RIGIDITY OF THE STRONGLY SEPARATING CURVE GRAPH

BRIAN H. BOWDITCH

Abstract. We define the strongly separating curve graph to be the full subgraph of the curve graph of a compact orientable surface, where the vertex set consists of all separating curves which do not bound a three-holed sphere. We show that, for all but finitely many surfaces, any automorphism of the strongly separating curve graph is induced by an element of the mapping class group.

1. Introduction

The main aim of this paper is to prove a rigidity result (Theorem 1.1) for certain curve graphs associated to compact orientable surfaces. It is a variation on some well known results in this direction. The main motivation for this particular statement is its application to the quasi-isometric rigidity of the Weil-Petersson metric.

Let $\Sigma$ be a compact orientable surface. We write $g(\Sigma)$ for its genus, and $p(\Sigma)$ for the number of boundary components. Write $\xi(\Sigma) = 3g(\Sigma) + p(\Sigma) - 3$ for its “complexity”.

Let $G(\Sigma)$ be the curve graph associated to $\Sigma$; that is, the 1-skeleton of the curve complex as defined in [H]. It has vertex set $C(\Sigma)$, the set of non-trivial non-peripheral simple closed curves in $\Sigma$, defined up to homotopy. Two elements of $C(\Sigma)$ are deemed adjacent if they can be homotoped to be disjoint. Note that the mapping class group, Map($\Sigma$), acts cofinitely on $G(\Sigma)$. The rigidity theorems of [Iv, Ko, L] tell us (in particular) that if $\xi(\Sigma) \geq 2$, then any automorphism of $G(\Sigma)$ is induced by an element of Map($\Sigma$). (Note that, since the curve complex is a flag complex, this is equivalent to the same statement for the curve complex.)

There are a number of variations of this. Given a subset, $A \subseteq C(\Sigma)$, we write $G(\Sigma, A)$ for the full subgraph of $G(\Sigma)$ with vertex set $A$. If $A$ is Map($\Sigma$) invariant, then Map($\Sigma$) also acts on $G(\Sigma, A)$. We say that $G(\Sigma, A)$ is rigid if every automorphism is induced by an element of Map($\Sigma$).

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For example, if $C_s(\Sigma)$ is the set of separating curves, we refer to $G_s(\Sigma) = G(\Sigma, C_s(\Sigma))$ as the \textit{separating curve graph}. (Note that if $g = 0$, then this is the same as $G(\Sigma)$.) The results of [BrM, Ki], together with that cited above for planar surfaces, tell us (in particular) that if $\xi(\Sigma) \geq 5$, then $G_s(\Sigma)$ is rigid.

We remark that, the non-separating curve graph, $G(\Sigma, C(\Sigma) \setminus C_s(\Sigma))$ of a large class of surfaces of genus at least 2 are is also rigid [Ir], though this is not directly relevant to the present paper.

Let $C_0(\Sigma) \subseteq C_s(\Sigma)$ be the set of curves which bound some three-holed sphere in $\Sigma$. Let $C_{ss}(\Sigma) = C_s(\Sigma) \setminus C_0(\Sigma)$, and let $G_{ss}(\Sigma) = G(\Sigma, C_{ss}(\Sigma))$. We refer to elements of $C_{ss}(\Sigma)$ as \textit{strongly separating} curves and to $G_{ss}(\Sigma)$ as the \textit{strongly separating curve graph}.

We will show here that $G_{ss}(\Sigma)$ is rigid in all but finitely many cases:

\textbf{Theorem 1.1.} If $g(\Sigma) + p(\Sigma) \geq 7$, then $G_{ss}(\Sigma)$ is rigid.

Note that if $p(\Sigma) \leq 1$, then $G_{ss}(\Sigma) = G_s(\Sigma)$, and so this is covered by the result of [BrM, Ki].

This still leaves unresolved about a dozen cases, which I suspect are also rigid. One can probably deal with a few more cases with some elaboration on the arguments here, though a complete answer may require new ideas.

It is natural to ask more generally for what classes of subsets $A \subseteq C_s(\Sigma)$ is $G(\Sigma, A)$ rigid. (Note there are only finitely many possibilities for $A$ for any given topological type.)

The motivation for studying this particular case is the application given in [Bo] to the Weil-Petersson metric on Teichmüller space. There it was shown that the rigidity of $G_{ss}(\Sigma)$ implies the quasi-isometric rigidity of the Weil-Petersson metric associated to $\Sigma$. In view of Theorem 1.1, this holds whenever $g(\Sigma) + p(\Sigma) \geq 7$.

In particular, Theorem 1.1, together with the results of that paper, shows that in all but at most finitely many cases, the Weil-Petersson space is quasi-isometrically rigid.

We will first give a proof of Theorem 1.1 in the case where $\Sigma = S_{0,7}$ (in Section 3). This forms the basis for dealing with the more general situation. The key observation, proven in Section 2 is that any heptagon in in $G_{ss}(S_{0,7})$ has a standard form up to the action of $\text{Map}(S_{0,7})$. Given that one can identify the topological types of multicurves in a more general $\Sigma$ in the terms of $G_{ss}(\Sigma)$ (as discussed Section 4) one can then adapt this to the general case (in Section 5). The particular case of $S_{0,8}$, however, requires a special argument, and will be discussed in Section 6.
An earlier draft of this paper (with a quite different proof and few cases) was written at the Tokyo Institute of Technology, and I am grateful for the hospitality of that institution, and for the invitation of Sadayoshi Kojima. I thank Javier Aramayona for his interest and suggestions.

2. Heptagons in the 7-holed sphere

We begin with a description of 7-cycles (or “heptagons”) in the separating curve graph of \( S_{0,7} \). In general, by an \( n \)-cycle in \( Gss(\Sigma) \), we mean a cyclically ordered sequence of \( n \) vertices, where consecutive vertices are adjacent. We refer to it as an odd or even cycle depending on whether \( n \) is odd or even. Note that any shortest odd cycle is necessarily isometrically embedded.

For the purposes of this and the next section, it will be convenient to view \( S_{0,7} \) as (the complement of) the 2-sphere, \( S \), with a set of 7 preferred points, \( \Pi \subset S \), which we refer to as punctures.

Note that if \( \gamma \in Css \), then \( \gamma \) bounds a disc, \( B(\gamma) \) with \( |B(\gamma) \cap \Pi| = 3 \). We write \( \pi(\gamma) = B(\gamma) \cap \Pi \). Note that, \( \alpha,\beta \in Css \) are adjacent if and only if we can realise \( \alpha,\beta \) so that \( B(\alpha) \cap B(\beta) = \emptyset \). We say that two curves are \( n \)-distant if they are a distance exactly \( n \) apart in \( Gss \).

In general, we will always assume that elements of \( Css \) are realised to have minimal pairwise intersection. In this case, we write \( \iota(\alpha,\beta) = |\alpha \cap \beta| \) for the geometric intersection number. We say that two curves, \( \alpha,\beta \) have simple intersection if \( \iota(\alpha,\beta) = 2 \). In this case, \( \alpha \cup \beta \) cuts \( S \) into four discs. In particular, \( B(\alpha) \cap B(\beta), B(\alpha) \cup B(\beta), B(\alpha) \setminus B(\beta) \) and \( B(\beta) \setminus B(\alpha) \) are all discs.

It is easily seen that \( Gss(S_{0,7}) \) has no 3-cycles. Also it has no 4-cycles. (For if \( \gamma_1,\gamma_2,\gamma_3,\gamma_4 \) were a 4-cycle, \( B(\gamma_1) \cup B(\gamma_2) \) and \( B(\gamma_2) \cup B(\gamma_4) \) would be disjoint connected subsurfaces of \( S \), each containing at least three points of \( \Pi \), which clearly is not possible.) We will also see shortly that \( Gss(S_{0,7}) \) has no 5-cycles. This implies that any 7-cycle must be isometrically embedded.

