of Lemma 8.7. Now $\rho_{X_\zeta}(\phi_\zeta c, \phi_\zeta d) \to \infty$, and $\rho_{X_\zeta}(\phi_\zeta c', \phi_\zeta d') \to \infty$. Thus we have the following for almost all
Let \( x \) so then using Lemmas 8.6, 8.7 and 8.8, we again see that \( D \) curve, \( \tau \) Lemma 8.9. Let \( D \gamma \) of a single curve \( \gamma \) \( \Xi \) and write \( \xi \). Therefore, if \( \xi \) \( \zeta \) \( \rho \zeta \) \( \varphi \zeta \) \( \mu \zeta \) \( \lambda \zeta \) \( \delta \zeta \) \( \gamma \zeta \) \( \delta \zeta \) \( \gamma \zeta \) \( \delta \zeta \) \( \gamma \zeta \) \( \delta \zeta \) \( \gamma \zeta \). Thus \( \mathcal{U}X(\tau) = \{Y_1, \ldots, Y_\xi\} \) as claimed.

It remains to show that \( Y_i \in D_i(Q) \). Suppose that \( x, y \in [c, d] \). Let \( x_\xi, y_\xi \in \mathcal{R}(\Sigma) \) with \( x_\xi \to x \) and \( y_\xi \to y \). After replacing \( x_\xi \) by \( \mu(c_\zeta, d_\zeta, x_\xi) \) and \( y_\xi \) by \( \mu(c_\zeta, d_\zeta, y_\xi) \), we can assume that \( \mu(c_\zeta, d_\zeta, x_\xi) \sim x_\xi \) and \( \mu(c_\zeta, d_\zeta, y_\xi) \sim y_\xi \). By Lemmas 8.2 and 8.1, it follows that \( \rho(x_\xi, y_\xi) \sim \rho_i(x_\xi, y_\xi) \). But \( \rho(x_\xi, y_\xi) \to \rho(x, y) \) and \( \rho_i(x_\xi, y_\xi) \to \rho_i(x, y) \). Therefore, if \( \phi_i^* x = \phi_i^* y \), then \( \rho_i(x, y) = 0 \) so \( \rho(x, y) = 0 \), so \( x = y \). In other words, this shows that \( \phi_i^*[c, d] \) is injective, so \( Y_i \in D_i(Q) \) as claimed. \( \square \)

Let \( I \) be the set of \( i \) such that \( Y_i = X(\gamma_i) \) is an annulus. Thus, \( \gamma_i \in D_i^0(Q) \), and \( \tau = \{\gamma_i \mid i \in I\} \). By Lemmas 8.7 and 8.9, we see that, in fact, \( D_i^0(Q) = \{\gamma_i\} \). Moreover, if \( D_j^0(Q) \neq \emptyset \) for some \( j \notin I \), then using Lemmas 8.6, 8.7 and 8.8, we again see that \( D_j^0(Q) \) consists of a single curve \( \gamma_j \), with \( \gamma_j \prec Y_j \). Now write \( I(Q) = \{i \mid D_i^0(Q) \neq \emptyset\} \), and let \( \tau(Q) = \{\gamma_i \mid i \in I(Q)\} \). We see that \( \tau(Q) \) is a big multicurve, containing \( \tau \), and that it also satisfies the conclusion of Lemma 8.9 (since \( \mathcal{U}X(\tau(Q)) \subseteq \mathcal{U}X_N(\tau) \)). Therefore retrospectively, we could have taken \( \tau = \tau(Q) \) in Lemma 8.9. (In fact, one can see easily that \( Q \subseteq L^*(\tau(Q)) \).

We also note the following strengthening of Lemma 8.8 below.

First, we note:

**Lemma 8.10.** Suppose \( a, b \in \mathcal{R}(\Sigma) \), \( \gamma \in \mathcal{G}_0(\Sigma) \) and \( \sigma_\xi(a, b) \sim 0 \) for all \( X \in \mathcal{X} \setminus \{X(\gamma)\} \). Then \( a, b \) differ by at most a power of Dehn twist about \( \gamma \), and a redecoration of \( \gamma \).

(The redecorating may be necessary if \( \gamma \in \hat{a} \cap \hat{b} \).)

**Proof.** Apply such a Dehn twist and/or redecoration, we can assume that \( \sigma_\xi(a, b) \sim 0 \), and so \( a \sim b \) by Lemma 4.2. \( \square \)
Note that it follows that \( \rho(a, b) \asymp \sigma_\gamma(a, b) \). (This statement is also a direct consequence of the distance formula of Rafi, given as Proposition 4.9 here.)

**Lemma 8.11.** Suppose that \( \gamma \in D^0_i(Q) \) and that \( X \in D_j(Q) \) is a complexity-1 subsurface. If \( i \neq j \), then either \( \gamma \cap X \), or there is some (unique) \( \beta \in D^0_j(Q) \) with \( \beta \prec X \).

