SOME PROPERTIES OF MEDIAN METRIC SPACES

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Abstract. We describe a number of properties of a median metric space. In particular, we show that a complete connected median metric of finite rank admits a canonical bilipschitz equivalent CAT(0) metric. Metric spaces of this sort arise, up to bilipschitz equivalence, as asymptotic cones of certain classes of finitely generated groups, and the existence of such a structure has various consequences for the large scale geometry of the group.

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1. Introduction

In this paper, we discuss some basic properties of median metric spaces. One of the main objectives will be show that, under certain hypotheses, such a metric is bilipschitz equivalent to a CAT(0) metric. Median metrics have been studied by a number of authors, see for example, [Ve, ChaDH] and the references therein. They arise, up to bilipschitz equivalence, as asymptotic cones of certain classes of groups, and the fact that they admit such a metric has various consequences for the structure of the group. We begin with some basic definitions (cf. [Ve, ChaDH]), which will be elaborated upon in later sections.

Let $(M, \rho)$ be a metric space. Given $a, b \in M$ let $I(a, b) = I_\rho(a, b) = \{x \in M \mid \rho(a, x) + \rho(x, b) = \rho(a, b)\}$.

Definition. We say that $(M, \rho)$ is a median metric space if, for all $a, b, c \in M$, $I(a, b) \cap I(b, c) \cap I(c, a)$ consists of exactly one element of $M$.

We will denote this element by $\mu(a, b, c)$. We refer to $\mu$ as the median induced by $\rho$. It turns out that $(M, \mu)$ is a median algebra. This follows by a result of Sholander [S] (see [ChaDH], and Section 2 here). For a general discussion of median algebras, see for example, [I, BaH, Ro]. Some further discussion relevant to this paper can be found in [Bo1, Bo2]. Examples of median metric spaces include $\mathbb{R}^n$ in the $l^1$-metric, $\mathbb{R}$-trees, $l^1$-products of $\mathbb{R}$-trees, and median subalgebras of such spaces.

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We define the rank of median algebra, $M$, to be the maximal $n$ such that $M$ contains a subalgebra isomorphic to the $n$-cube, $\{-1,1\}^n$. If such cubes exist for all $n$, we say that the rank is infinite.

In the case where rank($M$) \leq n, we will construct a new metric, $\sigma = \sigma_{\rho}$, canonically associated to $\rho$, and satisfying $\rho/\sqrt{n} \leq \sigma_{\rho} \leq \rho$. Among other things, we will show:

**Theorem 1.1.** If $(M, \rho)$ is a complete connected median metric space, then $(M, \sigma_{\rho})$ is a CAT(0) space.

The definition of a CAT(0) space will be given in Section 8. For a general discussion of such spaces, see for example, [BrH]. Connections with median algebras in a more combinatorial setting are described in [Che].

I suspect that the assumption of completeness in Theorem 1.1 is unnecessary, and we will give a variation, namely Theorem 8.2, without this assumption.

Median algebras arise in various contexts. We give some examples in Section 3. Our main motivation arises from geometric group theory, in particular, asymptotic cones of certain finitely generated groups.

An asymptotic cone of a finitely generated group is a complete metric space which captures much of the large scale geometry of the group (see [VaW, G]). It is known, for example, that any asymptotic cone of the mapping class group of a compact surface is bilipschitz equivalent to a median metric space [BeDS]. More generally, the notion of a “coarse median group” was proposed in [Bo]. When such a group is “finitely colourable”, any asymptotic cone admits a bilipschitz embedding into a finite product of $\mathbb{R}$-trees [Bo2]. Its image is a connected median metric space. It follows that the asymptotic cone is bilipschitz equivalent to a CAT(0) space (though the CAT(0) metric might not be canonically determined by the metric on the asymptotic cone). In fact, we can relax “finitely colourable” to “finite rank”, as we discuss in Section 5. In particular, it follows that the asymptotic cone is contractible. From this one can deduce that the group is $FP_{\infty}$ and has polynomial isoperimetric functions in all dimensions, [Ri]. (For the mapping class groups, these follow by automaticity [M]. See also [BeD] for some more refined results.) We aim to explore other consequences of this construction elsewhere.

Finally, we note that median metric spaces are special cases of those discussed in [Bo2], and so the results there also apply here. However, apart from some of the basic theory, these papers are largely independent.
Conventions. For clarity we collect together some conventions of terminology, whose precise definitions will be given later. As above a “cube” will refer to a finite median algebra (or subalgebra) of the form described earlier. A “face” of a finite median algebra (such as a cube) will be a cube in the algebra that is convex in the median sense. An “edge” will be a face consisting of exactly two points. The endpoints of an edge are termed “adjacent”. A “real cube” will be a finite product of compact real intervals, thought of as a topological or metric space. A “cube complex” will be a topological space built out of real cubes, termed “cells” of the complex. A “side” of the complex will be a 1-dimensional cell. (Note that the cells of a finite CAT(0) cube complex are in natural correspondence with the faces of the vertex set, as we will explain later.) An “interval” will refer to an interval in a median algebra. (In the case of the real line, this will be a compact real interval, though the notion is much more general.)

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2. Median metrics and median algebras

In this section, we give some basic definitions and review formulations of the notions of a median metric. Some of this can be found, expressed a little differently, in [Ve, ChaDH]. We refer to [I, BaH, Ro, BaC, Bo1] for further background.

First, we recall that a median algebra is a set, $M$, with a ternary relation $\mu : M^3 \rightarrow M$ satisfying the following for all $a,b,c,d,e \in M$.

(M1): $\mu(a, b, c) = \mu(b, c, a) = \mu(b, a, c)$,
(M2): $\mu(a, a, b) = a$, and
(M3): $\mu(a, b, \mu(c, d, e)) = \mu(\mu(a, b, c), \mu(a, b, d), e)$.

Given $a, b \in M$ we will write $[a, b] = [a, b]_M = \{x \in M \mid \mu(a, b, x) = x\}$ for the median interval from $a$ to $b$. One verifies that:

(I1): $(\forall a \in M) \ [a, a] = \{a\}$
(I2): $(\forall a, b \in M) \ [a, b] = [b, a]$
(I3): $(\forall a, b \in M)$, if $c \in [a, b]$, then $[a, c] \subseteq [a, b]$
(I4): $(\forall a, b, c \in M)$ there is a unique $m \in M$ such that $[a, b] \cap [b, c] \cap [c, a] = \{m\}$. 
In fact, in (I4), we have \( m = \mu(a, b, c) \).

It turns out that (I1)–(I4) provide an alternative way of defining a median algebra, as follows. If we have a set \( M \), and a map \([(a, b) \mapsto [a, b]]\) which assigns to any pair \( a, b \in M \) a subset \([a, b] \subseteq M\) satisfying the above properties, then \((M, \mu)\) is a median algebra, where \( \mu : M^3 \to M \) is the ternary operation defined by setting \( \mu(a, b, c) = m \) in (I4). This follows from work of Sholander [S].

In fact, one only needs part of Sholander’s paper for this. Since the logic might not be immediately apparent, a few comments are in order. To begin, we have included axiom (I2), so as not to worry about the order of \( a \) and \( b \). (With judicious formulation of the remaining axioms, this can probably be circumvented.) We will ignore this issue henceforth. Note that Postulate \( \Sigma_1 \) of [S] is the conjunction of Postulates \( D, B_1 \) and \( F \), which are respectively implied by our axioms (I4), (I1) and (I3). Now Paragraph (4.9) of [S] tells us that \( \Sigma_1 \) implies Postulate \( I \) of that paper. From Paragraph (3.8) we now get that \( \Sigma_1 \) implies Postulates \( M \) and \( N \), which are essentially the axioms of a median algebra (or a “median semilattice” in the terminology of that paper). This is observed in Paragraph (4.10) of [S]. Moreover, if we wish, we could strengthen our axiom (I3) to say in addition that if \( c \in [a, b] \) and \( d \in [a, c] \) then \( c \in [d, b] \). This property is formulated as (3.4) in [S], where its derivation is cited from a paper of Pitcher and Smiley. However, this assertion is almost immediate in the situation where we will want to apply it, that is in the case of a median metric. (Some further discussion of this can be found in [Ve].)

Note, in particular, that the median structure is completely determined by the set of intervals.

A subalgebra of \( M \) is a subset closed under \( \mu \). Any finite subset of \( M \) is contained in a finite subalgebra. (This follows from the fact that the free median algebra on a finite set is finite.) A subset \( C \subseteq M \) is convex if \([a, b] \subseteq C\). The convex hull \( \text{hull}(A) = \text{hull}_M(A) \), of a subset \( A \subseteq M \) is the intersection of all convex subsets of \( M \) containing \( A \). Clearly \( \text{hull}(A) \) is convex. A wall in \( M \) is an (unordered) partition of \( M \) into two disjoint non-empty convex subsets. A homomorphism between median algebras is a map respecting medians. Note also that a direct product of median algebras is itself a median algebra.