In fact, \( Gss(S_{0,7}) \) does contain 6-cycles. We will however focus on the 7-cycles, since these are more symmetrical, and will serve for our purposes.

Here is a description of a 7-cycle. Let \( \lambda \subset \Sigma \) be an embedded circle with \( \Pi \subset \lambda \). This determines a cyclic ordering on \( \Pi \) where we index the punctures as \( p_1, p_3, p_5, p_7, p_2, p_4, p_6 \). Let \( l_{13} \) be the segment between \( p_1 \) and \( p_3 \) etc. Thus \( \lambda = l_{13} \cup l_{35} \cup l_{57} \cup l_{72} \cup l_{24} \cup l_{46} \cup l_{61} \). Let \( B_i \) be a regular neighbourhood of \( l_{i-2,i} \cup l_{i,i+2} \) with \( B_i \cap \Pi = \{p_{i-2}, p_i, p_{i+2}\} \), and let \( \gamma_i = \partial B_i \). Thus, \( B_i = B(\gamma_i) \). Note that \( B_i \cap B_{i+1} = \emptyset \), and
so $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$ is a 7-cycle. Non-adjacent pairs of curves have simple intersection.

Now $\text{Map}(S_{0,7})$ acts on $G_{ss}(S_{0,7})$. A principal result of this section is to show:

**Proposition 2.1.** There is exactly one 7-cycle in $G_{ss}(S_{0,7})$ up to the action of $\text{Map}(S_{0,7})$.

We begin with an analogous statement for 3-sets in a 7-set. Given any set, $\Pi$, let $\Theta = \Theta(\Pi)$ be the graph whose vertex set, $V(\Theta)$ consists of subsets of cardinality 3 in $\Pi$, and whose edge set, $E(\Theta)$, consists of pairs of disjoint such 3-sets. If $|\Pi| = 7$, this is a connected 4-regular graph on thirty-five vertices. Note also that there is an edge colouring, $\chi : E(\Theta) \rightarrow \Pi$, given by $\Pi \setminus (P \cup Q) = \{\chi(e)\}$, where $e$ is the edge from $P$ to $Q$.

Writing $\Pi = \{1, 2, 3, 4, 5, 6, 7\}$, there is a 7-cycle in $\Pi$ given by $613–724–135–246–357–461–572$. We aim to show this is the only one up to the action of the symmetric group, $\text{Sym}(\Pi)$.

First, note that if $P, Q \in V(\Theta)$ are 2-distant, the $|P \cap Q| = 2$. Moreover, it is easily checked that if $P, Q$ are 3-distant, then $|P \cap Q| = 1$.

Clearly $\Theta$ has no 3-cycles. One also easily sees that there are no 5-cycles in $\Theta$. (For if $P_1, P_2, P_3, P_4, P_5$ were such a cycle, then $P_3$ and $P_4$ are both 2-distant from $P_1$, so $|P_1 \cap P_3| = |P_1 \cap P_4| = 2$, and we get the contradiction that $P_3 \cap P_4 \neq \emptyset$.)

Suppose, now that $P_1, P_2, P_3, P_4, P_5, P_6, P_7$ is a 7-cycle. Since there is no smaller odd cycle, this must be isometrically embedded in $\Theta$. Suppose that two edges have the same colour, $p$, say. Since the $P_i$ are all distinct, these edges cannot meet or contain adjacent vertices. Thus, up to cyclic reordering, the only possibilities for these edges are $P_1, P_2$ and $P_3, P_4$. Now $P_2$ and $P_4$ are 2-distant, and so $|P_2 \cap P_4| = 2$. Now, $P_1$ and $P_5$ are the complements of these sets in $\Pi \setminus \{p\}$, and so we also have $|P_1 \cap P_5| = 2$. But these are 3-distant, so this contradicts the earlier observation.

Therefore, each colour occurs exactly one around the cycle. Up to $\text{Sym}(\Pi)$, we can assume they occur in the cyclic order 1234567, starting with the edge $P_1, P_2$. Now consider the sequence $P_1, P_3, P_5, P_7$. We must proceed by replacing 2 by 1, then 4 by 3, then 6 by 5. So we must have started with $p_1$ being 246 (and ended with $P_7$ as 135). But now the whole sequence is completely determined by the colours on the edges. In fact, it must be precisely the cyclic sequence given above.

We have shown:

**Lemma 2.2.** If $|\Pi| = 7$, there are no 3-cycles or 5-cycles in $\Theta(\Pi)$. There is exactly one 7-cycle up to the action of $\text{Sym}(\Pi)$.
Note that the map $\pi$ defined above extends to a map $\pi : G_{ss} \rightarrow \Theta$, sending edges to edges. Composing with $\chi$, we also get a colouring of the edges of $G_{ss}$, which we also denote by $\chi : E(G_{ss}) \rightarrow \Pi$. Note that it is now an immediate consequence that there are no 5-cycles in $G_{ss}$ as stated earlier.

If $\alpha, \beta \in C_{ss}$ are adjacent, we set $A = A(\alpha, \beta)$ to be the closure of $S \setminus (B(\alpha) \cup B(\beta))$. This is an annulus, with $\partial A = \alpha \cup \beta$, and with $A(\alpha, \beta) \cap \Pi = \{p\}$, where $p = \chi(\alpha, \beta)$.

By an arc in $A$, we mean an arc $a \subseteq A \setminus \Pi$ with endpoints $\partial a = a \cap \partial A$. We generally regard an arc as defined up to homotopy in $A \setminus \Pi$, allowing ourselves to slide an endpoint of $a$ in $\partial A$. Up to homotopy, there are exactly three types of arc, depending on whether $a$ meets only $\alpha$, only $\beta$, or both $\alpha$ and $\beta$. We refer to these classes as $\alpha$-type, $\beta$-type or crossing arcs, respectively. Note that an $\alpha$-type arc and a $\beta$-type arc meet (minimally) in exactly two points.

Suppose that $\epsilon \in C_{ss}$ is 3-distance from both $\alpha$ and $\beta$. Then $|\pi(\epsilon) \cap \pi(\alpha)| = |\pi(\epsilon) \cap \pi(\beta)| = 1$, and so it follows that $p \in \pi(\epsilon)$. Let $C = C(\alpha, \beta; \epsilon)$ be (the closure of) the component of $A \cap B(\epsilon)$ containing $p$. Note that $C$ must intersect either $\alpha$ or $\beta$, possibly both. (In fact, given that there are only three classes of arc in $A$, one can easily see that $C$ can meet each of $B(\alpha)$ and $B(\beta)$ in at most a single arc, though we won’t explicitly need this.)

Suppose now that $\beta, \delta \in C_{ss}$ are 2-distance. Then $|\pi(\beta) \cap \pi(\delta)| = 2$, so $\pi(\delta) \setminus \pi(\beta) = \{q\}$ for some $q \in \Pi$. Let $D = D(\delta, \beta)$ be (the closure of) the component of $B(\delta) \setminus B(\beta)$ containing $q$. We claim that $D$ is a bigon:

**Lemma 2.3.** $D \cap B(\beta)$ consists of a single arc in $\beta$.

**Proof.** Let $\gamma \in C_{ss}$ be adjacent to both $\beta$ and $\delta$. Note $B(\beta), B(\delta) \subseteq S \setminus B(\gamma)$, and so $D \subseteq A = A(\beta, \gamma)$ and $D \cap \gamma = \emptyset$. Thus, $\partial D$ can contain only $\beta$-type arcs in $A$. There can only be one of these, so the statement follows easily. \hfill $\square$

Now suppose that $\sigma$ is a 7-cycle in $G_{ss}$.