**Proof.** If not, we must have \( \gamma \prec X \) and \( \gamma \cap X \), but the first possibility is ruled out by Lemma 8.8. Since \( X \) has complexity 1, either \( \gamma \cap Y \) for all \( Y \preceq X \), or else there is a unique \( \beta \prec X \) with \( \gamma \cap Y \) for \( Y \prec X \) and \( Y \neq X(\beta) \).

Let \( \phi_\zeta, c, d, c, d' \) be as in the proof of Lemma 8.8. In the first case above, we derive a contradiction as before. In the second case, let \( \beta_\zeta \to \beta \). We see that for almost all \( \zeta \), we have \( \beta_\zeta \prec X_\zeta \) and \( \gamma_\zeta \preceq X_\zeta \). For such \( \zeta \), if \( Y_\zeta \preceq X_\zeta \) with \( Y_\zeta \neq X(\beta_\zeta) \), and \( \sigma_\gamma(\phi c', \phi d') \) is bounded. If \( e, f \in [\phi c, \phi d] \), it follows that \( \sigma_\gamma(e_\zeta, f_\zeta) \) is bounded. It follows from the earlier observation, that \( \rho_\gamma(e_\zeta, f_\zeta) \) is bounded below by some increasing linear function of \( \rho_\gamma(e_\zeta, f_\zeta) \) and so also of \( \rho(e, f) \). \( \square \)

We can summarise what we have shown as follows. Recall that \( D_i(Q) \) is the set of points for which \( \psi_i^*[c, d] \) is injective for some (or equivalently any) \( i \)th side, \([c, d]\) of \( Q \).

**Proposition 8.12.** For any \( i \), the set \( D^0_i(Q) \) is either empty or consists of a single curve \( \gamma_i \in U^0 \). If it is empty, then there is a unique complexity-1 subsurface \( Y_i \in D_i(Q) \). If the \( \gamma_i \) are all disjoint, and they form a big multicurve \( \tau(Q) \). The \( Y_i \) are also disjoint, and are precisely the complexity-1 components of \( \tau(Q) \).

We note, in particular, that \( \gamma_i \) or \( Y_i \) is completely determined by any face of \( Q \), without reference to \( Q \) itself.

For the application to Theorem 1.4 in Section 9, we also note the following:

**Lemma 8.13.** Suppose that \( a, b, c \in R^*(\Sigma) \), with \( c \in [a, b] \setminus \{a, b\} \), and with \( \theta_\gamma^*a, \theta_\gamma^*b \) and \( \theta_\gamma^*c \) all distinct. Then \( c \notin R^\infty_\gamma(\Sigma) \).

**Proof.** Take \( \gamma_\zeta \in G^0(\Sigma) \) with \( \gamma_\zeta \to \gamma \) and \( a_\zeta, b_\zeta, c_\zeta \in R(\Sigma) \) with \( a_\zeta \to a, b_\zeta \to b \) and \( c_\zeta \to c \). If \( c \in R^\infty_\gamma \), then we could also take \( c_\zeta \in R_\gamma \). Let \( d_\zeta = \mu(a_\zeta, b_\zeta, c_\zeta) \). Thus \( d_\zeta \to c \). Since \( H(\gamma_\zeta) \) is quasi-isometric to a horodisc, it is easily seen that \( \min\{\sigma_\gamma(\gamma_\zeta, c_\zeta), \sigma_\gamma(b_\zeta, c_\zeta)\} \), is bounded above by a linear function of \( \sigma_\gamma(c_\zeta, d_\zeta) \). Passing to the limit, we see that \( \min\{\sigma_\gamma^\infty(a, c), \sigma_\gamma^\infty(b, c)\} = 0 \), giving a contradiction. \( \square \)
9. Conclusion of the proof of Theorem 1.4

We now proceed to the proof of the “only if” part of Theorem 1.4.

Suppose that \( \phi : \mathbb{R}^\xi \to \mathcal{R}(\Sigma) \) is a quasi-isometric embedding. Passing to an asymptotic cone with fixed basepoint, we get a map \( f = \phi_\infty : \mathbb{R}^\xi \to \mathcal{R}_\infty(\Sigma) \), which is bilipschitz onto its image, \( \Phi = f(\mathbb{R}^\xi) \). Note that the basepoint, \( o \), of \( \mathcal{R}_\infty(\Sigma) \) lies in \( \Phi \cap \mathcal{R}_\infty(\Sigma) \).

Now, by the result of [Bo4], \( \Phi \) is “cubulated”, here meaning that it is a locally finite union of convex hulls of \( \xi \)-cubes. (Such a convex hull is median isomorphic to \( [-1,1]^\xi \subseteq \mathbb{R}^\xi \).) In fact, we can find a neighbourhood of the basepoint \( o \in \Phi \), which has the structure of a finite cube complex, where each \( \xi \)-dimensional cell is the convex hull of a \( \xi \)-cube.

Now consider the link, \( \Delta \), of \( o \) in \( \Phi \). This is a simplicial complex which is a homology \((\xi - 1)\)-sphere. In particular, the \((\xi - 1)\)th dimensional homology of \( \Delta \) is non-trivial.

