An upper bound for injectivity radii in convex cores

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0. Introduction.

Let V be a complete hyperbolic 3-manifold with finitely generated fundamental group. By tameness [Bo,A,CalG], it is known that V is homeomorphic to the interior of a compact manifold, M. Let H be the compact core of M. (This is the minimal submanifold with locally convex boundary whose inclusion into V is a homotopy equivalence.)

The main result will be:

Theorem 0 : *There is some $r \geq 0$, depending only on the topology of M, such that any embedded hyperbolic ball in H has radius at most r.*

The question has been considered by a number of authors. The statement is proven in [Can] for product manifolds, and in [F1,F2] for books of I-bundles and acylindrical manifolds, and in [E] in the general irreducible case. A proof in the general (tame) case (different from that presented here) was worked out by Kleineidam and Souto, though never written down in detail.

Theorem 0 partially answers a question of McMullen (posed before the Tameness Theorem was proven in general) in the problem list [Bi]. That question asks whether one can find a radius bound that depends only on the minimal number of generators of $\pi_1(M)$.

In fact we prove a variation (Theorem 1.1) which is easily seen to imply Theorem 0. This says that any curve in H either lies in a small compressing disc in H, or else lies in an essential curve in H of bounded length. Indeed, by Ahlfors's Finiteness Theorem the boundary ∂H, of H is an intrinsically hyperbolic surface of bounded area, and so it's not hard to see that Theorem 0 also implies Theorem 1.1. In other words, they are essentially equivalent.

We also note that there are only finitely many possibilities for the topology of M once its homotopy class is determined. (This is a consequence of the characteristic submanifold construction.) Thus, in Theorem 0, one can take r to be a function of the fundamental group.

We will prove Theorem 1.1 in a series of more general situations. In Sections 3, 4 and 5, we deal with the convex co-compact case. Here, H is respectively a product, compression body, or general compact manifold. In section 6, we consider the case of a tame manifold without cusps. Finally we deal with the general case in Section 7.

1. Convex hulls.

In this Section we make a few relatively simple observations and give a reformulation of the main result, namely Theorem 1.1. We use $\mathbf{H}^3$ to denote hyperbolic 3-space, d for its metric, $N(Q,r)$ for the r-neighbourhood of $Q \subseteq \mathbf{H}^3$.

Let G be a non-cyclic infinite torsion-free group acting properly discontinuously on $\mathbf{H}^3$. We write $\tilde{H}$ for the convex hull of the limit set, and $H = \tilde{H}/G \subseteq V \subseteq \mathbf{H}^3/G$ for the convex core of V, which we assume to be compact. It follows by Ahlfors's Finiteness Theorem that in the induced path-metric, ∂H, is a finite disjoint union of complete hyperbolic surfaces. In the above, we should insert a clause about the "fuchsian" case, where H is a totally geodesic surface with totally geodesic boundary. It is then most natural to view ∂H as the double of H. Our statements are easily reinterpreted (though essentially elementary) in that case.

Given $t > 0$, let $\tilde{\Delta}(t) = \{x \in \mathbf{H}^3 \mid (\exists g \in G \setminus \{1\})(d(x,gx) \leq t)\}$, and set $\Delta(t) = \tilde{\Delta}(t)/G \subseteq V$. In other words, $\Delta(t)$ is the set of points of V contained in an essential curve of length at most t. If $\eta > 0$ is less than the Margulis constant, we write $\tilde{T}(\eta) = \tilde{\Delta}(\eta)$ and $T(\eta) = \tilde{T}(\eta)/G$. Thus $T(\eta)$ is the η-thin part of V. (We use different notation since we view $\Delta(t)$ and $T(\eta)$ differently — typically t and η will be large and small constants respectively.) Each component of $\tilde{T}(\eta)$ has the form $\tilde{T}_Z(\eta)$ where $Z \leq G$ is a maximal abelian subgroup. The quotient $\tilde{T}_Z(\eta)/Z \subseteq V$ is a Margulis region. This is either a cusp (rank 1 or rank 2) or a Margulis tube. In the last case, we will denote it by $T(\alpha,\eta)$, where α is a primative homotopy class of closed curves in V.

We can assume that η is also less than the 2-dimensional Margulis constant. If $\xi \in (0,\eta]$ we set $A_{\partial H}(\xi)$ to be the "ξ-thin" part of ∂H. We define this so that each component, A, of $A_{\partial H}(\xi)$ is an annulus with boundary curves α_1, α_2, say, each of length ξ. Note that if α is essential in V, then $A \subseteq T(\alpha,\eta) \subseteq T(\eta)$.

Suppose that α is trivial in V. Then α_i bounds an embedded disc $D_i \subseteq H$ of diameter at most ξ. We can assume D_1 and D_2 to be disjoint, so that $A \cup D_1 \cup D_2$ is a 2-sphere, and so it bounds a ball, $W \subseteq H$, with $W \cap \partial H = A$. We refer to W as a *ξ-handle* in H. Note that if δ is any path from D_1 to D_2 in W then $W \subseteq N(\delta,\xi)$.

We write $W(\xi)$ for the union of all ξ-handles in H (which we can assume to be disjoint). Note that, up to bounded Hausdorff distance, $W(\xi)$ can be described as the set of points of H contained in some compressing disc of circumference and diameter at most ξ. (This ties in with the informal description given in Section 0.)

We will write $\tau(M)$ for the minimal number of 3-simplices in a triangulation of M. This measures the topological compexity of M. As a "triangulation" we can allow the image of any G-invariant triangulation of $\tilde{H}$ as a simplicial complex, so that in H we can allow non-embedded simplices, but this makes no essential difference to the argument.

Note that $\tau(M)$ also bounds the complexity of ∂H (for example, as measured as the sum of the genera of its components).