Suppose that $\alpha, \beta$ is an edge of $\Sigma$. Let $\epsilon$ be the vertex of $\sigma$ opposite this edge. Thus, $\epsilon$ is 3-distance from both $\alpha$ and $\beta$, as above. Let $C = C(\alpha, \beta; \epsilon)$.

Suppose that $\beta \cap C \neq \emptyset$. This implies that any $\alpha$-type arc in $A = A(\alpha, \beta)$ must intersect $C$. Let $\gamma$ be the vertex of $\sigma$ adjacent to $\beta$ and distinct from $\alpha$ (so that $\gamma$ and $\epsilon$ are 2-distance). We claim:

**Lemma 2.4.** Let $\alpha, \beta, \gamma$ be consecutive vertices of $\sigma$, and suppose that $C \cap \beta \neq \emptyset$ (as above). Then $\alpha, \gamma$ have simple intersection.
Proof. Let δ be the vertex of σ between γ and ε (so that α, β, γ, δ, ε are consecutive vertices of σ). Let D = D(δ, β). By Lemma 2.3, this is a bigon; that is, ∂D = b ∪ d, where b and d are respectively arcs of β and δ. Now C ⊆ B(ε), D ⊆ B(δ) and B(ε) ∩ B(δ) = ∅, so C ∩ D = ∅. Let π(δ) \ π(β) = \{q\}. Note that q ≠ p, and so q ∈ π(α) \ B(α). Now \(d \cap C \subseteq D \cap C = ∅\), and so \(d \cap A\) contains no α-type arcs. Since \(d \cap β\) are the endpoints of d, it cannot contain any β-type arcs either. Thus, \(d \cap A\) consists only of crossing arcs, of which there must be exactly 2. This means that \(|d \cap α| = 2\) (with \(d \cap B(α)\) consisting of a single arc).

Let R = B(β) ∪ D. Now R is a disc with \(R \cap Π = π(β) \cup \{q\} = π(β) \cup π(α)\). Also \(R \subseteq B(β) \cup B(δ)\), so \(R \cap B(γ) = ∅\). It follows that γ and ∂R are homotopic in \(S \setminus Π\). In other words, they represent the same element of \(C_π\). But now, ∂R ⊆ β \∪ d, so \(|∂R \cap α| = 2\). Thus, \(γ, α, ε \leq 2\), so \(|∂R \cap α| = 2\). In other words, α, ε have simple intersection as claimed. □

Now, as already observed, at most one of \(C \cap α\) or \(C \cap β\) can be empty. If \(C \cap β = ∅\), we refer to β as a bad endpoint of the edge α, β. We say that a vertex of σ is bad if it a bad endpoint of both incident edges of σ. Thus, Lemma 2.4 tells us that if β is not bad, then the two vertices adjacent to β in σ have simple intersection.

Now no two bad vertices can be adjacent. It follows that there can be at most three bad vertices in total. In fact, we can index the vertices of σ consecutively (mod 7) as γ₁, γ₂, γ₃, γ₄, γ₅, γ₆, γ₇, so that none of γ₁, γ₃, γ₄ or γ₆ are bad. It then follows that each of the pairs, \{γ₇, γ₂\}, \{γ₂, γ₄\}, \{γ₃, γ₅\} and \{γ₅, γ₇\} has simple intersection.

We write \(B_i = B(γ_i)\), and label the points of Π as \(p_i\), so that \(π(γ_i) = B_i \cap Π = \{p_{i−2}, p_i, p_{i+2}\}\).

Consider first the discs \(B_2, B_3, B_4, B_5\). We have \(B_2 \cap B_3 = B_3 \cap B_4 = B_4 \cap B_5 = ∅\). Let \(A = A(γ₃, γ₄)\). Since γ₅ has simple intersection with γ₃, we see that γ₅ \∩ A consists of a single γ₃-type arc. Similarly, γ₂ \∩ A consists of a single γ₄-type arc. It follows that \(|γ₂ \cap γ₅| = 2\), in other words, γ₂, γ₅ have simple intersection. Thus, \(B_2 \cap B_5\) is a disc with \(B_2 \cap B_3 \cap Π = \{p_7\}\). Now \(B_3 \cap B_5\) and \(B_2 \cap B_4\) are also discs, with \(B_3 \cap B_5 \cap Π = \{p_3, p_5\}\) and \(B_2 \cap B_4 \cap Π = \{p_2, p_4\}\). We can therefore find an arc l with \(Π \subseteq l\), with endpoints \(p_1\) and \(p_6\) and with the points \(p_1, p_3, p_5, p_7, p_2, p_4, p_6\) occurring in this order along l, so that \(B_3, B_5, B_2, B_4\) are respectively regular neighbourhoods of \(l_{13} \cup l_{35}\), \(l_{35} \cup l_{57}\), \(l_{72} \cup l_{24}\) and \(l_{24} \cup l_{46}\), where we have cut l into six arcs, \(l = \cup l_{13} l_{35} l_{57} l_{72} l_{24} l_{46}\), connecting the points of Π. Note in particular, that \(R = B_2 \cup B_3\) is a disc with \(R \cap Π = \{p_3, p_5, p_7, p_2, p_4\}\). Also \(B_2 \cap B_3\) is a disc with \(B_2 \cap B_3 \cap Π = \{p_7\}\).
Now consider $B_7$. We have $B_7 \cap \Pi = \{p_5, p_7, p_2\}$. Now $\gamma_7$ has simple intersection with $B_5$, so $\gamma_7 \cap B_5$ consists of a single arc separating $p_3$ from $p_5$ and $p_7$ in $B_5$. We can therefore realise it so that it is disjoint from $B_2 \cap B_5$. In fact, we can take this arc to meet $l$ just once, in a point of the segment $l_{35}$. Similarly, we can realise $\gamma_7 \cap B_2$ as a single arc, also disjoint from $B_2 \cap B_5$, and meeting $l$ in a single point of $l_{24}$. In this way, $\gamma_7 \setminus R$ consists of exactly two arcs. Since $(B_7 \setminus R) \cap \Pi = \emptyset$, each of these arcs can be homotoped into $\partial R$ in $S \setminus \Pi$, fixing their endpoints. We can therefore realise $B_7$ as a regular neighbourhood of $l_{57} \cup l_{72}$.

Now let $l_{61}$ be any arc in $S$ meeting $l$ exactly at their common endpoints. Thus $\lambda = l \cup l_{61}$ is a circle containing $\Pi$. Now the homotopy classes of $\gamma_4$ and $\gamma_6$ are determined as $\partial(B_7 \cup B_2)$ and $\partial(B_7 \cup B_1)$. We can therefore realise $B_1$ and $B_6$ as regular neighbourhoods of $l_{46} \cup l_{61}$ and $l_{61} \cup l_{13}$.

We are therefore exactly in the situation of the example of a 7-cycle described earlier.

This proves Proposition 2.1.

We note the following immediate consequence.

**Definition.** We say that $\alpha, \beta \in C_{ss}$ form a bounding pair if they have simple intersection and $|\pi(\alpha) \cap \pi(\beta)| = 2$.

In view of Proposition 2.1, we see that $\alpha, \beta$ form a bounding pair if and only if they are 2-distant vertices in some 7-cycle in $G_{ss}$. Note that this is determined just by the structure of $G_{ss}$. We deduce:

**Lemma 2.5.** Any automorphism of $G_{ss}(S_{0,7})$ preserves the set of bounding pairs.

3. RIGIDITY FOR THE 7-HOLED SPHERE

Recall that $C_0 = C_0(S_{0,7})$ is the set of curves in $S \setminus \Pi$ which bound a disc containing exactly two points of $\Pi$. We extend the notation $B(\omega)$ and $\pi(\omega) = B(\omega) \cap \Pi$ to $\omega \in C_0$. If $\omega \in C_0$ and $a \in C_{ss}$, we write $\omega \preceq \alpha$ to mean that $B(\omega) \subseteq B(\alpha)$.