Given \( a, b \in M \), we can define a projection map, \([x \mapsto \mu(a, b, x)]\), from \( M \) to \([a, b]\). One can verify that this is a median homomorphism. Moreover, if \( c, d \in [a, b] \), then projection to \([a, b]\) followed by projection to \([c, d] \subseteq [a, b]\) agrees with projection directly to \([c, d]\).

The two-point set, \( \{-1, 1\} \), admits a unique median structure. By an \( n \)-cube in \( M \), we mean a subalgebra isomorphic to \( \{-1, 1\}^n \). The
rank of $M$ is the maximal $n$ such that $M$ contains an $n$ cube. The rank
is deemed infinite if it contains cubes of all finite dimensions. We will
refer to 2-cube as a square. (This is termed a “rectangle” in [ChaDH].)

The following is a trivial, though useful observation:

**Lemma 2.1.** A subset of $Q \subseteq M$ a square if and only if has exactly four
elements, and we can cyclically order them mod 4 as $Q = \{a_1, a_2, a_3, a_4\}$
so that $a_i \in [a_{i-1}, a_{i+1}]$ for all $i$.

We will generally simply say that “$a_1, a_2, a_3, a_4$ is a square”. We
refer to the pairs $a_i, a_{i+1}$ as its sides, and to the pairs $a_i, a_{i+2}$ as the
diagonals. We say that two ordered pairs, $a, b$ and $c, d$ are parallel if
either $a, b, d, c$ is a square, or else, $a = c$ and $b = d$. One can check that
the parallel relation is transitive, so one can speak of “parallel classes”.

More generally, if $Q$ is a cube, we can define an edge of $Q$ as a
two-element subset of $Q$, which is intrinsically convex in the subalge-
bra $Q$. A diagonal of $Q$ is a two-element subset, $\{a, b\} \subseteq Q$, with
$Q \subseteq [a, b]$. (Both these notions coincide with the obvious geometrical
interpretations.) We remark that if $a', b'$ is another diagonal of $Q$, then
$[a, b] = [a', b']$, since clearly each of these sets in included in the other.
In fact, this set is precisely hull$_M(Q)$.

Now suppose that $(M, \rho)$ is a metric space, with metric $\rho$. Given
$a, b, c \in M$, write $\langle a, b \rangle_c = \frac{1}{2}(\rho(a, c) + \rho(b, c) - \rho(a, b)) \in (0, \infty)$ for the
“Gromov product” of $a, b$ based at $c$. Thus, $I(a, b) = \{x \in M \mid \langle a, b \rangle_x = 0\}$. We write $S(a, b, c) = \frac{1}{2}(\rho(a, b) + \rho(b, c) + \rho(c, a)) = \langle a, b \rangle_c + \langle b, c \rangle_a +
\langle c, a \rangle_b$. If $a, b, c, d \in M$, we write $T(a, b, c; d) = \rho(d, a) + \rho(d, b) + \rho(d, c)$.
Clearly $T(a, b, c; d) \geq S(a, b, c)$.

It is easily verified that the following are equivalent for $a, b, c, m \in M$.

(C1): $m \in I(a, b) \cap I(b, c) \cap I(c, a)$

(C2): $T(a, b, c; m) = S(a, b, c)$

(C3): $\rho(a, m) = \langle b, c \rangle_a$, $\rho(b, m) = \langle c, a \rangle_b$ and $\rho(c, m) = \langle a, b \rangle_c$.

**Definition.** We say that $(M, \rho)$ is a median metric space, if, for all
$a, b, c \in M$, there is exactly one $m \in M$ such that any (hence all) of
the conditions (C1), (C2) or (C3) above hold.

In this case, we see easily that the maps $[(a, b) \mapsto I(a, b)]$ satisfy the
conditions (I1)–(I4) above, and so $(M, \mu)$ has the structure of a median
algebra on setting $\mu(a, b, c) = m$.

We will refer to a finite sequence $a_0, a_1, \ldots, a_p$ as a monotone se-
quence if $\rho(a_0, a_p) = \sum_{i=1}^p \rho(a_{i-1}, a_i)$. Note that this is equivalent to
the median condition that \( a_i \in [a_{i-1}, a_{i+1}] \) for all \( i \). (In [ChaDH], this is referred to as a geodesic sequence. We use the term “monotone” since we will eventually be dealing with more than one metric.)

As in [ChaDH], we note that squares are rectangular:

**Lemma 2.2.** Suppose that \((M, \rho)\) is a median metric space, and that \(a_1, a_2, a_3, a_4\) is a square. Then \(\rho(a_1, a_2) = \rho(a_3, a_4)\) and \(\rho(a_2, a_3) = \rho(a_4, a_1)\).

**Proof.** Let \( t_i = \rho(a_i, a_{i+1}) \). Then \( a_{i+1} \in [a_i, a_{i+2}] \) and so \( \rho(a_i, a_{i+2}) = t_i + t_{i+1} \), and we get \( t_1 + t_2 = t_3 + t_4 \) and \( t_2 + t_3 = t_4 + t_1 \). It follows that \( t_1 = t_3 \) and \( t_2 = t_4 \). \(\square\)

Note that the diagonals are also equal: \(\rho(a_1, a_3) = \rho(a_2, a_4)\).

Lemma 2.2 can, of course, be expressed by saying that if \(a, b, c, d\) are parallel pairs in \(M\), then \(\rho(a, b) = \rho(c, d)\).

One can reformulate the discussion of metrics starting instead with median algebras. Suppose that \((M, \mu)\) is a median algebra admitting a metric \(\rho\) with the property that \(\langle a, b \rangle_c = 0\) whenever \(c \in [a, b]\), that is, \([a, b] \subseteq I(a, b)\). It follows that \([a, b] = I(a, b)\). For suppose \(x \in I(a, b)\), and let \(m = \mu(a, b, x)\). Now \(\langle a, b \rangle_m = \langle a, x \rangle_m = \langle b, x \rangle_m = 0\), so \(\rho(x, m) = \langle a, b \rangle_x = 0\), and so \(x = m\). It follows that \(x \in [a, b]\) as claimed. In other words, we see that a median metric space is the same thing as a median algebra \((M, \mu)\) with a metric \(\rho\) satisfying \(\rho(a, b) = \rho(a, c) + \rho(c, b)\) whenever \(a, b, c \in M\) with \(c \in [a, b]\).

### 3. Examples of median metric spaces

In this section, we give some examples of median metric spaces which will feature in later discussions. Some related constructions have appeared elsewhere, or have analogues in a combinatorial setting. For a recent survey, see [BaC].

Let \(\Upsilon\) be a CAT(0) cube complex, thought of as the topological realisation of a combinatorial cell complex. It is known that the vertex set \(\Pi = V(\Upsilon)\) is a median algebra. We write \(\mathcal{W} = \mathcal{W}(\Pi)\) for the set of walls of \(\Pi\). (Alternatively, we can think of this as the set of hyperplanes of \(\Upsilon\).) We can also think of an element of \(\mathcal{W}\) as corresponding to a parallel class of edges of \(\Pi\). (The set of edges which cross any given wall will constitute such a parallel class.) Suppose we have a function, \(\lambda : \mathcal{W} \to (0, \infty)\). This gives rise to a path-metric on the 1-skeleton, so that if \(a, b \in \Pi = V(\Upsilon)\) then \(\rho(a, b) = \sum_{W} \lambda(W)\) where \(W\) ranges over the set of walls of \(\Pi\) which separate \(a\) from \(b\). This naturally extends to a path-metric, \(\rho\), on all of \(\Upsilon\), where each \(n\)-cell is given the structure of
a rectilinear parallelopided in $\mathbb{R}^n$ with the $l^1$-metric. Note that $(\Upsilon, \rho)$ is uniquely determined up to a cell-preserving isometry.

A more formal way to describe this is as follows. Let $Q(\Pi)$ be the cube consisting of the direct product $\prod W$, where each $W \in \mathcal{W}$ is viewed formally as a two-point median algebra. (We will only really need to consider the case where $\Pi$ is finite.) There is a natural embedding of $\Pi$ into $Q(\Pi)$. In this way, $\Upsilon$ can be seen as the full subcomplex of $Q(\Pi)$ with vertex set $\Pi$. Given our map $\lambda : \mathcal{W} \rightarrow (0, \infty)$, let $P = \prod_{W \in \mathcal{W}} [0, \lambda(W)]$. This is a median metric space in the $l^1$-metric, and $\Upsilon$ is a subalgebra, and itself a median metric in the induced path-metric.

Note that we could also put a euclidean structure on $\Upsilon$. For this, we start in the same way, putting the same metric on the 1-skeleton, but instead of taking the $l^1$-metric on each cell, we take the euclidean metric. (Or in the formulation of the previous paragraph, we take the path-metric induced from the euclidean metric on $P$.) This gives us a path-metric $\sigma$ with $\sigma \leq \rho$. It is also easy to see that it is $\text{CAT}(0)$. (Thus usual construction demands that we take all sides of unit length, but the same holds in this more general situation. The links are all $\text{CAT}(1)$ spaces.) Note also that, if $\Pi$ has rank $n$ (or equivalently, $\Upsilon$ has dimension $n$) then $\rho \leq \sigma \sqrt{n}$.