The main result of this paper is:

Theorem 1.1 : $(\forall \tau \in \mathbf{N})(\forall \xi > 0)(\exists t \geq 0)$ *if* $\tau(M) \leq \tau$, *then* $H \subseteq \Delta(t) \cup W(\xi)$.

We can refine Theorem 1.1 slightly. If we alter the conclusion to $H \subseteq N(\Delta(t), r) \cup W(\xi)$, then we can choose t to depend only on ξ and the complexity of ∂H, though r may also depend on $\tau(M)$. We will see that the proof automatically gives this. Note that $N(\Delta(t), r) \subseteq \Delta(t + 2r)$, and so this implies Theorem 1.1 as stated.

2. Some definitions and facts.

We begin by recalling or reformulating a few well known facts and constructions used in the proof. We assume here there are no parabolics. We describe how the relevant statements can be modified in Section 7.

Definition : By a *singular (hyperbolic) surface* in V, we mean a 1-lipschitz map, $\phi : \Sigma \longrightarrow V$, where Σ is a closed surface with a hyperbolic structure.

(We could allow uniformly lipschitz maps, or cone singularities with angles at least 2π without essential change.)

We say that a free homotopy class of closed curves in Σ is *compressing* if it is non-trivial in Σ but its image in V is trivial. Suppose we fix some $\eta > 0$. Then each component of the η-thin part of Σ is either compressing, or maps into the thin part, $T(\eta)$, of V. Also each component of the complement of the η-thin part of Σ has bounded diameter, depending only on η and the genus of Σ.

Definition : By a *multisurface* we mean a (possibly empty) disjoint union of closed surfaces.

We can generalise the above to *singular mutlisurfaces*. Note that the inclusion of ∂H into V is a singular multisurface (in fact a non-singular one).

Definition : A *multicurve* in a multisurface, Σ, is a disjoint union of homotopically distinct non-trivial closed curves. It is *complete* if each component of the complement is a three-holed sphere (or "3HS").

In other words, it gives a pants decomposition of each component of Σ. If $\phi : \Sigma \longrightarrow V$ is any homotopy class of maps, we say that γ is is *totally incompressible* if each component of $\phi(\gamma)$ is non-trivial in V.

Definition : By a *realisation* of γ in V we mean a singular hyperbolic multisurface, $\phi : \Sigma \longrightarrow V$, such that $\phi|\gamma$ maps each component of γ locally isometrically to the corresponding closed geodesic in V.

Lemma 2.1 : *Any totally incompressible multicurve, $\gamma \subseteq \Sigma$, admits a realisation. If γ is a complete multicurve, then any two such realisations are connected by a homotopy in V whose image lies in a bounded neighbourhood of the image of either of the realisations.*

Proof : These are fairly standard, see for example [Bo]. $\diamondsuit$

We also recall Bers's Lemma:

Lemma 2.2 : *Given any hyperbolic structure on a closed surface Σ, there is a complete geodesic multicurve total length is bounded above in terms of the complexity of Σ.* $\diamondsuit$

Clearly this also applies to multisurfaces.

We can assume that such a multicurve contains all closed geodesics in Σ of length at most η.

We also note the following:

Lemma 2.3 : *Suppose that $\phi : \Sigma \longrightarrow V$ is a singular hyperbolic multisurface, and that $\gamma \subseteq \Sigma$ is a totally incompressible complete multicurve of bounded length in Σ. Then we can homotope ϕ to a realisation of γ in V by a homotopy lying in $N(\phi(\Sigma) \cup T(\eta), r)$, where r depends only on η and the complexity of Σ.*

Proof : First homotope each component of $\phi(\gamma)$ to the corresponding closed geodesic. This can be done in a bounded neigbourhood of this curve union the corresponding Margulis tubes. We can then homotope so that each curve maps locally injectively. Now extend over each 3HS. $\diamondsuit$

We recall the notion of an "elementary move" on complete multicurves (corresponding an edge in the pants graph). We say that γ and δ are connected by an *elementary move* if there are components $\alpha \subseteq \gamma$ and $\beta \subseteq \delta$ such that $\gamma \setminus \alpha = \delta \setminus \beta$ and such that $\alpha \cup \beta$ has a regular neighbourhood that is either a four-holed sphere (4HS) or a one-holed torus (1HT). In these terms, the fact that the pants graph is connnected [HT] can be expressed as:

Lemma 2.4 : *If γ and γ' are complete muticurves, then there is a sequence $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_n = \gamma'$ of complete multicurves such that each γ_{i+1} is obtained from γ_i by an elementary move.* $\diamondsuit$

We can elaborate on the second part of Lemma 2.1:

Lemma 2.5 : *Suppose that γ and δ are complete multicurves related by an elementary move, and suppose that ϕ, ψ are realisations of γ and δ respectively. Then there is a homotopy from ϕ to ψ in V lying in a bounded neighbourhood of $\phi(\Sigma) \cup T(\eta)$.*

Proof : We've just got to worry about 1HT's and 4HS's. These are easily dealt with by cutting into simplices with all vertices in the curves. $\diamondsuit$

We will now move on to give proofs of Theorem 1.1, in increasing generality. First we will assume that the manifolds are convex cocompact, that is, the convex core, H, is compact.

This will be based on a simple homological principle. We will take $\mathbf{Z}_2$-coefficients. If ∂H bounds a singular cycle in V, then H is precisely the set of points to which the cycle maps with degree 1. We can construct such a 3-cycle for example as a continuous map, $\phi : H \longrightarrow V$, with $\phi|\partial H$ just inclusion of ∂H in V.

3. The convex cocompact product case.

We consider first the case where $H \cong \Sigma \times [-1, 1]$, where Σ is a closed surface. We write $\Sigma^\pm = \Sigma \times \{\pm 1\}$, so that $\partial H = \Sigma^- \cup \Sigma^+$. This case follows from well known constructions related to interpolating pleated surfaces. It was originally described in [Can]. We describe one such construction which we adapt later.