By an $\omega$-arc we mean an arc, $a$, in $S$ meeting $B(\omega)$ at one endpoint (the initial endpoint) and $\Pi$ at the other (terminal endpoint). We regard $a$ as being defined up to homotopy relative to $\Pi$, fixing the terminal endpoint, and allowing the initial endpoint to slide along $\omega$. Note that $a$ determines an element $\alpha \in C_{ss}$, with $\omega \preceq \alpha$, so that $B(\alpha)$ is a regular neighbourhood of $B(\omega) \cup a$. In fact, every $\alpha \in C_{ss}$ with $\omega \preceq \alpha$ arises in this way.
If \( \alpha, \beta \in C_{ss} \) is a bounding pair, then there is a unique \( \omega \in C_0 \) with \( \omega \leq \alpha \) and \( \omega \leq \beta \). In fact, \( \omega = \partial (B(\alpha) \cap B(\beta)) \). Note that \( \alpha, \beta \) correspond to disjoint \( \omega \)-arcs, \( a, b \) (i.e. we can realise \( a, b \) to be disjoint).

We say that \( \alpha, \beta \) bound \( \omega \).

**Definition.** A bounding triple consists of three curves, \( \alpha, \beta, \gamma \in C_{ss} \), such that \( \{\alpha, \beta\}, \{\beta, \gamma\} \) and \( \{\gamma, \alpha\} \) are all bounding pairs, and such that there is some \( \omega \in C_0 \) with \( \omega \leq \alpha, \beta, \gamma \).

In this case, \( \alpha, \beta, \gamma \) correspond to three pairwise disjoint \( \omega \)-arcs, \( a, b, c \).

Suppose we just know that \( \alpha, \beta, \gamma \in C_{ss} \) are such that any pair form a bounding pair. Then \( \pi(\alpha), \pi(\beta), \pi(\gamma) \) pairwise intersect in sets of two elements. It follows easily that \( |\pi(\alpha) \cap \pi(\beta) \cap \pi(\gamma)| = 1 \), and one sees easily that there is a curve (namely \( \partial (B(\alpha) \cup B(\beta) \cup B(\gamma)) \)), disjoint from each of \( \alpha, \beta, \gamma \). In the former case, \( |\pi(\alpha) \cap \pi(\beta) \cap \pi(\gamma)| = 2 \), and one sees easily that \( \alpha, \beta, \gamma \) form a bounding triple. We deduce:

**Lemma 3.1.** Suppose \( \alpha, \beta, \gamma \in C_{ss} \), then \( \alpha, \beta, \gamma \) form a bounding triple if and only if any pair of them form a bounding pair and there is no curve in \( C_{ss} \) which is disjoint from each of \( \alpha, \beta, \gamma \).

In view of Lemma 2.5, we see that we can recognise when three elements of \( C_{ss} \) form a bounding triple.

Let \( \mathcal{H} \) be the graph whose vertex set is the set of all bounding pairs in \( C_{ss} \), and where two vertices are deemed adjacent if the union of the two pairs is a bounding triple. Given any \( \omega \in C_0 \), let \( \mathcal{H}(\omega) \) be the full subgraph whose vertex set consists of those bounding pairs which bound \( \omega \). Now adjacent vertices of \( \mathcal{H} \) determine the same element of \( C_0 \), and so \( \mathcal{H}(\omega) \) is a union of components of \( \mathcal{H} \). In fact:

**Lemma 3.2.** \( \mathcal{H}(\omega) \) is connected.

**Proof.** This is best seen in terms of \( \omega \)-arcs. Recall that a vertex of \( \mathcal{H}(\omega) \) corresponds to a pair of disjoint \( \omega \)-arcs, and an edge of \( \mathcal{H}(\omega) \) corresponds to a triple of pairwise disjoint \( \omega \)-arcs.

Suppose that \( a, b, c, d \) are \( \omega \)-arcs with \( a \cap b = \emptyset \) and \( c \cap d = \emptyset \). We realise them in general position in \( S \). (They do not need to have minimal intersection in their homotopy classes.) We aim to connect the vertices \( a, b \) and \( c, d \) by a path in \( \mathcal{H}(\omega) \). Write \( I(a, b; c, d) = (a \cup b) \cap (c \cup d) \setminus \Pi \) for the set of interior intersection points. We proceed by induction on \( |I(a, b; c, d)| \).

The case where that \( I(a, b; c, d) = \emptyset \) is elementary, so we we assume there is some \( x \in I(a, b; c, d) \). After permuting \( a, b \) and \( c, d \), we can
assume that \( x \in a \cup c \), and that \( x \) is the first intersection point along \( c \), that is, the initial segment \( e \) of \( c \) ending from \( x \), meets \( a \cup b \) only at \( x \). Let \( f \) be the initial segment of \( a \) starting from \( a' = (a \setminus f) \cup e' \). We can move \( a' \) slightly so that \( a \cap a' = \emptyset \), while retaining disjointness from \( b \). In other words \( a, a', b \) are disjoint, and so correspond to a bounding triple. Note that \( |I(a', b; c, d)| < I(a, b; c, d) | \). We therefore replace \( a, b \) by \( a', b \) and proceed inductively. □

This shows that the elements of \( C_0 \) are in natural bijective correspondence with the connected components of \( H \). Also, if \( \omega \in C_0 \) and \( \alpha \in C_{ss} \), then \( \omega \leq \alpha \) if and only if \( \alpha \) occurs as a curve in a bounding pair of some vertex of the corresponding component of \( H \). Since \( H \) can be constructed out of \( G_{ss} \), we see that we can also reconstruct \( C_0 \) and the relation \( \leq \) out of \( G_{ss} \).

We can also recognise disjointness. If \( \omega \in C_0 \) and \( \alpha \in C_{ss} \), then \( \omega \) and \( \alpha \) are disjoint if and only if there is some \( \beta \in C_{ss} \) such that \( \omega \leq \beta \) and \( \alpha, \beta \) are either equal or disjoint. Similarly, if \( \omega, \omega' \in C_0 \), then \( \omega \) and \( \omega' \) are disjoint if and only if there are disjoint curves \( \alpha, \alpha' \in C_{ss} \) with \( \omega \leq \alpha \) and \( \omega' \leq \alpha' \).

We can therefore reconstruct the graph \( G(S_{0,7}) = G_{ss}(S_{0,7}) \) from the graph \( G_{ss}(S_{0,7}) \), and so any automorphism of the former extends to an automorphism of the latter. But we know that \( G(S_{0,7}) \) is rigid by [Ko], so we have shown:

**Proposition 3.3.** \( G_{ss}(S_{0,7}) \) is rigid.

4. Multicurves

In this section, we will explain how to identify classes of multicurves from the strongly separating curve graph. We assume that \( \Sigma \) is a compact orientable surface, with boundary \( \partial \Sigma \), which we view as a (possibly empty) set of curves. (Thus, \( S_{0,7} \) reverts to being a bona fide 7-holed sphere.) We will assume here that \( g(\Sigma) + p(\Sigma) \geq 7 \).

As before, we refer to a property or to a collection of curves in \( C_{ss} = C_{ss}(\Sigma) \) as being “determined” (by \( G_{ss} \)) if it can be seen just in terms of the graph structure of \( G_{ss} \); and so in particular, it is invariant under the action of Map(\( \Sigma \)). We similarly say that class of curves can be “recognised” or that we can “tell” whether or not a given property holds. We also say that another graph can be “constructed” (from \( G_{ss} \)) etc.