In fact, any finite median algebra, $\Pi$, can be identified with the vertex set of a finite $\text{CAT}(0)$ cube complex, as follows. Let $\mathcal{W} = \mathcal{W}(\Pi)$ be the set of walls in $W$. There is a natural embedding of $\Pi$ into $Q(\Pi)$, and we can think of $\Upsilon(\Pi)$ as the full subcomplex with vertex set $\Pi$. Note that $\Upsilon$ is unique up to isomorphism. We will denote it by $\Upsilon(\Pi)$. (Thus, $\Upsilon(\Pi)$ is subcomplex of $\Upsilon(Q(\Pi))$.) Note that each face of $\Pi$ corresponds to a cell of $\Upsilon(\Pi)$.

If $\Pi$ is a finite median metric space, then we also have a function $\lambda : \mathcal{W} \rightarrow (0, \infty)$ which assigns the distance between the vertices of any edge crossing a given wall. By Lemma 2.2, this is well defined. From this we see that any finite median metric space can be embedded in such a complex. We will return to this construction at the end of Section 7, see Lemma 7.6.

Here is a more general construction. To begin, let $Q = \{-1, 1\}^n$ be the standard $n$-cube. Its realisation, $\Upsilon(Q)$, is a real $n$-cube. Let $F(Q) = \{-1, 0, 1\}^n$. Then $F(Q)$ is also a median algebra, containing $Q$ as a subalgebra. We will think of $F(Q)$ as corresponding to the set of faces of $\Upsilon(Q)$. (It might also be thought of as the vertex set of the binary subdivision of $\Upsilon(Q)$.) Given $s, t \in F(Q)$ we will write $t \preceq s$ to mean that $t$ corresponds to a face of $s$. (Formally, this means that $s$ can be obtained from $t$ by resetting some of the $\pm 1$ co-ordinates equal
to 0.) Given $s \in F(Q)$, let $Q(s) = \{t \in Q \mid t \preceq s\}$. In other words, $Q(s)$ is the face of $Q$ corresponding to $s$. Note that if $r, s \in F(Q)$, then $r \preceq s$ if and only if $Q(r) \subseteq Q(s)$.

Now let $\Pi$ be any finite median algebra, and let $Q(\Pi) = \prod W$ be as above. Given $W \in \mathcal{W}$, we write $W = \{H^+(W), H^-(W)\}$, where the ± signs are assigned arbitrarily. In this way, the formal product, $Q(\Pi) = \prod W$, can be identified with $\{-1, 1\}^W$. Let $F(\Pi) \subseteq F(Q(\Pi))$ correspond to the set of faces of $\Pi$. (Formally, we can set $F(\Pi) = \{s \in F(Q(\Pi)) \mid Q(s) \subseteq \Pi\}$. Note that if $s \in F(\Pi)$ and $t \in F(Q(\Pi))$ with $t \preceq s$, then $t \in F(\Pi)$.

Suppose now that to each $W \in \mathcal{W}$, we have somehow associated a median algebra, $\Phi_W$, together with elements $p^+_W, p^-_W \in \Phi_W$ with $p^-_W \neq p^+_W$ and with $\Phi_W = \{p^+_W, p^-_W\}$. Let $P = \prod_{W \in \mathcal{W}} \Phi_W$ be the product median algebra. We can identify $Q(\Pi) \equiv \{-1, 1\}^W$ as a subalgebra $\prod_{W \in \mathcal{W}} \{p^+_W, p^-_W\} \subseteq P$. In this way, each element $s \in F(\Pi)$ gets canonically associated to a convex subset $P(s)$ of $P$, namely, $P(s) = \text{hull}_P(Q(s))$. Let $\Phi = \bigcup_{s \in F(\Pi)} P(s)$. We claim that $\Phi$ is a subalgebra of $P$.

To see this, define a map $h_W : \Phi_W \rightarrow \{-1, 0, 1\}$ for each $W \in \mathcal{W}$, by setting $h_W(p^+_W) = \pm 1$ and $h_W(x) = 0$ for all $x \in \Phi_W \setminus \{p^+_W, p^-_W\}$. This gives rise to a map $h : P \rightarrow F(Q(\Pi)) \equiv \{-1, 0, 1\}^W$, by taking $h_W$ on each co-ordinate. By construction, $\Phi = \Phi = \mathcal{Y}(\Pi)$.

As an example of this construction, if each $\Phi_W$ is a non-trivial compact real interval, we recover the realisation, $\Phi = \Phi(\Pi)$ of $\Pi$. In fact, if $\lambda : \mathcal{W}(\Pi) \rightarrow (0, \infty)$ is any map (as above), we can set $\Phi_W = [0, \lambda(W)]$ with $p^-_W = 0$ and $p^+_W = \lambda(W)$. This gives $\Phi = \Phi(\Pi)$ naturally equipped with the $l^1$-metric $\rho$ as described earlier.

The main purpose of the above construction will be described in Section 6, see Lemma 6.2.

4. Basic properties of median metric spaces

We discuss some of the basic properties of median metric spaces. We begin with the following general construction in a median algebra $(M, \mu)$.

Suppose that $a, b, c, d \in M$. Let $a' = \mu(b, c, d)$, $b' = \mu(c, d, a)$, $c' = \mu(d, a, b)$, $d' = \mu(a, b, c)$, $a'' = \mu(b', c', d')$, $b'' = \mu(c', d', a')$, $c'' =
If \( a, b, c, d \in M \), then \( \rho(\mu(a, b, c), \mu(a, b, d)) \leq \rho(c, d) \).

**Proof.** In the above notation, \( c, c'', d'', d \) is a monotone sequence, and the pairs \( c'', d'' \) and \( d', c' \) are parallel. Therefore \( \rho(c', d') = \rho(c'', d'') \leq \rho(c, d) \) as required. \( \square \)

In particular, we see that if \( c, d \in [a, b] \), then \( \rho(c, d) \leq \rho(a, b) \).

We also note that this implies that \( \mu : M^3 \rightarrow M \) is continuous. In other words, \( (M, \mu) \) is a topological median algebra.

Recalling the notation of Section 2, we also have:

**Lemma 4.2.** If \( a, b, c, d \in M \), then \( \rho(d, \mu(a, b, c)) \leq T(a, b, c; d) - S(a, b, c) \).

**Proof.** In the above notation, \( d' = \mu(a, b, c) \). Let \( A = \rho(a', d') \), \( B = \rho(b', d'') \) and \( C = \rho(c', d'') \); that is, \( A, B, C \) are the three side-lengths of the central cube. Let \( A_0 = \rho(a, a'') \), \( B_0 = \rho(b, b'') \), \( C_0 = \rho(c, c'') \) and \( D_0 = \rho(d, d'') \); that is, the lengths of the four free sides. Now, \( \rho(a, b) = A_0 + B_0 + A + B \) etc. and so \( S(a, b, c) = A_0 + B_0 + C_0 + A + B + C \). Also, \( \rho(d, a) = D_0 + A_0 + B + C \) etc. and so \( T(a, b, c; d) = 3D_0 + A_0 + B_0 + C_0 + 2(A + B + C) \). Finally, \( \rho(d, d'') = D_0 + A + B + C \leq 3D_0 + A + B + C = T(a, b, c; d) - S(a, b, c) \). \( \square \)

We obtain the following, proven in [Ve] and [ChaDH].
Lemma 4.3. The metric completion of a median metric space is a median metric space.

Proof. Let \((M, \rho)\) be a median metric space, and let \((\bar{M}, \bar{\rho})\) be its completion. Let \(a, b, c \in \bar{M}\), and choose sequences, \(a_i, b_i, c_i \in M\), with \(a_i \to a, b_i \to b\) and \(c_i \to c\). Let \(m_i = \mu(a_i, b_i, c_i)\). By Lemma 4.1, \(\rho(m_i, m_j) \leq \rho(a_i, a_j) + \rho(b_i, b_j) + \rho(c_i, c_j)\), so \((m_i)_i\) is Cauchy, so \(m_i\) tends to some \(m \in M\). By continuity, we see that \(m \in I_\bar{\rho}(a, b) \cap I_\bar{\rho}(b, c) \cap I_\bar{\rho}(c, a)\). Suppose that \(d \in I_\rho(a, b) \cap I_\rho(b, c) \cap I_\rho(c, a)\). Now \(T(a_i, b_i, c_i; d) \to T(a, b, c; d) = S(a, b, c)\). By Lemma 4.2, \(\rho(d, m_i) \leq T(a_i, b_i, c_i; d) - S(a_i, b_i, c_i) \to 0\), so \(\rho(d, m) = 0\), so \(d = m\).