Let $\gamma^\pm$ be a complete multicurve in $\Sigma^\pm$ of length bounded in terms of genus(Σ) (Lemma 2.2). Here all curves are incompressible. Let $\phi^\pm : \Sigma \rightarrow V$ be a realisation of $\gamma^\pm$. By Lemmas 2.1 and 2.3, we can find a homotopy from $\Sigma^\pm \hookrightarrow V$ to $\phi^\pm : \Sigma \longrightarrow V$ which lies in a bounded neighbourhood of $\Sigma^\pm \cup T(\eta)$. Now let $\gamma^- = \gamma_0, \gamma_1, \ldots, \gamma_n = \gamma^+$ be a sequence of complete multicurves as given by Lemma 2.4. Let $\phi_i : \Sigma \longrightarrow V$ be a realisation of γ_i. By Lemma 2.5, there is a homotopy from ϕ_i to ϕ_{i+1} lying in a bounded neighbourhood of $\phi_i(\Sigma) \cup T(\eta)$. (Note that we don't need a bound on the length of the geodesic realisation of γ_i for this.) Assembling all the homotopies gives us a homotopy from Σ^- to Σ^+ lying in a bounded neighbourhood of $\partial H \cup \bigcup_i \phi_i(\Sigma) \subseteq V$. We can view this as a map $\phi : H \longrightarrow V$, with $\phi|\partial H$ just inclusion $\partial H \hookrightarrow V$. Also, for each i, $\phi_i(\Sigma) \subseteq \Delta(t_0)$ where t_0 depends only on genus(Σ). (Since, for each i, each point of Σ in the intrinsic hyperbolic metric lies in a curve of bounded length.) Thus, $H \subseteq \phi(H) \subseteq \Delta(t)$, where t depends only on genus(Σ) as claimed.

4. The convex cocompact compression body case.

We next move on the case where H is a compact compression body, which we can assume not to be a product in the above sense. This has outer and inner boundaries $\partial^+ H$ and $\partial^- H$ respectively. Thus, $\partial^+ H$ is a compressible surface, and $\partial^- H$ is a (possibly empty) incompressible multisurface.

Our aim again will be to construct a map $\phi : H \longrightarrow V$, with $\phi|\partial H$ just inclusion, $\partial H \hookrightarrow V$, using a series of singular hyperbolic multisurfaces, $\phi_i : \Sigma_i \longrightarrow V$, for $i = 0, 1, \ldots, n$. This time, we will set ϕ_0 and ϕ_n to be respectively the inclusions of $\partial^+ H$ and $\partial^- H$ into V. Each ϕ_{i+1} will obtained from ϕ_i either by a compression, as defined below, or by a homotopy. For topological reasons, there can only be boundedly many compressions (in terms of genus($\partial^+ H$)). Each of these compressions and homotopies will lie in a bounded neighbourhood of $\phi_i(\Sigma_i) \cup T(\eta)$. (Here, the bound depends on genus($\partial^+ H$) and η.) The strategy will be to perform compressions along "short" compressing curves whenever we have the opportunity. In this way, by going back a bounded number of times, we see that each ϕ_i is obtained from some ϕ_j where $j \leq i$ by a series of consecutive such compressions and where either $j = 0$, or where ϕ_{j-1} has no short compressing discs. In the former case, we note that $\phi_0(\Sigma_0) = \partial^+ H \subseteq \Delta(t_0) \cup W(\xi)$, where t depends only on genus$(\partial^+ H)$ and ξ. In the latter case, we note that the intrinsically η-thin part of Σ_{j-1} maps into $T(\eta)$ (since there are no short compressions) and so $\phi_{j-1}(\Sigma_{j-1}) \subseteq \Delta(t_0)$, with t_0 depending only on genus$(\partial^+ H)$. Assembling ϕ out of these maps, we get $H \subseteq \phi(H) \subseteq \Delta(t) \cup W(\xi)$, where t depends only on genus$(\partial^+ H)$ and ξ as required.

We begin by describing the notion of compression in topological terms. By a *compression* of a multisurface, Σ, we mean a disjoint union of compression bodies, C, such that Σ is the union of their outer boundaries. We assume that there are no sphere or torus inner boundary components. We also assume that C is not a product, and so the inner boundary has strictly lower complexity than the inner boundary. Note that we can connect a sequence of such compressions into a single compression.

Suppose $\phi : \Sigma \longrightarrow V$ is any map, and $\beta \subseteq \Sigma$ is a totally compressing multicurve (i.e. each component is compressing). We can construct a compression as follows. First glue a disc to each component of β and then thicken up the resulting 2-complex into a compression body. This admits a natural map into V up to homotopy. We now cap off each spherical inner boundary component with a 3-ball. We also cap off each toroidal inner boundary component with a solid torus, in such a way that the map into of the torus into V extends over the solid torus. (This is always possible since there are no rank-2 free abelian subgroups of $\pi_1(V)$.) This gives another compression body, C, with inner boundary, Σ', say. We can extend ϕ over C and so in particular get a map of Σ' into V.

Definition : We say that such a manifold, C, together with the map, ϕ, is a *compression* Σ, in V, which *compresses* the multicurve, β.

Here is another way of describing a compression in V, which is topologically equivalent, and which is how we will carry it out geometrically.