Given a curve \( \alpha \in G(\Sigma) \), write \( B(\alpha) \) for the subsurface of \( \Sigma \) bounded by \( X \) which has lower complexity, \( \xi(X) \). (We will only use this notation where it unambiguous, and \( X \) is one of a few simple types.) Thus, \( \alpha \) is
“0-separating” if $B(\alpha)$ is an $S_{0,3}$, and “1-separating” if $B(\alpha)$ is either an $S_{0,4}$ or an $S_{1,1}$. We write $C_0$ and $C_1$ respectively for the sets of 0-separating and 1-separating curves. We say that two curves cross if they are not disjoint.

Suppose that $\alpha, \beta, \gamma \in C_{ss}$. We can tell if $\alpha$ separates $\beta$ from $\gamma$, for if not, there would be a fourth curve in $C_{ss}$ which crosses both $\beta$ and $\gamma$ but does not cross $\alpha$. Therefore, we can recognise elements of $C_1$: a curve in $C_{ss}$ is 1-separating if and only if it does not separate any two other elements of $C_{ss}$.

By a multcurve, $\tau$, in $C_{ss}$, we mean a non-empty set of pairwise disjoint curves in $C_{ss}$. We will sometimes abuse notation by regarding $\tau$ as a subset of $\Sigma$. We claim that we can identify the collection of sets $\tau$ together with the elements of $\tau$ which a bound a given component.

Let $X$ be a component of $\Sigma \setminus \tau$. Write $\partial X \subseteq \tau \cup \partial \Sigma$ for its intrinsic boundary, and $\partial_X X = \tau \cap \partial X$ for the relative boundary. We write $p(X) = |\partial X|$, $q(X) = |\partial_X X|$ and $g(X)$ for the genus of $X$. Write $C_{ss}(\Sigma, X)$ for the set of elements of $C_{ss}(\Sigma) \setminus \tau$ contained in $X$. Note that $C_{ss}(\Sigma, X)$ is either empty or infinite. In the former case, we refer to $X$ as small. It is either an $S_{0,3}$ (with $q(X) \geq 2$) or else has the form $B(\alpha)$ for some $\alpha \in C_1(\Sigma)$. In the latter case, we refer to $X$ as large.

Note that if $\alpha \in C_{ss}$, then $\alpha \in C_{ss}(\Sigma, X)$ for some (large) component $X$, if and only if it does not lie in $\tau$ and does not cross $\tau$ (i.e. is disjoint from each element of $\tau$). If $\alpha, \beta \in C_{ss}$ do not cross $\tau$, then they lie in the same set $C_{ss}(\Sigma, X)$ if and only if they are not separated by any element of $\tau$. Thus, from $\tau$ we can identify the collection of sets $C_{ss}(\Sigma, X)$ which arise from the large components, $X$, of $\Sigma \setminus \tau$. We next want to recognise their topological type.

To this end, we define a chain in $C_{ss}$ to be a sequence, $\gamma_0, \gamma_1, \ldots, \gamma_n$ of disjoint curves such that $\gamma_j$ separates $\gamma_i$ from $\gamma_k$ whenever $i < j < k$.

Suppose that $X$ is a component of $\Sigma \setminus \tau$ with $q(X) \geq 2$. Choose any distinct $\alpha, \beta \in \partial_X X$, and let $n$ be maximal so that there is a chain $\alpha = \gamma_0, \ldots, \gamma_n = \beta$ in $C_{ss}$ (thus, $\gamma_i \in C_{ss}(\Sigma, X)$ for all $i \neq 0, n$). Each component of $X \setminus \bigcup \gamma_i$ is either an $S_{0,3}$ or an $S_{1,2}$. Moreover, we can tell if the component between $\gamma_i$ and $\gamma_{i+1}$ is an $S_{1,2}$, since in that case, there will be some $\delta \in C_{ss}$ such that $\gamma_i$ separates $\delta$ from $\alpha$, and $\gamma_{i+1}$ separates $\delta$ from $\beta$. Thus, we know the number, $m$, of such components. We see that $g(X) = m$ and $p(X) = n - m + 2$, so we have determined the type of $X$ in this case.

Now suppose that $q(X) = 1$, and that $X$ is large (not an $S_{0,4}$ nor an $S_{1,1}$). Write $\partial_X X = \{\alpha\}$. Suppose that $\beta \in C_{ss}(\Sigma, X) \cap C_1(\Sigma)$ (which
we can recognize). Let \( Y = X \setminus B(\beta) \), so that \( Y \) is a component of \( \Sigma \setminus (\tau \cup \beta) \), and \( \partial_\Sigma Y = \{\alpha, \beta\} \). By the previous paragraph, we know the type of \( Y \) (given \( \beta \)). Therefore we know the collection of types of all such \( Y \) which can arise in this way. Given that \( B(\beta) \) is either an \( S_{0,4} \) or an \( S_{1,1} \), there are at most two such types. We now see easily that this data determines the topological type of \( X \), except that we may not be able to distinguish the pairs \( \{S_{0,4}, S_{1,2}\} \) or \( \{S_{0,5}, S_{1,3}\} \).

Now suppose that \( \delta \in C_1(\Sigma) \) is any 1-separating curve. Let \( Z = \Sigma \setminus B(\delta) \). Suppose, for the moment that \( \Sigma \neq S_{0,7} \). Since we are assuming that \( g(\Sigma) + p(\Sigma) \geq 7 \), we see that \( Z \neq S_{0,4}, S_{0,5}, S_{1,2}, S_{1,3} \).

Thus, by the previous paragraph, we can determine the type of \( Z \). Since this holds for all elements of \( C_1(\Sigma) \), we can now easily determine the topological type of \( \Sigma \), and also tell whether an element of \( C_1(\Sigma) \) bounds an \( S_{0,4} \) or an \( S_{1,1} \).

Retrospectively, we can now go back to the earlier set up and distinguish an \( S_{0,4} \) from an \( S_{1,2} \), or an \( S_{0,5} \) from an \( S_{1,3} \) in the complement of \( \tau \). We therefore now know the types of all large components. From this one can easily determine \( \tau \) up to the action of \( \text{Map}(\Sigma) \). We assumed earlier that \( \Sigma \neq S_{0,7} \), but this also holds fairly trivially if \( \Sigma = S_{0,7} \).

In summary, we have shown:

**Lemma 4.1.** Suppose \( g(\Sigma) + p(\Sigma) \geq 7 \). Suppose that \( \tau, \tau' \subseteq C_{ss}(\Sigma) \) are two multicurves, and that there is an automorphism of \( \mathcal{G}_{ss}(\Sigma) \) taking \( \tau \) to \( \tau' \). Then there is an element of \( \text{Map}(\Sigma) \) taking \( \tau \) to \( \tau' \).

Given a component, \( X \), of \( \Sigma \setminus \tau \), set \( C_{ss}(X) \) and \( \mathcal{G}_{ss}(X) \) as defined intrinsically to \( X \). Thus, \( \mathcal{G}_{ss}(X) \) is the full subgraph of \( \mathcal{G}_{ss}(\Sigma, X) \) with vertex set \( C_{ss}(X) \). Note that we can tell whether a curve \( \gamma \in C_{ss}(\Sigma, X) \) lies in \( C_{ss}(X) \) (since it does not bound an \( S_{0,3} \) component of \( \Sigma \setminus \{\tau \cup \gamma\} \)). Thus, we can construct \( \mathcal{G}_{ss}(X) \) out of \( \mathcal{G}_{ss}(\Sigma) \), given \( \tau \).

5. **Rigidity of other surfaces**

In this section we will prove Theorem 1.1, except in the case where \( \Sigma = S_{0,8} \).

We assume that \( g(\Sigma) + p(\Sigma) \geq 7 \). If \( \alpha, \beta \in C_{ss}(\Sigma) \), we write \( \alpha \leq \beta \) to mean that \( B(\alpha) \subseteq B(\beta) \).