Finally, we consider connectedness of a median metric space, \(M\). If \(J \subseteq \mathbb{R}\) is an interval, we say that a continuous path, \(\gamma : J \to M\) is **monotone** if \(\gamma(v) \in [\gamma(t), \gamma(u)]\) whenever \(t, u, v \in J\) with \(t \leq v \leq u\). (This ties in with the notion of a “monotone sequence” defined in Section 2.) Note that if such a path exists, we can assume that it is injective, and we can reparametrise so that it is a \(\rho\)-geodesic, that is, \(\rho(\gamma(t), \gamma(u)) = |t - u|\) for all \(t, u \in J\). Conversely, any \(\rho\)-geodesic will be monotone. Recall that a **geodesic space** is a metric space in which any pair of points can be connected by a geodesic. We see that a metric median space is geodesic if and only if every pair of points can be connected by a monotone path.

The following is an easy consequence of the fact that the projection \([x \mapsto \mu(a, b, x)]\) from \(M\) to \([a, b]\) is continuous for all \(a, b \in M\), and the fact that \([c, d] \subseteq [a, b]\) for all \(c, d \in [a, b]\) (that is, intervals are convex).

Lemma 4.4. Let \((M, \rho)\) be a median metric space. Then \((M, \rho)\) is connected (respectively, path connected; respectively geodesic) if and only if, for all \(a, b \in M\), the interval \([a, b]\) is connected (respectively, path connected; respectively geodesic).

We suspect that these notions are all equivalent. We can certainly make the following observation:

Lemma 4.5. Let \((M, \rho)\) be a connected median metric space. Suppose that \(a, b \in M\) and that \(0 < t < \rho(a, b)\). Then there exists \(c \in M\) with \(\rho(a, c) = t\) and \(\rho(b, c) = \rho(a, b) - t\).

Proof. Suppose not. Let \(U = \{x \in M \mid \rho(a, \mu(a, b, x)) < t\}\) and \(V = \{x \in M \mid \rho(b, \mu(a, b, x)) \leq \rho(a, b) - t\}\). Then \(U\) and \(V\) are open, \(M = U \cup V\), \(a \in U\) and \(b \in V\), contradicting the fact that \(M\) is connected.

In particular, setting \(t = \frac{1}{2}\rho(a, b)\), we see that any pair of points of \(M\) must have a midpoint. A standard completion argument now shows:
Lemma 4.6. Any complete connected median metric space is geodesic.

In fact, as was pointed out to me by Hans Bandelt, a complete median metric space, \( M \), is geodesic if and only if it has the “Menger property”, which in this context means that \([a, b] \neq \{a, b\}\) for all distinct \( a, b \in M \). Indeed, any complete metric space \((M, \rho)\) is geodesic if and only if \( I_\rho(a, b) \neq \{a, b\}\) for all distinct \( a, b \in M \).

5. Cubes in median algebras

In the next two sections we describe some general median algebra constructions which we apply to median metric spaces in Sections 7 and 8. In the present section we make some observations about cubes in median algebras. It seems that some related statements can found in, or derived from, the literature on distributive lattices. However, since much of it is written in a form not no readily accessible to geometers, we give a self-contained account here.

Let \((M, \mu)\) be a median algebra, and let \( a, b \in M \). In this section, we will adopt the convention that the interval denoted \([a, b]\) has a preferred “initial point”, \( a \), and “terminal point”, \( b \). Given \( x, y \in [a, b] \), we write \( x \land y = \mu(a, x, y) \) and \( x \lor y = \mu(b, x, y) \). Then \(([a, b], \land, \lor)\) is a distributive lattice. We write \( x \leq y \) to mean that \( x \land y = x \), or equivalently \( x \lor y = y \). Then \( \leq \) is a partial order on \([a, b]\), with minimum \( a \) and maximum \( b \). We can recover the median on \([a, b]\) from the lattice structure as \( \mu(x, y, z) = (x \land y) \lor (y \land z) \lor (z \land x) \). In particular, any lattice homomorphism will be a median homomorphism.

Suppose that \( e_1, e_2, \ldots, e_n \in [a, b]\). We write \( I = \{1, 2, \ldots, n\} \). Given \( J \subseteq I \) write \( e_J = \bigvee_{i \in J} e_i \), with the convention that \( e_\emptyset = a \). We say that \((e_i)\) spans \([a, b]\) if \( e_I = b \). In this case, if \( x \in [a, b] \), then \( x = \lor_{i \in I}(x \land e_i) \).

Let \( P = \prod_{i \in I} [a, e_i] \) be the product median algebra. We define a map \( \theta : [a, b] \to P \), by setting \( \theta(x) = (x \land e_1, x \land e_2, \ldots, x \land e_n) \), and a map \( \phi : P \to [a, b] \) by setting \( \phi(x_1, x_2, \ldots, x_n) = \lor_{i \in I} x_i \). Now, by the above, if \( (x_i) \) spans \([a, b]\), then \( \phi \circ \theta \) is the identity on \([a, b]\). Note that \( P \) is itself intrinsically an interval, namely \( P = \{\theta(a), \theta(b)\} \), and hence also a distributive lattice. Indeed the lattice structure is the same as that induced from those on \([a, e_i]\) by defining \( \land \) and \( \lor \) co-ordinatewise.

Note that for all \( x, y \in [a, b] \), we have \( \theta(x \land y) = \theta(x) \land \theta(y) \) and \( \theta(x \lor y) = \theta(x) \lor \theta(y) \). It follows that \( \theta \) is a monomorphism from \([a, b]\) into \( P \).

We say that \((e_i)\) is independent if \( e_i \land e_j = a \) whenever \( i \neq j \). In this case, if \( x \in [a, e_i] \) and \( y \in [a, e_j] \), then \( x \land y = (x \land e_i) \land (y \land e_j) \).
$e_j = x \wedge y \wedge (e_i \wedge e_j) = a$. Now, if $x = (x_1, \ldots, x_n) \in P$, then 
$\phi(x) \wedge e_j = (\bigvee_{i \in I} x_i) \wedge e_j = x_j$. We now see that $\theta \circ \phi(x) = x$, and so 
$\theta$ and $\phi$ are inverse homomorphisms. We have shown:

**Lemma 5.1.** Suppose $a, b \in M$ and that $e_1, \ldots, e_n \in [a, b]$, where $(e_i)_i$ are independent and span $[a, b]$. Then $[a, b]$ is naturally isomorphic to the product median algebra $\prod_{i \in I} [a, e_i]$. 

In particular, this gives a monomorphism $\phi : \prod_{i \in I} [a, e_i] \rightarrow [a, b]$, with image $Q = \{ e_J \mid J \subseteq I \}$. Thus, $Q$ is an $n$-cube in $[a, b] \subseteq M$. Intrinsically, $Q$ is the interval $[a, b] \cap Q$. In other words, $a, b$ is a diagonal of the cube. Note that $e_J \land e_K = e_{J \land K}$ and $e_J \lor e_K = e_{J \lor K}$ for $J, K \subseteq I$. 

Suppose that $e_i = c \lor d$, with $c \land d = a$, for some $c, d \in [a, b] \setminus \{a\}$. Then $a, c, e_i, d$ is a square, and we could replace $e_i$ by $c, d$ to obtain a larger independent spanning set for $[a, b]$. With this in mind, we say that $(e_i)_i$ is maximal if no $\{a, e_i\}$ is the diagonal of a square.

**Definition.** A basis for $[a, b]$ is a maximal independent spanning set for $[a, b]$.

**Lemma 5.2.** Suppose that $e_1, \ldots, e_n$ is a basis for $[a, b]$, and that $e'_1, \ldots, e'_m$ is an independent spanning set. Then we can partition $I$ into non-empty subsets as $I = I_1 \uplus \cdots \uplus I_m$ such that for all $i$ we have 
$e'_i = e_i \wedge \bigvee_{j \in I_i} e_j$.

**Proof.** Given $j \in I$, we have $e_j = \bigvee_{i=1}^m (e_j \wedge e'_i)$. In particular, there must be some $i \in \{1, \ldots, m\}$ with $e_j \wedge e'_i \neq a$. Note that $e_j = c \lor d$ and $a = c \land d$, where $c = e_j \wedge e'_i$ and $d = \bigvee_{k \neq i} (e_j \wedge e'_k)$. Since $c \neq a$, we must have $d = a$, and so $e_j \wedge e'_k = a$ for all $k \neq i$. It now also follows that $e_j = c$. In other words, for all $j \in I$, there is a unique $i(j) \in \{1, \ldots, m\}$ with $e_j \wedge e'_{i(j)} = e_j$ and with $e_j \wedge e'_i = a$ whenever $i \neq i(j)$. 

Given $i \in \{1, \ldots, m\}$, let $I_i = \{ j \in I \mid i(j) = i \}$. Clearly $I_i \land I_k = \emptyset$ if $i \neq k$. Moreover, if $i \in I$, then since $e_j \wedge e'_i = a$ for all $j \notin I_i$, we have 
$e'_i = \bigvee_{j \in I_i} (e_j \wedge e'_i) = \bigvee_{j \in I_i} (e_j \wedge e'_i) = e_i$.