Suppose that $\gamma \subseteq \Sigma$ is a complete multicurve. Let $\beta \subseteq \gamma$ be the union of all compressing curves in γ. Let $\mathcal{P}$ be the set of components of $\Sigma \setminus \gamma$. We write $\mathcal{P} = \mathcal{P}_0 \sqcup \mathcal{P}_1 \sqcup \mathcal{P}_3$ where $\mathcal{P}_i$ has exactly i boundary components in β. (We count as 2 any pair of boundary components that get identified to the same curve of γ.) We first construct a multisurface Σ^1 by cutting Σ along β and gluing in a disc to each boundary component arising. Thus, each $P \in \mathcal{P}_3$ gives rise to a 2-sphere component of Σ^1. Each element of $\mathcal{P}_1$ turns into an annulus in Σ^1. There may be torus components of Σ^1 each of which consists of a closed circuit of such annuli. The remaining "hyperbolic" components of Σ^1 all have genus at least 2. Now the union of all closed annuli arising from $\mathcal{P}_1$ will give us, in addition to tori, a number (possibly 0) of closed annuli in the hyperbolic components. For each such annulus, A, we perform another surgery by cutting along the boundary components and regluing in pairs, so as to give a torus (arising from A) and a disjoint surface homeomorphic to the original. We thus arrive at another multisurface, Σ^2. We finally throw away all the sphere and torus components of Σ^2 to give us a multisurface, Σ'. Each of the above multisurfaces comes with a natural homotopy class of map to V. Also, the multicurve γ in Σ gives rise to a complete multicurve γ' in Σ'. (In constructing γ', we throw away the components of β, and identify certain pairs of curves in $\gamma \setminus \beta$.) It's not hard to see that $\Sigma \sqcup \Sigma^2$ bounds a compression body in a natural way. To get to Σ' it remains to cap off the 2-spheres with 3-balls, and tori with solid tori. (Note that if F is a torus component of Σ^2, then the image of $\pi_1(F)$ in $\pi_1(V)$ is non-trivial by construction, and hence contained in a unique maximal cyclic subgroup. There is thus a canonical way to glue in the solid torus so as to kill the kernel.) By construction, we get a compression of Σ to Σ' in V, in the same sense as described earlier. Note also by construction, each curve of γ' is non-compessing on Σ'.

Before applying the above we need a couple of geometrical observations. The proofs are straightforward.

Lemma 4.1 : *Any map $\phi : F \longrightarrow V$ of a 2-sphere, F, into V extends to a map $\phi : B \longrightarrow V$ of the 3-ball, with the diameter of $\phi(B)$ bounded above by the intrinsic diameter of F in the induced metric.*

Any map $\phi : F \longrightarrow V$ of a torus F, into V extends to a map of a solid torus, D, into V such that $\phi(D)$ lies in a bounded neighbourhood of $\phi(F) \cup T(\eta)$, where the bound depends on the intrinsic diameter of F and η. ◊

In our constructions, these intrinsic diameters will all be bounded in terms of genus($\partial^+ H$).

Suppose now that $\phi : \Sigma \longrightarrow V$ is a singular multisurface (that is 1-lipschitz with respect to some hyperbolic structure on Σ). Suppose that $\gamma \subseteq \Sigma$ is an intrinsically geodesic complete multicurve of bounded length (Lemma 2.2). It is assumed to contain all closed geodesics of length at most η. Let $\beta \subseteq \gamma$ be union of all compressing curves in γ. We say that Σ "admits a short compression" if there is some such γ for which β is non-empty. In this case, let $\gamma' \subseteq \Sigma'$ be as described above, and let $\phi' : \Sigma' \longrightarrow V$ be a realisation of γ'. (Recall that no component of γ' is compressing.) We can now interpolate between ϕ and ϕ' by a compression whose image lies in a bounded neighbourhood of $\phi(\Sigma) \cup T(\eta)$. (If α is a component of β, then span $\phi(\alpha)$ by disc of bounded diameter. If δ is a component of $\gamma \setminus \beta$ giving rise to a component of γ', then homotope $\phi(\delta)$ to its geodesic realisation in V by a homotopy lying in a bounded neighbourhood of $T(\eta)$. To extend to a compression it remains to cap off a collection of 2-spheres and tori, all of which have bounded intrinsic diameter. This can be achieved using Lemma 4.1.

Terminology : We refer to the above process as a "short compression" of $\phi(\Sigma)$.

Note that if $\phi(\Sigma)$ admits no short compression, then each compressing curve of Σ must have length at least η. As in the product case above, we see that in this case, that $\phi(\Sigma) \subseteq \Delta(t_0)$, where t_0 depends only on the complexity of Σ, hence in our set up, t_0 will only depend on genus($\partial^+ H$).

We can now set about constructing our map $\phi : H \longrightarrow V$ as follows. Start with $\Sigma_0 = \partial^+ H$ and let $\phi_0 : \Sigma_0 \longrightarrow V$ be inclusion. We construct a map $\phi_1 : \Sigma_1 \longrightarrow V$ as follows. If ϕ_0 admits a short compression then we carry it out to obtain a map $\phi_1 : \Sigma_1 \longrightarrow V$, with realises some multicurve, $\gamma_1 \subseteq \Sigma_1$. If it admits no such compression, then we choose a multicurve, γ_0, in Σ_0 of bounded length. We set $\Sigma_1 = \Sigma_0$ and $\gamma_1 = \gamma_0$. We let $\phi_1 : \Sigma_1 \longrightarrow V$ be a singular surface realising γ_1 (cf. the product case, Section 3).

If ϕ_1 admits a short compression we carry it out. (The complete multicurve used for this just depends on the hyperbolic structure on Σ_1 and need not bear any relationship to γ_1.) After a bounded number of steps, we will arrive at a singular map $\phi_p : \Sigma_p \longrightarrow V$ realising a complete multicurve γ_p, and which admits no short compression. Note that by construction, γ_p has no compressible components. (It is possible that Σ_p is empty.)

Suppose that $\phi_p(\Sigma_p)$ is not incompressible in V. Let $\beta \subseteq \Sigma_p$ be any compressing curve. Let $S \subseteq \Sigma_p$ be the component containing β. We extend β to a complete multicurve, and connect this to $\gamma_p \cap S$ by a path $\gamma_p \cap S = \delta_0, \delta_1, \ldots, \delta_q \supseteq \beta$ in the pants graph (Lemma 2.4). We extend each δ_j to a complete muticurve in Σ_p, also denoted δ_j by setting $\delta_j \setminus S = \gamma_p \setminus S$ for all $j \in 0, 1, \ldots, q$. (In other words, we leave the other components of Σ_p alone.) We can assume that δ_j is totally incompressible for all $j < p$ (otherwise we just stop there, and reset $p = j$).