First, consider the case where \( p(\Sigma) \geq 5 \). In this case, we will also assume that \( \Sigma \neq S_{0,8} \).

In this case, we define a **bounding pair** to be a pair of elements \( \alpha, \beta \in C_{ss} \) such that \( B(\alpha) \) and \( B(\beta) \) are both \( S_{0,4} \)'s and \( B(\alpha) \cap B(\beta) \) is an \( S_{0,3} \). (This also accords with the definition given for an \( S_{0,7} \) in Section 2.) Note that there is a unique \( \omega \in C_0 \) with \( \omega \leq \alpha \) and \( \omega \leq \beta \), namely \( \omega = \partial_\Sigma(B(\alpha) \cap B(\beta)) \).
We claim:

**Lemma 5.1.** If $\gamma(\Sigma) + p(\Sigma) \geq 7$ and $\Sigma \neq S_{0,8}$, then the class of bounding pairs is in invariant under the action of $\text{Map}(\Sigma)$ on $G_{ss}(\Sigma)$.

**Proof.** We can recognise a bounding pair, $\alpha, \beta$ by the following criterion:

(*) $B(\alpha)$ and $B(\beta)$ are both $S_{0,4}$'s. There is some multicurve, $\tau$, in $\Sigma$ such that $\alpha, \beta$ both lie in some component, $X$, of $\Sigma \setminus \tau$, of type $S_{0,7}$. Moreover, (necessarily) $\alpha, \beta \in C_{ss}(X)$ and $\alpha, \beta$ form a bounding curve intrinsically in $X$.

First note that (*) is recognisable in $G_{ss}(\Sigma)$. We can certainly recognise a multicurve, $\tau$, in $C_{ss}(\Sigma)$, and by Lemma 4.1, we can tell if $\alpha, \beta$ lie in a component $X$ of $\Sigma \setminus \tau$ of type $S_{0,7}$. As explained at the end of Section 4, we can also construct $G_{ss}(X)$. By Lemma 2.5, we can tell if $\alpha, \beta$ form a bounding pair intrinsically to $X$.

Suppose that $\alpha, \beta$ is a bounding pair in $\Sigma$. Note that $Y = \Sigma \setminus (B(\alpha) \cup B(\beta))$ satisfies $g(Y) = g(\Sigma)$ and $p(Y) = p(\Sigma) - 4$. In particular, $g(Y) + p(Y) \geq 3$ and $Y$ is not an $S_{0,5}$. Thus, we can find a multicurve, $\tau \subseteq C_{ss}(Y)$ so that $X = Y \setminus \bigcup_{\gamma \in \tau} B(\gamma)$ is an $S_{0,7}$. Now $B(\alpha), B(\beta) \subseteq X$, and we see that $\alpha, \beta$ is a bounding pair in $X$. Thus, $\alpha, \beta$ satisfies (*).

Conversely, if $\alpha, \beta$ satisfies (*), then again we must have $B(\alpha), B(\beta) \subseteq X$ and so $B(\alpha) \cap B(\beta)$ is an $S_{0,3}$, so $\alpha, \beta$ is a bounding pair in $\Sigma$. □

We now define a **bounding triple** in the same way as for an $S_{0,7}$, namely a triple $\alpha, \beta, \gamma \in C_{ss}$ such that each pair is a bounding pair, and such that there is some $\omega \in C_0$ with $\omega \leq \alpha, \beta, \gamma$.

To see that this is again determined by $G_{ss}(\Sigma)$, we make the following general observation, which holds for any surface $\Sigma$.

**Lemma 5.2.** Suppose that $B_1, B_2, B_3$ are connected subsurfaces (in general position) such that $\partial B_i$ are all connected, and such that $|B_i \cap B_j| = 2$ whenever $i \neq j$. If $B_1 \cap B_2 \cap B_3$ and $\Sigma \setminus (B_1 \cup B_2 \cup B_3)$ are both nonempty, then they are both connected with connected boundary.

**Proof.** This is a simple exercise on noting that a regular neighbourhood of $\partial B_1 \cup \partial B_2 \cup \partial B_3$ is an $S_{0,8}$. □

Given this, we can now recognise a bounding triple, as a triple $\alpha, \beta, \gamma$, with each pair forming a bounding pair, and such that there is a fourth curve $\delta$ disjoint from $\alpha, \beta, \gamma$ and such that $\alpha, \beta, \gamma$ all lie in an $S_{0,6}$ component of $\Sigma \setminus \delta$. Note that this implies that $B(\alpha) \cap B(\beta) \cap B(\gamma)$ must me non-empty (it must contain a boundary component of $\Sigma$). It
now follows easily using Lemma 5.2, that \( B(\alpha) \cap B(\beta) \cap B(\gamma) \) is in fact an \( S_{0,3} \), and so we can set \( \omega \) to be its relative boundary in \( \Sigma \).

The remainder of the proof now follows exactly as for an \( S_{0,7} \). We define the graphs \( H \) and \( H(\omega) \) in the same way. It is sufficient to show that \( H(\Sigma) \) is connected. The argument follows exactly as with that of Lemma 3.2. We can define an \( \omega \)-arc to be an arc connecting \( B(\omega) \) to \( \partial \Sigma \setminus B(\omega) \). Taking a regular neighbourhood of \( B(\omega) \cup a \cup \epsilon \), where \( \epsilon \) is the boundary component, we get an \( S_{0,4} \), namely \( B(\alpha) \), where \( \alpha = \partial_S B(\alpha) \in C_{ss} \), so that \( \omega \leq \alpha \). A bounding pair corresponds to a disjoint pair of \( \omega \)-arcs, terminating in distinct boundary components. Similarly a bounding triple corresponds to a disjoint triple of \( \omega \)-arcs to distinct boundary components. The we can now copy the argument of Lemma 3.2 (after collapsing each component of \( \partial \Sigma \setminus B(\omega) \) to a point).

This allows us to reconstruct \( G_s(\Sigma) \), and so, using [Ko, BrM, Ki] we see that \( G_{ss}(\Sigma) \) is rigid in these cases.

We now move on to the cases where \( p(\Sigma) \leq 4 \). We can assume that \( p(\Sigma) \geq 2 \) (otherwise \( G_{ss}(\Sigma) = G_s(\Sigma) \), and we are covered by [Ko, BrM, Ki]). Note that, \( g(\Sigma) \geq 3 \) in these cases.

We will use the following construction. Let \( T(\Sigma) \) be the graph whose vertex set consists of those \( \alpha \in C_1(\Sigma) \) for which \( B(\alpha) \) is an \( S_{1,1} \), and where \( \alpha, \beta \) are deemed adjacent if \( B(\alpha) \cap B(\beta) = \emptyset \). (This is a full subcomplex of \( G_{ss}(\Sigma) \).) We note:

**Lemma 5.3.** If \( g(\Sigma) \geq 3 \), then \( T(\Sigma) \) is connected.

**Proof.** Suppose \( \alpha, \beta \) are vertices of \( T(\Sigma) \). Since the separating curve graph of a closed surface of genus at least 3 is connected, we can connect \( \alpha, \beta \) by a vertex path \( \alpha = \gamma_0, \gamma_1, \ldots, \gamma_n = \beta \) in \( T(\Sigma) \), so that no complementary component of any \( \gamma_i \) is planar. Taking \( n \) to be minimal, we see that \( \gamma_{i-1} \) must cross \( \gamma_{i+1} \) for all \( i \neq 0, n \). Let \( \delta_i \) be a vertex of \( T(\Sigma) \) in the component of \( \Sigma \setminus \gamma_i \) not containing \( \gamma_{i-1}, \gamma_{i+1} \). We see that \( \alpha, \delta_1, \ldots, \delta_{n-1}, \beta \) is a path in \( T(\Sigma) \) connecting \( \alpha \) to \( \beta \). \( \square \)

Let \( \hat{T}(\Sigma) \) be the flag simplicial complex with 1-skeleton \( T(\Sigma) \), so that every complete subgraph of \( T(\Sigma) \) is contained in a simplex of \( \hat{T}(\Sigma) \). Given \( n \geq 1 \), let \( S_n(\Sigma) \) be the graph whose vertex set consists of \( n \)-simplices of \( \hat{T}(\Sigma) \), and where two such simplices are deemed adjacent of they have a common \((n-1)\)-face.