Finally note that $b = \bigvee_{i=1}^m e'_i = \bigvee_{i=1}^m e_{I_i} = e_{I_1} \uplus \cdots \uplus e_{I_m}$. It follows that $I = \bigcup_{i=1}^m I_i$, for if $k \in I \setminus \bigcup_{i=1}^m I_i$, we would get the contradiction that $e_k = b \land e_k = e_{i(k)} \cap I_i = e_b = a$. \hfill \qed 

**Corollary 5.3.** Any two bases for $[a, b]$ agree up to permutation.

**Proof.** Let $(e_i)_{i=1}^n$ and $(e'_i)_{i=1}^m$ be bases. By Lemma 5.2, we see that $m = n$, and that each $I_i$ is a singleton. \hfill \qed 

Suppose now that $a, b \in M$, and that $Q \subseteq [a, b]$ is a cube with $a, b \in Q$, that is $a, b$ is a diagonal of $Q$. If $e_1, \ldots, e_n \in Q$ are the points adjacent to $a$ in $Q$ (that is the sets $\{a, e_i\}$ are the edges of $Q$
containing \(a\), then \((e_i)_i\) is an independent spanning set of \([a, b]\) in \(M\). Conversely, we have already observed that if \(e_1, \ldots, e_n\) are independent and span \(a, b\), then \(Q\) is isomorphic to \(\prod_{i=1}^n \{a, e_i\}\). We see that the cube \(a, b\) is not the diagonal of a strictly larger cube. (In other words, if \(Q \subseteq Q'\) where \(Q'\) is a cube containing \(a, b\), then \(Q = Q'\).) This is the same as saying that \((e_i)_i\) is a basis for \([a, b]\). As a consequence of Corollary 5.3, we have:

**Lemma 5.4.** Given \(a, b \in M\), there is at most one maximal cube in \(M\) with diagonal \(a, b\).

If \(M\) has finite rank, then such a cube must always exist — take a cube of maximal rank with diagonal \(a, b\).

If we start with a cube, \(Q\), then \(\text{hull}_M(Q) = [a, b]\) where \(a, b\) is any diagonal of \(Q\). We can associate to each wall \(W \in \mathcal{W}(Q)\), a median algebra \(\Phi_W\), well defined up to isomorphism (and inclusion into \(M\) up to the relation of parallelism). In fact, we can let \(\Phi_W = [a, e_i] = \text{hull}\{a, e_i\}\), where \(\{e_1, \ldots, e_n\}\) is the basis for \(Q = [a, b]\). From the above, we see:

**Lemma 5.5.** The convex hull, \(\text{hull}(Q)\), of the cube \(Q \subseteq M\) is naturally isomorphic to the product median algebra \(\prod_{W \in \mathcal{W}(Q)} \Phi_W\).

Although it is not needed for the proof of the main theorem, it is worth noting that the convex hulls of cubes of maximal dimension are isomorphic to real cubes.

To this end, note that an interval \([c, d] \subseteq M\) has (intrinsic) rank 1 if and only if the partial order on \([c, d]\) is a total order. In this case, we will say that the interval \([c, d]\) is linear. If \(c, d \in [a, b]\), then \([c, d] \subseteq [a, b]\). Moreover, if \(c = a\) then the order on \([a, d]\) is precisely the restriction of the partial order on \([a, b]\).

Suppose that \((e_i)_i\) is an independent spanning set of \([a, b]\) and that \([a, e_i]\) is linear for each \(i\). Clearly, this implies that \((e_i)_i\) is a basis.

In fact, suppose \(\text{rank} M = n < \infty\), and \(e_1, \ldots, e_n\) in an independent basis for \([a, b]\), then \([a, e_i]\) is linear for each \(i\). For if not, we can find a square \(S \subseteq [a, e_i]\) for some \(i\). Using Lemma 5.1, we see that \(S \times \prod_{j \neq i} \{a, e_j\}\) is a cube of dimension \(n + 1\), giving a contradiction.

Putting the above facts together we conclude:

**Lemma 5.6.** Suppose that \(M\) is a connected metrisable topological median algebra. Suppose that \(Q \subseteq M\) is a cube in \(M\) whose dimension equals \(\text{rank}(M) < \infty\). Then \(\text{hull}(Q)\) is isomorphic to a real \(n\)-cube.

**Proof.** This follows from the topological characterisation of a real interval as a connected metrisable space with exactly two non-cut points. \(\square\)
Note that if $M$ is a connected median metric space, then this cube will be isometric to a compact real $n$-cube with an $l^1$-metric. (We have already noted that a median metric space is a topological median space, by Lemma 4.1.)

6. Finite subalgebras of median algebras

In this Section, we show how to associate, to any finite subalgebra, $\Pi$, of any median algebra, $M$, a larger subalgebra, $M(\Pi) \supseteq \Pi$, which has a kind of “cellular” structure, where the “cells” are product median spaces in bijective correspondence with the faces of the complex $\Pi$ (or equivalently, the cells of $\Upsilon(\Pi)$). This construction will be used in the proof of Lemma 7.5.

First we make the following definition. Suppose that $A, B \subseteq M$. A parallel map between $A$ and $B$ is a bijection $\psi : A \rightarrow B$ such that, for all $x, y \in A$, $\psi x, \psi y$ is parallel to $x, y$. Clearly its inverse is also parallel. It is not hard to see that, if $A$ is a subalgebra, then so is $B$, and $\psi$ is a median isomorphism between them (though this will be clear in the case of interest here). In particular, if $a, b, d, c$ is a square in $M$, then the projection of $M$ to $[a, b]$ defined by $\phi(x) = \mu(a, b, x)$ restricted to $[c, d]$ is a parallel map from $[c, d]$ to $[a, b]$. Moreover, the projection of $M$ to $[c, d]$ composed with $\psi|_{[c, d]}$ is also equal to $\psi$.

Suppose that $\Pi \subseteq M$ is a finite subalgebra, and let $F(\Pi)$ be the set of faces as described in Section 3. Let $M(\Pi) = \bigcup_{s \in F(\Pi)} \hull(Q(s))$. We claim:

**Lemma 6.1.** $M(\Pi)$ is a subalgebra of $M$.

In fact, we can describe the structure of $M(\Pi)$ in terms of the construction of Section 5. Given $W \in W(\Pi)$, let $\{p_W^-, p_W^+\}$ be an edge of $\Pi$ which crosses $W$. Let $\Phi_W = [p_W^-, p_W^+]$. Note that $\Phi_W$ is well defined up to a parallel map in $M$, and that there is a well defined projection, $\theta_W : M \rightarrow \Phi_W$ (namely, $\theta_W(x) = \mu(p_W^-, p_W^+, x)$). Note that $\theta_W$ is an epimorphism. Let $\Phi$ be the subalgebra of $P = \prod_{W \in W(\Pi)} \Phi_W$, as defined in Section 3.

We also get a natural map $\phi : \Phi \rightarrow M$ as follows. Note that if $s \in F(\Pi)$, then by Lemma 5.5, there is a natural isomorphism, $\phi_s : P(s) \rightarrow \hull(Q(s))$. In fact, this is naturally isomorphic to $\prod_{W \in W_s} \Phi_W$, where $W_s = W(Q(s)) \subseteq W$ is the set of walls crossing $s$. Note that if $t \preceq s$, then $P(t) \subseteq P(s)$ and $\phi_t = \phi_s|P(s)$. Assembling these, we get a map $\phi : \Phi \rightarrow M$. By construction, $\phi(\Phi) = M(\Pi)$. We claim:

**Lemma 6.2.** $\phi : \Phi \rightarrow M$ is a monomorphism.
Clearly this implies Lemma 6.1.

Lemma 6.2 will follow from the following observation. Note that if $W \in \mathcal{W}(\Pi)$, we have a projection, $\psi_W : P \to \Phi_W$, namely the median projection (defined as for $M$), or equivalently, simply the co-ordinate projection to the factor $\Phi_W$ of $P$. We claim:

**Lemma 6.3.** If $W \in \mathcal{W}(\Pi)$, then $\psi_W = \theta_W \circ \phi$.

**Proof.** Let $x \in \Phi$, and set $s = h(x) \in F(\Pi)$, where $h : \Phi \to F(\Pi)$ is the map defined in Section 3. We distinguish two cases.

Suppose $W \in \mathcal{W}_s$. In this case, the statement follows from the fact that $\phi_s : P(s) \to \text{hull}(Q(s))$ is an isomorphism.

Suppose that $W \notin \mathcal{W}_s$. In this case, without loss if generality, we have $Q(s) \subseteq H^-(W) \cap \Pi$. Let $\{p^-_W, p^+_W\}$ be an edge of $\Pi$ crossing $W$ with $p^+_W \in H^+(W)$. Since $\Pi$ is a subalgebra of $M$, we see that $\theta_W(Q(s)) = \{p^-_W\}$ (since the projection of $Q(s)$ to $\{p^-_W, p^+_W\}$ in $\Pi$ is $\{p^-_W\}$). Now $\phi(x) = \phi_s(x) \in \text{hull}(Q(s))$, by the construction of $\phi$, and so $\theta_W(\phi(x)) = p^-_W$. (This follows from the fact that $\theta_W$ is a monomorphism.) But, by the construction of $\Phi \subseteq P$, we also have $\psi_W(x) = p^-_W$, so the result follows.