Suppose that $q = 1$. Then δ_1 is obtained from $\gamma_p = \delta_0$ by replacing some component α of δ_0 with the compressing curve $\beta \subseteq \delta_1$. The curves α and β must lie in a 4HS component, F, of $\Sigma_p \setminus (\delta_0 \setminus \beta) = \Sigma_p \setminus (\delta_1 \setminus \beta)$, and intersect minimally there. We now perform a compression on β. This involves removing F, and reconnecting its boundary curves in pairs homotopic in V to give us a new multisurface, Σ_{p+1}. This multisurface cannot have any torus or sphere components. (Though it is possible that a pair of boundary curves of F already identified to a single curve in Σ_p, in which case we just discard it.) Since the pairs of boundary components were already mapped to the same geodesic realisation in V, we get a map $\phi_{p+1} : \Sigma_{p+1} \longrightarrow V$ realising a complete multicurve $\gamma_{p+1} \subseteq \Sigma_{p+1}$. The maps ϕ_p and ϕ_{p+1} are connected by a compression whose image lies in $\phi_p(\Sigma_p)$. (In this construction, the lengths of the multicurves need not be bounded, but this does not matter.)

Terminology : We refer to the above process as a "long compression".

We now continue with ϕ_{p+1} playing the role of ϕ_1 before. If there is a short compression, we carry it out and repeat, until there is no more short compression. We then set about the process of finding another long compression via Lemma 2.4, as above.

Suppose that $q > 1$, then δ_1 is totally incompressible, and we set $\Sigma_{p+1} = \Sigma_p$ and $\phi_{p+1} : \Sigma_{p+1} \longrightarrow V$ to be a realisation of δ_1. If ϕ_{p+1} admits a short compression we carry it out, and start again. If not we move on, with δ_1 playing the role of δ_0. If $q = 2$, we carry out a long compression. If $q > 2$, we set $\Sigma_{p+2} = \Sigma_p$ and realise δ_2 by a map $\phi_{p+2} : \Sigma_{p+2} \longrightarrow V$ and then check whether this admits a short compression. We continue in this way, performing a short compression whenever we can, and searching for a long compression, whenever we cannot.

After a bounded number of compressions, we eventually arrive at some map $\phi_m : \Sigma_m \longrightarrow V$, with $\phi_m(\Sigma_m)$ incompressible in V. (Possibly $\Sigma_m = \emptyset$.)

Now $\phi_m(\Sigma_m)$ is homotopic to $\partial^- H$. We now proceed as in the product case (for each component of Σ_m) so as to construct maps $\phi_{m+i} : \Sigma_{m+i} \longrightarrow V$, with $\Sigma_{m+i} = \Sigma_m$ for all i, terminating with a map $\phi_n : \Sigma_n \longrightarrow V$, which is just the inclusion of Σ^- into V.

Assembling all the ϕ_i, we get a map $\phi : H \longrightarrow V$, with $\phi | \partial H$ just inclusion, as discussed earlier. This proves Theorem 1.1, in the case where H is a compact compression body.

5. General cocompact case.

We explain how to reduce the general cocompact case to the previous cases dealt with. A similar principle is used in [E].

Let V be a complete hyperbolic manifold with compact convex core, H. The inclusion $H \hookrightarrow V \hookrightarrow M$ is homotopic to a homeomorphism. Indeed, $M \setminus H \cong \partial M \times [0, \infty)$.

Let S be a boundary component of $H \cong M$. Let V_S be the cover of V corresponding to $S \hookrightarrow V$. We lift S to an injective map $S \longrightarrow V_S$. This is a boundary component of the convex core, $H_S \subseteq V_S$.

Now V_S is also convex cocompact. This is a standard argument due to Thurston. (By Ahlfors's Finiteness Theorem, ∂H_S is compact. Since H is compact, every point must lie a bounded distance from its boundary, and so the same is true of H_S. Thus, H_S is compact.) Since $S \hookrightarrow H_S$ is surjective on fundamental groups, H_S is a compression body.

Suppose we are given a triangulation of S. Then we can construct a piecewise straight map, ψ of S into V_S, homotopic to inclusion. By "piecewise straight" we mean that each simplex gets mapped to a totally geodesic simplex in V. We can do this by mapping in vertices, edges and triangles in turn. In fact, by choosing this so as to minimise the total length of the 1-skeleton, we will have $\psi(S) \subseteq H_S$. (Otherwise there will be a vertex x in S, whose link gets mapped strictly into a hemisphere in the unit tangent space in V_S. This would allow us to shorten the 1-skeleton by pushing the vertex in the direction of the centre of the hemisphere.) We refer to piecewise straight maps of this sort as "balanced".

Let $W_S(\xi)$ be the union of all ξ-handles in H_S, and $\Delta_S(t)$ be the set of points contained in an essential loop of length at most t in V_S. By the earlier cases (Sections 3 and 4), we have $H_S \subseteq \Delta_S(t) \cup W_S(\xi)$. In particular, we get a $\mathbf{Z}_2$-homology cycle bounded by $\psi(S) \cup S$ in V_S and whose image lies in $\Delta_S(t_S) \cup W_S(\xi)$, where t_S depends only on the genus of S. Mapping down to V, we get a map $\psi_S : S \longrightarrow V$ with $\psi_S(S) \cup S$ bounding a homology cycle lying in $\Delta(t_S) \cup W(\xi)$. Let P_S be the set of point to which this maps with degree 1.