Let \( \Delta^0 \) be its vertex set. Each \( x \in \Delta^0 \) corresponds to a 1-cell of \( \Phi \), with one vertex \( o \) and the other denoted \( a(x) \in \Phi \subseteq \mathcal{R}_\infty(\Sigma) \). Note that this 1-cell is precisely the median interval, \([o, a(x)]\).

Let \( \gamma(x) \in \mathcal{U}\mathcal{G}^0(\Sigma) \) or a complexity-1 subsurface, \( Y(x) \in \mathcal{U}\mathcal{X} \). Any \((\xi - 1)\)-simplex in \( \Delta \) corresponds to a \( \xi \)-cube, and so, by Proposition 8.12, we see that the curves \( \gamma(x) \) or subsurfaces \( Y(x) \) are all disjoint, as \( x \) ranges over the vertices of the simplex.

**Lemma 9.1.** Suppose that \( x, y \in \Delta^0 \) are distinct, and that both correspond to curves, \( \gamma(x) \) and \( \gamma(y) \). Then \( \gamma(x) \neq \gamma(y) \).

**Proof.** Suppose, for contradiction, that \( \gamma(x) = \gamma(y) = \gamma \); say. Since the intervals \([o, a(x)]\) and \([o, a(y)]\) meet precisely in \( o \), it follows that \( o = \mu(a(x), a(y), o) \); in other words, \( o \in [a(x), a(y)] \). Moreover, \( \theta^\infty_\gamma[o, a(x)] \) and \( \theta^\infty_\gamma[o, a(y)] \) are both injective (by construction of \( \gamma(x) \) and \( \gamma(y) \)), and so, in particular, \( \theta^\infty_\gamma o, \theta^\infty_\gamma a(x) \) and \( \theta^\infty_\gamma a(y) \) are all distinct. We now apply Lemma 8.13 to give the contradiction that \( o \notin \mathcal{R}_\infty^\infty \).

**Lemma 9.2.** Each \( x \in \Delta^0 \) corresponds to some \( \gamma(x) \in \mathcal{U}\mathcal{G}^0(\Sigma) \).

**Proof.** Suppose, to the contrary, that \( x \in \Delta^0 \) corresponds to a complexity-1 subsurface, \( Y(x) \). Let \( Q \) be any \( \xi \)-cube of the cubulation containing \( a(x) \), so that \( \{o, a(x)\} \) is a face. (This corresponds to a \((\xi - 1)\)-simplex in \( \Delta \).) Let \( \tau = \tau(Q) \) be the big multicurve described by Proposition 8.12. Let \( \gamma \in \tau \) be a boundary curve of \( Y(x) \) in \( \Sigma \). Thus, \( \gamma = \gamma(y) \) for some \( y \in Q \) (adjacent to \( x \) in \( \Sigma \)). Let \( Q_0 \subseteq Q \) be the \((\xi - 1)\)-face containing \( o \) but not containing \( a(x) \). (This corresponds to a \((\xi - 2)\)-simplex of \( \Delta \).)
Now, given that $\Phi$ is homeomorphic to $\mathbb{R}^\xi$, it is easily seen that there must be a (unique) $\xi$-cube, $Q'$, of the cubulation with $Q \cap Q' = Q_0$. Let $z \in \Delta^0$ be the unique vertex with $a(z) \in Q' \setminus Q_0$. Let $\tau' = \tau(Q')$. Now $\tau'$ is obtained from $\tau$ by replacing $\gamma(y)$ by $\gamma(z)$ and leaving all other curves alone. Note also that all complexity-1 components of the complement also remain unchanged (since these are the subsurfaces $Y(w)$ for those vertices $w$ which do not correspond to curves). It therefore follows that, in fact, we must have $\gamma(z) = \gamma(y) = \gamma$, contradicting Lemma 9.1. In other words, this situation can never arise, and so each $x \in \Delta^0$ must correspond to a curve $\gamma(x)$. 

By Lemmas 9.2 and 9.1, we therefore have an injective map $[x \mapsto \gamma(x)] : \Delta^0 \rightarrow \mathcal{UC}^0(\Sigma) = \mathcal{UC}(\Sigma)$. Now (again since $\Phi$ is homeomorphic to $\mathbb{R}^\xi$), every edge of $\Delta$ lies inside some $(\xi - 1)$-simplex. So if $x, y \in \Delta^0$ are adjacent, $a(x), a(y)$ lie in some cube of the cubulation, and so $\gamma(x) \wedge \gamma(y)$. Now the ultraproduct, $\mathcal{UC}(\Sigma)$, of the curve complex $C(\Sigma)$ is a flag complex (since $C(\Sigma)$ is) so we get an injective simplicial map of $\Delta$ into $\mathcal{UC}(\Sigma)$. This gives us an injective map of $\Delta$ into $C(\Sigma)$. Since $C(\Sigma)$ has dimension $\xi - 1$, it follows that $C(\Sigma)$ has non-trivial homology in dimension $\xi - 1$. But by the result of [Har] referred to above, the homology is trivial in all dimensions at least $\xi'$. It now follows that $\xi = \xi'$, and so $\Sigma$ has genus at most 1, or is a closed surface of genus 2.

This proves the “only if” part of Theorem 1.4.

**References**


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