**Proof of Lemma 6.2.** First, to see that $\phi$ is injective, suppose that $a, b \in \Phi$ with $\phi a = \phi b$. By Lemma 6.3, we get $\psi_W a = \psi_W b$ for all $W \in \mathcal{W}$, so $a = b$.

To see that $\phi$ is a homomorphism, suppose that $a, b, c \in \Phi$ with $c \in [a, b]_\phi$. Suppose, for contradiction, that $\phi c \notin [\phi a, \phi b]_M$. Let $W_0 \in \mathcal{W}(M)$ be a wall of $M$ separating $\phi c$ from $\mu_M(\phi a, \phi b, \phi c)$. Without loss of generality, we have $\phi a, \phi b \in H^-(W_0)$ and $\phi c \in H^+(W_0)$.

Note that $Q(f(a)) \cap H^-(W_0) \neq \emptyset$, and $Q(f(c)) \cap H^+(W_0) \neq \emptyset$. In particular, $H^+(W) \neq \emptyset$, where $H^+(W) = H^+(W_0) \cap \Pi$, and so $W = \{H^-(W), H^+(W)\}$ is a wall of $\Pi$. Let $\Phi_W = [p^-_W, p^+_W]$, with $p^+_W \in H^+(W)$, and with $\{p^-_W, p^+_W\}$ an edge of $\Pi$. Now, $\psi_W a = \theta_W \phi a \in [p^-_W, \phi a] \subseteq H^-(W_0)$, so $\psi_W a \in H^-(W_0)$. Similarly, $\psi_W b \in H^-(W_0)$ and $\psi_W c \in H^+(W_0)$. But $\psi_W$ is projection to an interval, hence a homomorphism, so $\psi_W c \in \psi_W([a, b]) \subseteq [\psi_W a, \psi_W b] \subseteq H^-(W_0)$, giving a contradiction.

**7. CUBES IN MEDIAN METRIC SPACES**

In this section, we will describe how to define a CAT(0) metric associated to a median metric space of finite rank.

Let $(M, \rho)$ be a median metric space. Suppose that $Q \subseteq M$ is an $m$-cube with edge-lengths $t_1, \ldots, t_m$. Set $\omega(Q) = \sqrt{\sum_i t_i^2}$. Note that, if $a, b$ is a diagonal of $Q$, then $\rho(a, b) = \sum_i t_i$, and so $\rho(a, b)/\sqrt{m} \leq \omega(Q)$. Suppose that $W \in \mathcal{W}(\Pi)$, we have a projection, $\psi_W : P \to \Phi_W$, namely the median projection (defined as for $M$), or equivalently, simply the co-ordinate projection to the factor $\Phi_W$ of $P$. We claim:
\( \omega(Q) \leq \rho(a, b) \). Suppose that \( Q' \subseteq Q \) is a cube in \( Q \) containing \( a, b \) (that is, a subalgebra isomorphic to a cube.) Its edge-lengths have the form \( \sum_{j \in I_i} t_j \), where \( I = \bigcup_i I_i \) is a partition of \( I = \{1, \ldots, m\} \). It follows that \( \omega(Q') \geq \omega(Q) \).

Suppose that rank \( M = n < \infty \). Given \( a, b \in M \), write \( Q(a, b) \subseteq M \) for the unique maximal cube in \( M \) with diagonal \( a, b \). We write \( \omega(a, b) = \omega(Q(a, b)) \). Thus, \( \rho(a, b)/\sqrt{n} \leq \omega(a, b) \leq \rho(a, b) \). Moreover, if \( Q' \subseteq M \) is any other cube with diagonal \( a, b \), then Lemma 5.2 tells us that \( Q' \) is a subcube of \( Q \), and so \( \omega(Q') \geq \omega(Q) \).

Suppose \( a, b, c, d \in M \), and \( Q \) is a cube with diagonal \( a, b \). Then \( \pi(Q) \) is a cube of diagonal \( \pi(a), \pi(b) \), where \( \pi \) is the projection map \( x \mapsto \mu(a, b, x) \). Since \( \pi \) is 1-lipschitz, the edge-lengths of \( \pi Q \) are at most those of \( Q \) and so \( \omega(\pi Q) \leq \omega(Q) \). By taking \( Q \) to be maximal, and applying the above, we see that \( \omega(\pi a, \pi b) \leq \omega(a, b) \).

We now define a metric \( \sigma = \sigma_{\rho} \) on \( M \) as follows. If \( \underline{a} = a_0, a_1, \ldots, a_p \) is a sequence in \( M \), we write \( \omega(\underline{a}) = \sum_{i=1}^{\rho} \omega(a_{i-1}, a_i) \). Given \( a, b \in M \), let \( \sigma(a, b) = \inf \omega(\underline{a}) \) as \( \underline{a} \) ranges over all sequences (of any finite length) with \( a_0 = a \) and \( a_p = b \).

**Lemma 7.1.** \( \sigma(a, b) = \inf \omega(\underline{a}) \) as \( \underline{a} \) ranges over all monotone sequences from \( a \) to \( b \).

**Proof.** Let \( \underline{c} = c_0, \ldots, c_p \) be any sequence with \( a = c_0 \) and \( c_p = b \). First, project \( \underline{c} \) to \([a, b]\) to give another sequence, \( \underline{d} \), from \( a \) to \( b \); that is, set \( d_i = \mu(a, b, c_i) \). From the above observation, we have \( \omega(\underline{d}) \leq \omega(\underline{c}) \). Now project to \( d_1, \ldots, d_p = b \) to give a sequence \( e_1, \ldots, e_p \) from \( d_1 \) to \( b \). Project \( e_2, \ldots, e_p \) to \([e_2, b]\) to give \( f_2, \ldots, f_p \), etc. After \( p \) steps, we arrive at a monotone sequence, \( \underline{a} = a, d_1, e_2, f_3, \ldots, b \), with \( \omega(\underline{a}) \leq \omega(\underline{c}) \).

If \( \underline{a} \) is monotone, then \( \rho(a, b) = \sum_{i=1}^{\rho} \rho(a_{i-1}, a_i) \), and so it follows that \( \rho(a, b)/\sqrt{n} \leq \sigma(a, b) \leq \rho(a, b) \).

From the above, we now see easily that \( \sigma \) is a metric on \( M \).

**Lemma 7.2.** Suppose \( a, b \in M \) and \( c \in [a, b] \), then \( \sigma_{\rho}(a, c)^2 + \sigma_{\rho}(b, c)^2 \leq \sigma_{\rho}(a, b)^2 \).

**Proof.** Let \( \delta > 0 \), and let \( a = d_0, d_1, \ldots, d_p = b \) be a monotone sequence with \( \sum_i \omega(d_{i-1}, d_i) \leq \sigma_{\rho}(a, b) + \delta \). Let \( Q_i = Q(d_{i-1}, d_i) \) be the maximal cube with diagonal \( d_{i-1}, d_i \). Let \( w_i = \omega(Q_i) \). Let \( d_i^-, \mu(a, c, d_i) \) and \( d_i^+ = \mu(c, b, d_i) \). Thus \( a = d_0^-, d_1^-, \ldots, d_p^+ = c \) and \( c = d_0^+, d_1^+, \ldots, d_p^+ = b \) are monotone sequences. Let \( Q_i^- \) and \( Q_i^+ \) be the projections of \( Q_i \) to \([a, c]\) and \([c, b]\) respectively. These are also cubes with diagonals \( d_i^-, d_i^+ \) and \( d_i^-, d_i^+ \) respectively. Let \( \omega_i^- = \omega(Q_i^-) \). Then \( \sigma_{\rho}(a, c) \leq \sum_i \omega_i^- \) and \( \sigma_{\rho}(c, b) \leq \sum_i \omega_i^+ \). Let \( c_i = \mu(c, d_{i-1}, d_i) \). Now \( d_{i-1}, c_i \) is
parallel to $d_{i-1}, d_i^-$. (In the notation of Section 5, note that $d_{i-1} \leq d_i$, $c_i = (c \lor d_{i-1}) \land d_i = (c \land d_i) \lor d_{i-1}$, $d_i^- = c \land d_i$ and $d_i^+ = c \lor d_i$.