We now take a triangulation of $M \cong H$ with at most τ 3-simplices. This induces a triangulation on ∂H. We perform the above construction for each component of ∂H, to give a piecewise straight map $\psi : \partial H \longrightarrow V$ (mapping in the remaining vertices, edges, triangles and simplices in turn). We extend this to a piecewise straight map $\psi : H \longrightarrow V$. Let $Q \subseteq V$ be the set of points to which it maps with degree 1 (in $\mathbf{Z}_2$). The volume of Q is bounded above by τ times the volume of the regular ideal 3-simplex. This places a bound, say r, depending only on τ, on the radius of the largest hyperbolic ball we can embed in Q.

Now $H = Q \cup \bigcup_S P_S$ as S ranges over the boundary components. It follows that $H \subseteq \Delta(t) \cup W(\xi)$, where $t = 2r + \max_S t_S$. This proves Theorem 1.1 in the general convex cocompact case. (Note that this includes the cocompact case, where $M = H$ is a closed manifold, and $\partial H = \emptyset$.)

6. Geometrically tame without parabolics.

Let V be a complete hyperbolic 3-manifold with without parabolics. By tameness

[Bo,A,CalG], V is homeomorphic to the interior of a compact manifold, M. We can embed M into V so that $V \setminus M \cong \partial M \times [0,\infty)$. Let S be a component of ∂M, so that $E_S \equiv S \times [0,\infty)$ is an end of M. This end is either geometrically finite, in which case, we can assume that S is a boundary component of ∂H, or else simply degnerate, in which case, we can assume that $E_S \subseteq H$.

In the simply degenerate case, there is a sequence, $\omega^n : S \longrightarrow E_S$, of singular hyperbolic surfaces, each homotopic in E_S to the inclusion of $S = \partial E_S$, and tending out the end. Indeed we can take ω^n to be the realisation of some complete multicurve, γ^n, in S. Suppose β is any curve S such that $\omega^n(\beta)$ is compressing in V. There is a compressing disc in V whose diameter is bounded above by $\mathrm{length}(\omega^n(\beta))$. This disc must meet H. It follows that the length of the shortest compressing curve in $\omega^n(S)$ (if there is any) must tend to ∞ as $n \to \infty$. In particular, we can assume that none of the surfaces $\omega^n(S)$ admits any short compression (in the sense described in Section 4).

Now suppose that $x \in H$ is any point in the convex core. For each simply degenerate end, E_S, we choose a singular $\omega_S : S \longrightarrow E_S$ with x lying in the same component of $E_S \setminus \omega_S(S)$ as $S = \partial E_S$. (Set $\omega_S = \omega^n$ for large enough n.) Note in particular, that $\omega_S(S)$ homologically separates x from the end. If S is geometrically finite, we set ω_S to be the inclusion $S \hookrightarrow V$. Combining these gives us a map $\omega : \partial M \longrightarrow V$ which we can extend to a homology cycle with boundary $\omega(\partial M)$. This maps with degree 1. For this, we use a piecewise straight map, $\psi : M \longrightarrow V$, with $\psi|\partial M$ balanced, constructed as in Section 5. We need to find a cycle with boundry $\psi(S) \cup \omega(S)$ for each component, S, of ∂M. We follow the procedure of previous sections.

Suppose, for example, that E_S is a simply degenerate end. As in Section 5, we lift to V_S, which is the interior of a compact product manifold or compession body, M_S, with outer boundary $\partial^+ M \cong S$. The outer end is simply degenerate and isometric to E_S. We can lift $\omega_S : S \longrightarrow E_S$ to a map to V_S, also realising the complete multicurve γ_S. Since $\omega_S(S)$ has no short compression, we can immediately set about searching for a long compression similarly as in Section 4, using Lemma 2.4, starting with γ_S. Moreover, if $\omega_S(S)$ is sufficently far out the end, then $\omega_S(S)$ lies in the convex core, H_S, of V_S. The lift of $\psi(S)$ to V_S also lies in H_S (since it is a balanced piecewise straight map). We can therefore find a homology cycle with boundary $\psi(S) \cup \omega_S(S)$ in V_S. As in Section 4, this lies in $\Delta(t_S)$ where t_S depends only on $\mathrm{genus}(S)$. If E_S is geometrically finite, we similarly obtain a homology cycle lying in $\Delta(t_S) \cup W(\xi)$.

We end up constructing a $\mathbf{Z}_2$-homology 3-cycle, with boundary $\omega(\partial M)$, and supported on $\Delta(t) \cup W(\xi)$, where t depends only on τ and ξ. This cycle also maps to x with degree 1, and so $x \in \Delta(t) \cup W(\xi)$. Since $x \in H$ was arbitrary, we deduce that $H \subseteq \Delta(t) \cup W(\xi)$, as required.

7. The general case.

To deal with the general case, where we allow for cusps, a few modifications are necessary. We need to adapt the definition of *singular surface* to allow for nodal surfaces. We need to consider $\mathbf{Z} \oplus \mathbf{Z}$-cusps. Also, the $\mathbf{Z}$-cusps may cut the topological ends of M

into pieces, some of which may be geometrically finite while others simply degenerate.

Let V be a complete hyperbolic 3-manifold with $\pi_1(V)$ finitely generated. Again by tameness, we can embed M into V, so that $V \setminus \operatorname{int} M \cong \partial M \times [0,\infty)$. The $\mathbf{Z} \oplus \mathbf{Z}$-cusps are in bijective correspondence to the toroidal components of ∂M, and each can be assumed to be the boundary of the corresponding η-Margulis cusp. Let $\partial_0 M$ be the union of all the components of genus at least 2. Tameness also tells us that there is a multicurve, $\pi \subseteq \partial_0 M$, with the $\mathbf{Z}$-cusps in bijective correspondence with the components of π.