**Lemma 5.4.** If \( g(\Sigma) \geq n + 2 \), then \( S_n(\Sigma) \) is connected.

**Proof.** If \( 0 \leq m \leq n - 2 \), then the link of any \( m \)-simplex in \( \hat{T}(\Sigma) \) is isomorphic to \( T(\Sigma') \), where \( \Sigma' \) is obtained by removing \( m + 1 \) disjoint copies of \( S_{1,1} \) from \( \Sigma \). Thus \( g(\Sigma') = g(\Sigma) - m - 1 \geq 3 \), and so this is
connected by Lemma 5.3. Since $\hat{T}(\Sigma)$ is itself connected, the statement now follows easily.

We now consider the case where $g(\Sigma) \geq 5$. (This will cover all cases except $S_{3,4}, S_{4,3}$ and $S_{4,4}$, which we discuss later.)

In this case, we define a bounding pair to be a pair, $\alpha, \beta \in G_{ss}(\Sigma)$ such that $B(\alpha), B(\beta)$ are both $S_{1,3}$'s and where $B(\alpha) \cap B(\beta)$ is an $S_{0,3}$. This implies that $|\partial B(\alpha) \cap \partial B(\beta)| = 2$. Again, there is a unique $\omega \in C_0$ with $\omega \leq \alpha, \beta$, namely $\omega = \partial(B(\alpha) \cap B(\beta))$. This property is also recognisable:

**Lemma 5.5.** If $G(\Sigma) \geq 5$, then the class of bounding pairs is invariant under any automorphism of $G_{ss}(\Sigma)$.

**Proof.** The argument follows exactly as with Lemma 3.2. The criterion (*) still holds, except that $B(\alpha), B(\beta)$ are now $S_{1,3}$'s instead of $S_{0,4}$'s.

We also define a bounding triple in the same way. Given Lemma 5.5, we see easily that $\alpha, \beta, \gamma$ form a bounding triple if and only if they pairwise form bounding pairs, and there is some $\delta \in C_{ss}(\Sigma)$ such that $\alpha, \beta, \gamma$ all lie in an $S_{3,3}$ component of $\Sigma \setminus \delta$.

We now proceed, as usual, to define the graphs $H$ and $H(\omega)$ for $\omega \in C_0$. We claim that $H(\omega)$ is connected. This is a bit more involved in this case.

Suppose that $\alpha \in C_{ss}(\Sigma)$ with $B(\alpha)$ and $S_{1,3}$ and with $B(\omega) \subseteq B(\alpha)$. Let $\epsilon \in C_{ss}(\Sigma)$ be any curve with $B(\epsilon)$ and $S_{1,1}$ with $B(\epsilon) \subseteq B(\alpha) \setminus B(\omega)$. Thus, $B(\alpha) \setminus (B(\omega) \cup B(\epsilon))$ is an $S_{0,3}$, so there is (up to homotopy) a unique arc, $a$, in $B(\alpha)$ from $B(\omega)$ to $B(\epsilon)$, meeting $B(\omega)$ and $B(\epsilon)$ precisely at its endpoints. Note that $B(\alpha)$ is a regular neighbourhood of $B(\omega) \cup a \cup B(\epsilon)$. Conversely, given any $\epsilon \in C_{ss}(\Sigma)$ with $B(\epsilon)$ an $S_{1,1}$ disjoint from $B(\omega)$, we can obtain such an $\alpha$ as the boundary of a regular neighbourhood of $B(\omega) \cup a \cup B(\epsilon)$. (Of course, such a representation of $\alpha$ is not unique, but that will not matter.)

Now, a vertex of $H(\omega)$ arises from a pair of disjoint such curves, $\epsilon, \eta, \zeta$, and disjoint arcs, $a, b, c$, connecting $B(\omega)$ respectively to $B(\epsilon)$ and $B(\eta)$, in $\Sigma \setminus (B(\omega) \cup B(\epsilon) \cup B(\eta))$. Similarly, a bounding triple arises from three disjoint such curves, $\epsilon, \eta, \zeta$, and three disjoint arcs, $a, b, c$ in the complement of $B(\omega) \cup B(\epsilon) \cup B(\eta) \cup B(\zeta)$.

Suppose we fix $\epsilon, \eta, \zeta$, and let $H(\omega; \epsilon, \eta, \zeta)$ be the full subgraph of $H(\omega)$, where all the vertices arise (as above) from some pair of curves in $\{\epsilon, \eta, \zeta\}$. Now $H(\omega; \epsilon, \eta, \zeta)$ is connected. This can be seen by the same argument as in the previous case (applied to the surface $\Sigma \setminus$
(\(B(\epsilon) \cup B(\eta) \cup B(\zeta)\)), where the notion of "bounding pair" reverts to the previous case).

Now if \(\epsilon, \eta, \zeta, \theta\) are all disjoint such curves, we can connect \(B(\omega)\) to each of \(B(\epsilon), B(\eta), B(\zeta), B(\theta)\) in the complement of \(B(\omega) \cup B(\epsilon) \cup B(\eta) \cup B(\zeta) \cup B(\theta)\). It then follows that \(H(\omega; \epsilon, \eta, \zeta) \cap H(\omega; \epsilon, \eta, \theta) \neq \emptyset\).

But now, by Lemma 5.4, we can get between any two triples \(\epsilon, \eta, \zeta\) and \(\epsilon', \eta', \zeta'\) by a sequence of such moves, replacing one curve at a time. Since \(H(\alpha)\) is a union of such \(H(\omega; \epsilon, \eta, \zeta)\), it follows that \(H(\omega)\) is connected as claimed.

This allows us to construct \(G_s(\Sigma)\) out of \(G_{ss}(\Sigma)\), and so rigidity follows as before.

Finally we should discuss the cases where \(\Sigma \in \{S_{3,4}, S_{1,3}, S_{4,4}\}\). In these cases we take a bounding pair to be any pair \(\alpha, \beta \in G_{ss}(\Sigma)\) with \(B(\alpha), B(\beta)\) each either an \(S_{0,4}\) or an \(S_{1,3}\). Similarly, for a bounding triple, \(\alpha, \beta, \gamma\), we can allow each of \(B(\alpha), B(\beta), B(\gamma)\) to be an \(S_{0,4}\) or an \(S_{1,3}\).

Note that \(B(\alpha)\) is now determined by a curve which is either a component of \(\partial \Sigma \setminus B(\omega)\), or a curve \(\epsilon \in C_{ss}(\Sigma)\) with \(B(\epsilon)\) an \(S_{1,1}\) disjoint from \(B(\omega)\), together with an arc, \(a\), from \(B(\omega)\) to \(\epsilon\). Thus, \(B(\alpha)\) is a regular neighbourhood of \(B(\omega) \cup a \cup \epsilon\) or \(B(\omega) \cup a \cup B(\epsilon)\), respectively. Bounding pairs and triples then arise for disjoint curves and arcs similarly as before.

We can define \(H(\omega; \epsilon, \eta, \zeta)\) similarly as in the previous case, where \(\epsilon, \eta, \zeta\) are disjoint curves, each either a boundary curve or in \(C_{ss}(\Sigma)\) as above. The same argument shows that the graph is connected.