Now $d_i^- \land d_{i-1} = (c \land d_i) \land d_{i-1} = d_{i-1}^- = d_i^- \land d_{i-1}$ and $d_i^+ \lor d_{i-1} = c_i$, and so also, $d_{i-1}^- \leq d_i \leq c_i$ and $d_i^+ \leq d_{i-1} \leq c_i$. It follows that $d_{i-1}, c_i, d_i^-, d_i^+$ is a square.) We see that $d_{i-1}, c_i$ is the diagonal of a (not necessarily maximal) cube parallel to $Q_i^-$. Similarly, $c_i, d_i$ is the diagonal of a cube parallel to $Q_i^+$. Note that $Q_i^-$ and $Q_i^+$ are the projections of $Q_i$ respectively to $[d_{i-1}^+, c]$ and to $[c, d_i^+]$. We therefore see that, $(\omega_i^-)^2 + (\omega_i^+)^2 \leq \omega_i^2$. Now $(\sum_i \omega_i^-)^2 + (\sum_i \omega_i^+)^2 \leq (\sum_i \omega_i)^2$. (Note that $(\omega_i^- \omega_j^- + \omega_i^+ \omega_j^+)^2 \leq ((\omega_i^-)^2 + (\omega_i^+)^2)((\omega_j^-)^2 + (\omega_j^+)^2) \leq \omega_i^2 \omega_j^2$, so $\omega_i^- \omega_j^- + \omega_i^+ \omega_j^+ \leq \omega_i \omega_j$, and the inequality follows on expanding both sides.) We get that $\sigma_\rho(a, c)^2 + \sigma_\rho(b, c)^2 \leq (\sigma_\rho(a, b) + \delta)^2$. Since this holds for all $\delta > 0$, the statement follows.

\begin{corollary}
For all $a, b \in M$, $I_{\sigma_\rho}(a, b) \subseteq I_\rho(a, b)$.
\end{corollary}

\begin{proof}
Suppose $x \in I_{\sigma_\rho}(a, b)$. Applying Lemma 7.2 with $c = \mu(a, b, x)$ we deduce that $\sigma_\rho(x, c) = 0$, so $x = c \in I_\rho(a, b)$ as required.
\end{proof}

As an example of the above construction, if $\rho$ is the $l^1$-metric on $\mathbb{R}^n$, then cubes are the vertex sets of rectilinear parallelopipeds, and we see that $\sigma$ is the euclidean metric on $\mathbb{R}^n$. More generally, if $(\Upsilon, \rho)$ is a CAT(0) cube complex, and $\lambda : \mathcal{W}(\Upsilon) \to (0, \infty)$, we get a median metric on $\Upsilon$ as discussed in Section 2. In this case, $\sigma_\rho$ coincides with the corresponding euclidean structure, $\sigma$, on $\Upsilon$. (First note that one easily sees that $\sigma_\rho$ and $\sigma$ agree on each cell of $\Upsilon$. From the fact that $(\Upsilon, \sigma)$ is geodesic, we can deduce that $\sigma_\rho \leq \sigma$, and directly from the definition of $\sigma_\rho$, we see that $\sigma \leq \sigma_\rho$.)

Now, as discussed in Section 2, any finite median metric space arises as the vertex set of such a complex. We therefore get:

\begin{lemma}
Suppose that $(\Pi, \rho)$ is a finite median metric space. Then we can canonically identify $\Pi$ as the vertex set of a CAT(0) cube complex, $\Upsilon(\Pi)$, admitting a CAT(0) metric $\sigma_\Pi$. Moreover, $\Upsilon(\Pi)$ also canonically admits a median metric $\rho_\Pi$, which agrees with $\rho$ on $\Pi$, and is such that $\sigma_\Pi = \sigma_{\rho_\Pi}$. In fact, any cell, $P$, of $\Upsilon(\Pi)$, of any dimension $n$, can be embedded into $\mathbb{R}^n$ as a rectilinear parallelopiped in such a way that $\rho_\Pi$ and $\sigma_\Pi$ on $P$ respectively agree with the usual $l^1$-metric and the euclidean metric induced from $\mathbb{R}^n$.
\end{lemma}

Now suppose that $(M, \rho)$ is a median metric space. Given any finite subalgebra $\Pi \subseteq M$, the metric restricted to $\Pi$ is an intrinsic median metric, and so we can construct $\Upsilon(\Pi)$, as in Lemma 7.4.
For the rest of this section, we will be assuming that \((M, \rho)\) is a geodesic space. Note that Corollary 7.3 implies that any \(\sigma_-\rho\text{-geodesic in } M\) can be reparameterised to give a \(\rho\text{-geodesic.}

**Lemma 7.5.** If \((M, \rho)\) is a geodesic median metric space, and \(\Pi \subseteq M\) a finite subalgebra, then there is a median monomorphism \(f : \Upsilon(\Pi) \to M\) extending the inclusion of \(\Pi\) into \(M\). Moreover, \(f\) is an isometric embedding as a map \((\Upsilon(\Pi), \rho_{\Pi}) \to (M, \rho)\). Also, if \(M\) has finite rank, then \(f\) is 1-lipschitz as a map \((\Upsilon(\Pi), \sigma_{\Pi}) \to (M, \sigma_{\rho})\).

Here, in general, \(f\) is not canonically defined. Note that, in the last clause, we need that \(M\) is finite rank just so that \(\sigma_{\rho}\) is defined.

**Proof.** We begin by describing how to construct \(f\). If \(Q \subseteq \Pi \subseteq M\) is a face of \(\Pi\), we can choose a diagonal, \(a, b\), and let \(e_1, \ldots, e_n\) be the adjacent vertices to \(a\) in \(Q\). By Lemma 5.1, \(\text{hull}_M(Q)\) is isomorphic as a median algebra to the direct product \(\prod_i [a, e_i]\). Now the cell, \(P_Q\), of \(\Upsilon(\Pi)\) is isometric to \(\prod_i [0, r_i] \subseteq \mathbb{R}^n\) in the \(l^1\) metric, where \(r_i = \rho(a, e_i)\). Since \(M\) is a geodesic space, we can find a geodesic \(\gamma_i : [0, r_i] \to [a, e_i]\), with \(\gamma_i(0) = 0\), and \(\gamma_i(r_i) = e_i\). Combining these, we get a distance-preserving map \(f_Q : P_Q \to \text{hull}_M(Q)\). In fact, we can assume that all the paths of the form \(\gamma_i\) crossing any given wall of \(\Pi\) are parallel, and so the maps \(f_Q\) fit together to give a map \(f : \Upsilon(\Pi) \to M\). We need to check that this is a monomorphism. To do this, we start again with a more formal description of \(f\) in terms of the constructions of Sections 5 and 6.

By Lemma 6.2, there is an isomorphism, \(\phi : \Phi \to M(\Pi)\), where \(\Phi\) is the subalgebra of the product \(P = \prod_{W \in W} \Phi_W\), as described in Section 5, and where \(M(\Pi)\) is the subalgebra of \(M\) described in Section 5. We can write \(\Phi_W = [p_W^+, p_W^-]\), and set \(\delta_W : [0, \lambda(W)] \to \Phi_W\) to be a \(\rho\text{-geodesic from } p_W^+\) to \(p_W^-\). Doing this on every co-ordinate, we get a product map \(\prod_{W \in W} [0, \lambda(W)] \to P\). Restricting to \(\Upsilon(\Pi)\), we get a median homomorphism, \(\delta : \Upsilon(\Pi) \to \Pi\). Let \(f = \phi \circ \delta : \Upsilon(\Pi) \to M(\Pi)\). From this description, it is clear that \(f\) is a median monomorphism. This in turn implies that \(f\) is an isometric embedding from \((\Upsilon(\Pi), \rho_{\Pi})\) to \((M, \rho)\).

Finally, to see that \(f : (\Upsilon(\Pi), \sigma_{\Pi}) \to (M, \sigma_{\rho})\) is 1-lipschitz, let \(a, b \in \Pi\) and let \(a = a_0, a_1, \ldots, a_p = b\) now be a sequence of points, in order, along the geodesic from \(a\) to \(b\) in \((\Upsilon(\Pi), \sigma_{\Pi})\), and with \(a_{i-1}, a_i\) lying in some cell of \(\Upsilon(\Pi)\) for all \(i\). (This is also a monotone sequence.) Let \(Q_i\) be the maximal cube in \(\Upsilon(\Pi)\) with diagonal \(a_{i-1}, a_i\). Now \(f(Q_i)\) is a cube in \(M\) with diagonal \(f(a_{i-1}), f(a_i)\), so by the earlier observation, and the fact that \(f\) is a median monomorphism,
we have $\omega(f(a_{i-1}), f(a_i)) \leq \omega(f(Q_i)) = \omega(Q_i) = \sigma_1(a, b)$. Thus, $\sigma_\rho(f(a), f(b)) \leq \sum_{i=1}^n \omega(f(a_{i-1}), f(a_i)) \leq \sum_{i=1}^n \sigma_\rho(a_{i-1}, a_i) = \sigma_\rho(a, b)$. \hfill \Box

**Lemma 7.6.** Suppose that $M$ is a geodesic median metric space of finite rank. Suppose that $A \subseteq M$ is any finite subset, and $\delta > 0$. Then there is a finite subalgebra, $\Pi \subseteq M$ with $A \subseteq \Pi$, such that if $(\Upsilon(\Pi), \sigma_\Pi)$ is the CAT(0) cube complex with vertex set $\Pi$ as given by Lemma 7.4, then $\sigma_\Pi(a, b) \leq \sigma(a, b) + \delta$ for all $a, b \in A$.