More precisely, if $\alpha \subseteq \partial_0 M$ is a component of π, then there is an η-Margulis cusp, $T(\alpha,\eta)$, homotopic to α in V. We can assume that ∂M meets $T(\alpha,\eta)$ in an annulus $A(\alpha)$ with core curve α. We write $\Psi(V) \subseteq V$ for the complement of the interiors of all the η-Margulis cusps.

Definition : We refer to $\Psi(v)$ as the *non-cuspidal part* of V.

We write $\mathcal{F}$ for the set of components of $\partial_0 M \setminus \bigcup_{\alpha \subseteq \pi} \operatorname{int} A(\alpha)$. Each component of $\Psi(V) \setminus M$ has the form $E_F \cong F \times [0,\infty)$, where F is identified with $F \times \{0\}$ — the relative boundary of E_F in $\Psi(V)$. These are the non-cuspidal "geometric" ends of V. Each is either geometrically finite or simply degenerate, and we partition $\mathcal{F} = \mathcal{F}_F = \mathcal{F}_D$ accordingly. If $F \in \mathcal{F}_D$ then $E_F \subseteq H$.

By Ahlfors's Finiteness Theorem, ∂H is an intrinsically hyperbolic multisurface of finite area. Suppose that ∂H meets a $\mathbf{Z} \oplus \mathbf{Z}$ cusp, T. Then T must lie on the "inside" of ∂H. This means that each component of $T \cap \partial H$ together with an annulus in ∂T bounds a solid torus in V outside H. Such annuli play no role in the following discussion. If T is $\mathbf{Z}$-cusp, then $T \cap \partial H$ is a (possibly empty) union of compact annuli (with T on the inside of ∂H as above) together with at most two totally geodesic cusps whose boundary curves are horocyclic curves in ∂H and meeting ∂H orthogonally. (For this we need to choose η sufficiently small depending on the topological type of ∂H). Again the compact annuli intersections play no role in the following discussion.

We will need to generalise the notion of a singular hyperbolic surface in V, to allow for curves to be sent off to infinity in parabolic cusps.

Let Σ be a multisurface. By a *nodal structure* in Σ, we mean a (possibly empty) multicurve, ζ, together with a complete finite-area hyperbolic structure on $\Sigma \setminus Z$, where Z is a closed regular neighbourhood of ζ. We also note that Bers's Lemma also applies to finite area surfaces, and therefore also to nodal structures. In otherwords, we can extend ζ to a complete multicurve in Σ so that the length of each other component is bounded above in terms of genus(Σ).

To define a singular nodal surface, we embed V as a submanifold of a manfold $V \cup C(V)$ with boundary $C(V)$, where each component of $C(V)$ corresponds to a cusp. Formally, we can view $C(V)$ as a disjoint copy of $\partial\Psi(V)$, and topologise $V \cup C(V)$ in such a way that $V \cup C(V) \setminus \operatorname{int} \Psi(V)$ is a product, $\partial\Psi(V) \times [0,\infty]$ with $\partial\Psi(V) \equiv \partial\Psi(V) \times \{0\}$ and with $C(V) \equiv \partial\Psi(V) \times \{\infty\}$, and such that each $\{x\} \times [0,\infty)$ is a geodesic ray.

Definition : A *(singular) nodal surface* is a map $\phi : V \cup C(V)$ such that $Z = \phi^{-1}(C(V))$ is a regular neighbourhood of a multicurve in Σ, and such that $\phi|(\Sigma \setminus Z)$ is 1-lipschitz with respect to a nodal structure corresponding to the multicurve.

Only $\phi|(\Sigma \setminus Z)$ is relevant to geometric arguments. We need ϕ defined on Z to enable us to define homotopy and homology classes.

The observations of Section 2 go through with little change, with "singular hyperbolic surface" replaced by "nodal surface".

To explain how the proof is modified, we work backwards through the paper.

In Section 6, we need to take account to the possibility that an end of M may have both geometrically finite and simply degenerate parts. Suppose that $F \in \mathcal{F}_D$, i.e. E_F is simply degnerate. In this case we have a sequence, $(\omega^n)_n$ of 1-lipschitz maps into M where the domain of each is a finite area surface homeomorphic to $\operatorname{int} F$. The image of ω^n meets $\Psi(V)$ in a compact subset of E_F. This homologically separates F from the end of E_F. Moreover, these compact subsets tend out the end as $n \to \infty$. On the other hand, if $F \in \mathcal{F}_F$, we can set each ω^n to be equal to the inclusion of the corresponding component of ∂H into V. Piecing together these maps with annuli in $C(V)$, we get a nodal surface $\omega^n : \partial_0 M \longrightarrow V \cup C(V)$, based on the multicurve π determined by the $\mathbf{Z}$-cusps. We extend this topologically to a map $\omega^n : \partial M \longrightarrow V \cup C(V)$, taking each (toroidal) component of $\partial M \setminus \partial_0 M$ to a component of $C(V)$.

Let $Q^n \subseteq V$ be the set of points $\mathbf{Z}_2$-homologically seperated from infinity by $\omega^n(\partial_0 M)$, in other words, the set of points of V to which $\omega^n : \partial M \longrightarrow V \cup C(V)$ maps with degree 1. By construction, $\Psi(V) \cap H \subseteq \bigcup_n Q^n$.

Let S be a component of $\partial_0 M$. Let V_S be the corresponding cover. We define $C(V_S)$ similarly as for $C(V)$. There is a natural map $V_S \cup C(V_S) \longrightarrow V \cup C(V)$ which bijective on each component of $C(V_S)$. As before, by tameness, V_S is the interior of a compact manifold, which must be a compression body (or product) with outer boundary correponding to S. This time there might be toroidal inner boundary components corresponding to $\mathbf{Z} \oplus \mathbf{Z}$-cusps. Let H_S be the convex core of V_S. If we assume the compression body case (which will be explained below) then $H_S \subseteq \Delta(t_S) \cup W_S(\xi)$, where t_S depends only on the genus of S.