Moreover, we claim we can get between any two such triples, \(\epsilon, \eta, \zeta\) and \(\epsilon', \eta', \zeta'\), replacing each curve at time by a disjoint curve. In the case where \(p(\Sigma) \geq 4\), there are at least two components of \(\partial \Sigma \setminus B(\omega)\), so this follows easily applying Lemma 5.3 to \(\Sigma \setminus B(\omega)\). If \(p(\Sigma) = 3\), then \(g(\Sigma \setminus B(\omega)) = g(\Sigma) \geq 4\), and so the statement follows since \(S_2(\Sigma \setminus B(\omega))\) is connected by Lemma 5.4.

The argument can now be completed as before, proving Theorem 1.1 in all cases except \(S_{0,8}\).

6. The 8-holed sphere

In this section we outline the proof of rigidity for \(G_{ss}(S_{0,8})\). The argument is essentially the same as for \(S_{0,7}\), except we need to start by finding a different rigid subgraph, in order to recognise curves with simple intersection. We will revert to thinking of \(S_{0,8}\) as \(S \setminus \Pi\), where \(\Pi\) is a subset of the 2-sphere, \(S\), with \(|\Pi| = 8\).
Let $\Delta$ be the graph obtained by adding four diagonal edges to an octagon. More formally, we write $V(\Delta) = \{v_1, \ldots, v_8\}$, where $v_i$ is deemed adjacent to $v_j$ whenever $|i - j| = 1$ or $|i - j| = 4$ (taking indices mod 8).

One can realise $\Delta$ as follows. Recall from $\Theta(\Pi)$ is the graph whose vertex set consists 3-sets in $\Pi$ and where two such sets are deemed adjacent if they are disjoint subsets of $\Pi$. If $\Pi = \{1, \ldots, 8\}$, then the full subgraph of $\Theta(\Pi)$ with vertex set $\{P_1, \ldots, P_8\}$ is isomorphic to $\Delta$.

In fact, we claim that all copies of $\Delta$ arise in $\Theta(\Pi)$ arise in this way. First note that there are no 3-cycles in $\Theta(\Pi)$, and so any 5-cycle in $\Delta$. It follows that any map of $\Delta$ into $\Theta(\Pi)$ sending edges to edges must be injective.

Thus, $\lambda$ constructed as follows. Let $\lambda$ be a regular neighbourhood of the arc $l_{i-3,i} \cup l_{i,i+3}$ and let $\gamma_i = \partial B_i$. The map $[v_i \mapsto \gamma_i]$ now gives an embedding of $\Delta$ into $\mathcal{G}(S_{0,8}, C_1)$. We claim:
Lemma 6.2. There is exactly one embedded copy of $\Delta$ in $G(S_{0,8}, C_1)$ up to the action of $\text{Map}(S_{0,8})$.

So, let $\Delta \subseteq G(S_{0,8}, C_1)$ be such a copy. After composing with $\pi$, we get a map of $\Delta$ into $\Theta(\Pi)$ sending edges to edges, which as we have already noted, must also be an embedding. By Lemma 6.1, we can now label the elements of $\Pi$ as $\{p_1, \ldots, p_8\}$ so that $\pi(\gamma_i) = \{p_{i-3}, p_i, p_{i+3}\}$ for all $i$. (In other words, as in the given example.) Write $B_i = B(\gamma_i)$.

Now $\gamma_1, \gamma_2, \gamma_6, \gamma_5$ is a square in $\Delta$, and so $(B_1 \cup B_6) \cap (B_5 \cup B_2) = \emptyset$. Also $|(B_1 \cup B_5) \cap \Pi| = |(B_1 \cup B_2) \cap \Pi| = 4$. It follows that we can find disjoint discs, $B_{16}, B_{52} \subseteq S$ with $B_1 \cup B_6 \subseteq B_{16}$ and $B_5 \cup B_2 \subseteq B_{52}$. Note that $\partial B_{16}$ and $\partial B_{52}$ are homotopic in $S \setminus \Pi$, and so determine a curve, $\gamma_{16} = \gamma_{52}$. We have similar pairs of discs, $\{B_{63}, B_{27}\}$, $\{B_{38}, B_{74}\}$ and $\{B_{85}, B_{41}\}$. Note that we have similar curves, $\gamma_{ij}$, defined whenever $|i - j|$ is $3$ or $5$.

Now consider the curves $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Let $A = S \setminus (B_1 \cup B_2)$. This is an annulus with $A \cap \Pi = \{p_3, p_8\}$. Now $B_3 \cap B_2 = \emptyset$, $B_3 \cap B_1 \neq \emptyset$ (since $p_6 \in B_3 \cap B_1$), and $p_3, p_8 \in B_3 \cap A$. Thus, we can find (disjoint) arcs, $e, f$ in $B_3 \cap A$, respectively connecting $p_3$ and $p_8$ to $\partial B_1$.

Let $D$ be (the closure of) the component of $B_1 \setminus B_1$ containing $p_7$. Now $B_4 \cup B_1 \subseteq B_{41}$ and $(B_{41} \setminus B_4) \cap \Pi = \{p_7\}$. It follows (cf. Lemma 2.3) that $D \cap B_1$ is a single arc, $b$, say. Thus, $\partial D = b \cup d$, where $d \subseteq \gamma_4$ is an arc with endpoints $d \cap B_1 = d \cap b$.

Now, $d \cap (e \cup f) \subseteq B_4 \cap B_3 = \emptyset$. It follows that any arc of $d \cap A$ which does not include an endpoint could be homotoped into $B_2$. It follows that $d \cap B_2$ must consist of a single arc, and so $|d \cap B_2| = 2$. Now $R = B_1 \cup D$ is a disc with $R \cap \Pi = B_{41} \cap \Pi$, and so $\partial R$ is homotopic in $S \setminus \Pi$ to $\gamma_{41} = \gamma_{52}$. Also $|\partial R \cap \gamma_2| = 2$, and so $\iota(\gamma_{41}, \gamma_2) = 2$. In other words, $\gamma_{41}$ and $\gamma_2$ have simple intersection (cf. Lemma 2.4).

By symmetry, it follows that each curve $\gamma_i$ is either disjoint from, or has simple intersection with, each curve of the form $\gamma_{jk}$.

It is now fairly straightforward to see that $\{\gamma_1, \ldots, \gamma_8\}$ must be precisely the set of curves described in our example. For example, note that $B_{41} \cap \Pi = \{p_7, p_4, p_1, p_6\}$, and $B_{41} \cap \gamma$ is an arc which cuts off a disc, namely $B_{41} \cap B_2$, containing $p_7$. Now $B_1 \cap \Pi = \{p_4, p_1, p_6\}$, and so $\gamma_4 = \partial B_1$ is homotopic to $\partial(B_{41} \setminus B_2)$. Similarly, $\gamma_4$ is homotopic to $\partial(B_{41} \setminus B_3)$. It follows that $B_1 \cap B_4$ is a disc containing $p_1, p_4$. One can now proceed to show that any pair of the curves $\gamma_i$ are either disjoint or have simple intersection, and that this determines them completely.

This proves Lemma 6.2.

The remainder of the argument is now essentially identical to that for $S_{0,7}$. We define bounding pairs in the same way. Such a pair,
\(\alpha, \beta\) can be recognised by the fact that \(\alpha, \beta \in C_1(S_{0,8})\), there is a curve \(\gamma \in C_{ss}(S_{0,8}) \setminus C_1(S_{0,8})\) disjoint from both \(\alpha\) and \(\beta\), and \(\alpha, \beta\) are vertices of some embedded copy of \(\Delta\) in \(G(S_{0,8}, C_1)\). We now proceed as before.

This shows that \(G_{ss}(S_{0,8})\) is rigid, completing the proof of Theorem 1.1.

References


Mathematics Institute, University of Warwick, Coventry, CV4 7AL, Great Britain