**Proof.** For each pair, $a, b \in A$, let $a = a_0, a_1, \ldots, a_p = b$ be a monotone sequence with $\sum_{i=1}^p \omega(a_{i-1}, a_i) \leq \sigma(a, b) + \delta$. Let $Q(a_{i-1}, a_i)$ be the maximal cube in $M$ with diagonal $a_{i-1}, a_i$. (Thus, $\omega(a_{i-1}, a_i) = \omega(Q(a_{i-1}, a_i))$.) Let $B(a, b) = \bigcup_{i=1}^p Q(a_{i-1}, a_i)$, and let $B = \bigcup_{a, b \in A} B(a, b)$. Let $\Pi \subseteq M$ be a finite subalgebra containing $B$.

Now suppose that $a, b \in A$. Let $a = a_0, a_1, \ldots, a_p = b$ be as above. Now $Q(a_{i-1}, a_i) \subseteq \Pi$, and $\rho_\Pi$ agrees with $\rho$ on $\Pi$. Since $\sigma_\Pi = \sigma_\rho|_\Pi$, we have $\sigma_\Pi(a_{i-1}, a_i) \leq \omega(Q(a_{i-1}, a_i)) = \omega(a_{i-1}, a_i)$, and so $\sigma_\Pi(a, b) \leq \sum_{i=1}^p \sigma_\Pi(a_{i-1}, a_i) \leq \sum_{i=1}^p \omega(a_{i-1}, a_i) \leq \sigma(a, b) + \delta$. \hfill \Box

Putting Lemmas 7.4, 7.5 and 7.6 together, we have shown:

**Lemma 7.7.** Suppose that $(M, \rho)$ is a geodesic median metric space of finite rank. Given any finite $A \subseteq M$, and any $\delta > 0$, there is a compact CAT(0) space $(\Upsilon, \sigma_\Upsilon)$ with $A \subseteq \Upsilon$, and a 1-lipschitz map, $f : (\Upsilon, \sigma_\Upsilon) \to (M, \sigma_\rho)$ extending the inclusion of $A$ into $M$, such that for all $a, b \in A$, we have $\sigma_\Upsilon(a, b) \leq \sigma_\rho(a, b) + \delta$.

8. CAT(0) spaces

Let $(M, \sigma)$ be a geodesic metric space. Suppose $\delta \geq 0$. By a $\delta$-kite in $M$ we mean an ordered quadruple of points, $K = \{a, b, c, d\}$ with $\rho(a, d) + \rho(d, b) \leq \rho(a, b) + \delta$. Given $\epsilon \geq 0$, an $\epsilon$-comparison of $K$ is a map $\xi : K \to \mathbb{R}^2$, into the plane with euclidean metric $\sigma_0$ such that $|\sigma_0(\xi(x), \xi(y)) - \sigma(x, y)| \leq \epsilon$, whenever $x, y \in K$ and $\{x, y\} \neq \{c, d\}$. Note that $\xi(K)$ is then a $(\delta + 3\epsilon)$-kite in $\mathbb{R}^2$.

If $K$ is a $0$-kite, then we can always find a $0$-comparison, $\xi$, of $K$, and $\xi(K)$ is then a $0$-kite. Moreover, the image is uniquely determined up to isometry of $\mathbb{R}^2$. In this case, we can speak of “the” $0$-comparison to $\mathbb{R}^2$. In this terminology, a standard definition of a CAT(0) space is as follows:

**Definition.** A CAT(0) space is a geodesic metric space, $(M, \sigma)$, with the property that if $a, b, c, d$ is a $0$-kite in $M$, and $\xi$ is the $0$-comparison of $a, b, c, d$ in $\mathbb{R}^2$, then $\sigma(c, d) \leq \sigma_0(\xi(c), \xi(d))$. 
Lemma 8.1. Given \( \eta, r > 0 \), there is some \( \delta \geq 0 \) with the property that if \( (M, \sigma) \) is any \( \text{CAT}(0) \) space, \( K = \{a, b, c, d\} \subseteq M \) is any \( \delta \)-kite in \( M \) of diameter at most \( r \), and \( \xi: K \to \mathbb{R}^2 \) is any \( \delta \)-comparison of \( K \), then \( \sigma(c,d) \leq \sigma_0(\xi(c),\xi(d)) + \eta \).

We can now prove the following:

Theorem 8.2. Suppose that \( (M, \rho) \) is a median metric space of finite rank, and that \( (M, \sigma_\rho) \) is geodesic. Then \( (M, \sigma_\rho) \) is \( \text{CAT}(0) \).

Proof. First note that, using Corollary 6.2, any \( \sigma_\rho \)-geodesic can be reparameterised as a \( \rho \)-geodesic, and so \( (M, \rho) \) is also a geodesic space.

Suppose that \( K = \{a, b, c, d\} \subseteq M \) is a 0-kite (in the metric \( \sigma_\rho \)). Let \( \xi: K \to \mathbb{R}^2 \) be a 0-comparison. Let \( r = \text{diam} K \). Given any \( \eta > 0 \), let \( \delta \) be as given by Lemma 8.1. Let \( f: (\Upsilon, \sigma_\Upsilon) \to (M, \sigma_\rho) \) be as given by Lemma 7.7 with \( A = K \) and \( \delta \) as given. In particular, for each \( x, y \in K \), we have \( \sigma_\rho(x, y) \leq \sigma_\Upsilon(x, y) + \delta \). Therefore \( K \) is a \( \delta \)-kite in \( (\Upsilon, \sigma_\Upsilon) \) and \( \xi \) is a \( \delta \)-comparison with respect to the metric \( \sigma_\Upsilon \). Since \( (\Upsilon, \sigma_\Upsilon) \) is \( \text{CAT}(0) \), it follows from Lemma 8.1 that \( \sigma_\Upsilon(c,d) \leq \sigma_0(\xi(c),\xi(d)) + \eta \). It follows that \( \sigma_\rho(c,d) \leq \sigma_0(\xi(c),\xi(d)) + \eta \). Since this holds for all \( \eta > 0 \), and \( \xi \) is fixed, we have \( \sigma_\rho(c,d) \leq \sigma_0(\xi(c),\xi(d)) \). Thus, by definition, \( (M, \sigma_\rho) \) is \( \text{CAT}(0) \) as claimed. \( \square \)

If \( M \) is complete, then it is enough to assume that it is connected:

Lemma 8.3. Let \( (M, \rho) \) be a complete connected median metric space of finite rank, then \( (M, \sigma_\rho) \) is a geodesic space.

Proof. Note that, by Lemma 4.6, we already know that \( (M, \rho) \) is geodesic. Since the metrics \( \rho \) and \( \sigma_\rho \) are bilipschitz equivalent, \( (M, \sigma_\rho) \) is also complete. Therefore, it is enough to prove the existence of midpoints in \( (M, \sigma_\rho) \).

Let \( a, b \in M \). Given \( \delta > 0 \), we first claim that there is some \( c \in M \) with \( \sigma_\rho(a,c), \sigma_\rho(b,c) \leq \frac{1}{2}(\sigma_\rho(a, b) + \delta) \). To this end, let \( f: (\Upsilon, \sigma_\Upsilon) \to (M, \sigma_\rho) \) be the map given by Lemma 7.7, with \( A = \{a, b\} \). Let \( x \) be a midpoint of \( a, b \) in \( (\Upsilon, \sigma_\Upsilon) \), and let \( c = f(x) \). Now, \( \sigma_\Upsilon(a, c) \leq \sigma_\Upsilon(a, x) = \frac{1}{2} \sigma_\Upsilon(a, b) \leq \frac{1}{2}(\sigma_\rho(a, b) + \delta) \). Similarly, \( \sigma_\rho(b,c) \leq \frac{1}{2}(\sigma_\rho(a, b) + \delta) \) as claimed.

Suppose \( d \) is another such point. We can view \( K = \{a, b, c, d\} \) as a \( \delta \)-kite in \( M \). We have a \( \delta \)-comparison \( \xi: K \to \mathbb{R}^2 \), such that \( \xi(c) = \xi(d) \) is the midpoint of \( \xi(a), \xi(b) \). Therefore, if \( \eta > 0 \), then by choosing \( \delta > 0 \) sufficiently small depending on \( \eta, r \), Lemma 8.1 tells us that \( \sigma_\rho(c,d) \leq \eta \).
In this way we obtain a sequence of points \((c_i)_i\) in \(M\) with \(\sigma_\rho(a,c_i) \to \frac{1}{2}\sigma_\rho(a,b)\) and \(\sigma_\rho(b,c_i) \to \frac{1}{2}\sigma_\rho(a,b)\), and with \((c_i)_i\) Cauchy. Thus \(c_i\) converges to a midpoint of \(a,b\) in \((M,\sigma_\rho)\) as required. \(\square\)

It now follows by Theorem 8.2 that, under these assumptions, \((M,\sigma_\rho)\) is CAT(0). This proves Theorem 1.1.

References

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