As before, we can construct a balanced piecewise straight map, $\psi_S : S \longrightarrow V_S \cup C(V_S)$, which in this case may be a nodal surface. We will have $\psi_S(S) \cup V \subseteq H_S$. Let $\omega^n_S : S \longrightarrow V_S \cup C(V_S)$ be the lift of $\omega^n|S$. We can assume that $\omega^n_S(S) \cap V_S \subseteq H_S \cup T(V, \eta)$. Now ω^n_S is homotopic to ψ_S. Let $P^n_S \subseteq V_S$ be the set of points to which the homotopy maps with degree 1. Thus, $P^n_S \subseteq H_S \cup T(V, \eta) \subseteq \Delta(t_S) \cup W_S(\xi)$.

Mapping down to V, we get maps $\psi_S : S \longrightarrow V$, homotopic to $\omega^n|S$. Combining the maps ψ_S for each such component S, we get a map $\psi : \partial_0 M \longrightarrow V$. We extend this to a piecewise straight map $\psi : M \longrightarrow V \cup C(V)$, sending each toroidal boundary component to a component of $C(V)$. This map is homotopic to ω^n. As before, we set Q to be the set of points of V to which ψ maps with degree 1. Thus the volume of Q is bounded above in terms of $\tau(M)$.

Let $P^n \subseteq V$ be the set of points homologically between $\psi(\partial M)$ and $\omega^n(\partial M)$. Thus $R^n = Q \cup P^n$. Also, $P^n \subseteq \Delta(t_S) \cup W(\xi)$. It follows that $R^n \subseteq \Delta(t) \cup W(\xi)$, where t is the maximum of t_0 and t_S as S ranges over the components of $\partial_0 M$. Now $V \setminus \Psi(V) \subseteq \Delta(\eta) \subseteq \Delta(t)$ assuming $t \geq \eta$. Since $H \cap \Psi(V) \subseteq \bigcup_n R^n$, it follows that $H \subseteq \Delta(t) \cup W(\xi)$

as required.

To complete the proof, we need to explain how the product and compression body cases are adapted from Sections 3 and 4 respectively.

First suppose that $M \cong \Sigma \times [-1,1]$ and write $\partial^+ M = \partial_0 M = \partial^+ M \sqcup \partial^+ M$, where $\partial^\pm M \equiv \Sigma \times \{\pm 1\}$. Let $\pi \subseteq \partial M$ be the multicurve corresponding to the set of $\mathbf{Z}$-cusps of V. (There are no $\mathbf{Z} \oplus \mathbf{Z}$-cusps in this case.) We construct a sequence of maps $\omega^n : \partial M \longrightarrow V \cup C(V)$ as in the general case above. Thus, ω^n is a nodal surface based on π, and each complementary surface F is sent either to a component of ∂H (when $F \in \mathcal{F}_F$), or to a singular hyperbolic surface $\omega^n(F)$, realising a complete multicurve γ^n_F in a simply degenerate geometric end (when $F \in \omega^n(F)$). If $F \in \mathcal{F}_F$, we let $\gamma^n_F = \gamma_F$ be a complete multicurve in F as given by Bers's Lemma. We set $\gamma^n = \pi \cup \bigcup_{F \in \mathcal{F}} \gamma^n_F$. Then $\gamma^n \supseteq \pi$ is a complete multicurve in ∂M. Let $\gamma^{n\pm} = \gamma^n \cap \partial M$. By Lemma 2.4, we can connect γ^{n-} to γ^{n+} by a path in the pants graph. Realising these by singular surfaces as in Section 2, we construct a homotopy from $\omega^n | \partial^- M$ to $\omega^n | \partial^+ M$. We can view this as a map $\psi^n : M \longrightarrow V \cup C(V)$, with $\psi^n | \partial M = \omega^n | \partial M$, and with $\psi^n(M) \subseteq \Delta(t)$ where t depends only on genus(Σ). (As before, $W(\xi) = \emptyset$ in this case.) Now $H \subseteq \bigcup_n \psi^n(M)$, and so $H \subseteq \Delta(t)$ as required.

Finally, we consider the case where M is a compression body. Let $\partial^+ M \subseteq \partial_0 M$ be the outer boundary component, and let $\partial^- M$ be the inner boundary. This time, $\partial^- M$ may contain tori, namely the components of $\partial^- \setminus \partial_0 M$.

As before, we construct a sequence of nodal surfaces, $\omega^n : \partial^+ M \longrightarrow V \cup C(V)$. We can assume that there is no short compression in the simply degenerate parts. We follow the argument of Section 4 to construct a compression to a nodal multisurface homotopic to the inner boundary, and thence (via the product case) to another nodal multisurface, $\omega^n : \partial^- M \longrightarrow V \cup C(V)$, where $\omega^n | \partial^- M \cap \partial^0 M$ maps each complementary surface, F, either to a component of ∂H or to a singular surface, $\omega^n(F)$, in a simply degenerate part, and where $\omega^n(F) \cap \Psi(M)$ goes out the corresponding end as $n \rightarrow \infty$. Also $\omega^n | \partial^- M \setminus \partial_0 M$ maps each component homeomorphically to a toroidal component of $C(V)$. Once we have done this, we can conclude that $H \subseteq \Delta(t) \cup W(\xi)$, where t depends only on ξ and genus$(\partial^+ M)$.

There are a couple of complications we have to consider in the previous paragraph. Certain curves in our sequence of multicurves may get mapped to components of $C(V)$ rather than to a closed geodesic. In this case, we just use a nodal surface to realise it. Also, after performing a compression, we may end up with an essential torus inner boundary component. These all correspond to $\mathbf{Z} \oplus \mathbf{Z}$-cusps, and can be homotoped to the corresponding torus component of $C(V)$ by a homotopy in a bounded neighbourhood of $T(\eta)$.

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