

End invariants of hyperbolic 3-manifolds

Brian H. Bowditch

School of Mathematics, University of Southampton,
Highfield, Southampton SO17 1BJ, Great Britain.

<http://www.maths.soton.ac.uk/staff/Bowditch/>

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0. Introduction.

In the late 1970s, Thurston made a number of conjectures concerning the geometry of 3-manifolds. This paper is concerned with the Ending Lamination Conjecture which, in combination with other results now established, proposes a classification of hyperbolic 3-manifolds with finitely generated fundamental group, or equivalently, finitely generated torsion-free kleinian groups.

A proof of the Ending Lamination Conjecture for indecomposable 3-manifolds has been given by Minsky, Brock and Canary [Mi2, BroCM1]. A sequel [BroCM2] is promised to deal with the general case. Another argument for the indecomposable case is given in [Bow3]. In this paper we show how the arguments of that paper can be adapted to the general case. This also calls for some reinterpretation of arguments in [Bow2].

A precise statement of the Ending Lamination Conjecture is given in Section 1. We begin with an overview of the situation here. For simplicity of exposition, we shall assume that everything is orientable.

Let $M = \mathbf{H}^3/\Gamma$ be a complete orientable hyperbolic 3-manifold. We assume that $\Gamma \equiv \pi_1(M)$ is finitely generated. Let $\Psi(M)$ be the *non-cuspidal part* of M , i.e. M with open Margulis cusps removed. Thus $\partial\Psi(M)$ consists of euclidean tori and bi-infinite cylinders. They bound respectively \mathbf{Z} and $\mathbf{Z} \oplus \mathbf{Z}$ -cusps of M . The Tameness Theorem, conjectured by Marden [Mar], and proven by work of Bonahon [Bon], Agol [A] and Calegari and Gabai [CalG] tells us that $\Psi(M)$ is topologically finite (see also [So]). This means that there is a compact manifold, $\bar{\Psi}$, with boundary $\partial\bar{\Psi}$, and a closed subsurface $\partial_I\bar{\Psi} \subseteq \partial\bar{\Psi}$, such that $\Psi(M)$ is homeomorphic to $\bar{\Psi} \setminus \partial_I\bar{\Psi}$. Fixing some such (proper homotopy class of) homeomorphism, we can identify $\partial\Psi(M)$ with $\partial_V\bar{\Psi} = \partial\bar{\Psi} \setminus \partial_I\bar{\Psi}$, which we refer to as the *vertical boundary* of $\bar{\Psi}$. Note that the ends of $\bar{\Psi}$ are in bijective correspondence with the components of $\partial_I\bar{\Psi}$. Each end e has a neighbourhood homeomorphic to $\Sigma \times [0, \infty)$, where $\Sigma = \Sigma(e)$ is such a component. Note that this meets $\partial_V\bar{\Psi}$ in $\partial\Sigma \times [0, \infty)$.

Thurston gave a geometrical reinterpretation of tameness, which was shown to be equivalent by Canary [Can] (and is essentially what is proven in the above references). This tells us that there are two types of end of $\Psi(M)$, namely “geometrically finite” and “simply degenerate”. Moreover, each has an “end invariant” associated to it, respectively a Riemann surface or a lamination.

The Ending Lamination Conjecture asserts that M is determined up to isometry by its end invariants. More formally, suppose that $\Psi \rightarrow \Psi(M)$ and $\Psi \rightarrow \Psi(M')$ are two homeomorphisms, or “markings”, from the same “topological model”, Ψ , to the non-cuspidal parts of two complete hyperbolic 3-manifolds M and M' . Suppose that the end

invariants for corresponding ends of M and M' are equal, taking account of markings. Then there is an isometry $M \rightarrow M'$, restricting to an isometry $\Psi(M) \rightarrow \Psi(M')$, which when precomposed with the marking $\Psi \rightarrow \Psi(M)$ is properly homotopic to the given marking $\Psi \rightarrow \Psi(M')$. (We are, of course, assuming that the non-cuspidal parts have been defined in some canonical way.) We remark that, by the topological results of Waldhausen [W], it is enough to define a marking to be a (proper homotopy class of) proper homotopy equivalence from Ψ to the non-cuspidal part. (This is effectively what is actually used in the proof anyway.)

The “indecomposable” case is when there is no disc in $\bar{\Psi}$ whose boundary is an essential curve in $\partial_I \bar{\Psi}$. This is equivalent to saying that each end of Ψ is incompressible, i.e. π_1 -injective. This case was dealt with in [Mi2, BroCM2] using results from [MasM2], and subsequently also in [Bow3] using results from [Bow2]. Both arguments make extensive use of the curve complex defined by Harvey [Har]. Most of it boils down to an analysis of the geometry of simply degenerate ends. (The geometrically finite case of the Ending Lamination Conjecture follows from the now well established deformation theory of Ahlfors, Bers, Kra, Marden, Maskit, Sullivan etc.)

The decomposable case introduces a number of complications. Many of these are of a technical nature and can be dealt with by appropriately reinterpreting various constructions, though others are more subtle. Most of the work of this paper is to give a means of reducing the analysis of the geometry of a simply degenerate end to questions that are intrinsic to that end. The basic idea is that once one is far from the topological compact core of the manifold, one does not notice intrinsically whether an end is irreducible or not, and so essentially the same arguments can be applied.

Note that the Ending Lamination Conjecture (together with tameness) only addresses the issue of uniqueness of hyperbolic structures with particular end invariants. The question of existence is addressed through work of Thurston, Canary, Otal, Ohshika, Souto, Kleiner, Lecuire and Namazi. A general statement, and other references, are given in [KiLO]. We note, in particular, that there are constraints on the ending lamination of a degenerate end. In particular it must lie in the Masur domain [Mas]. This, information however is not needed here, since the relevant facts are implicit in the analysis. Another constraint, is that, in the case of doubly degenerate product (surface times the real line) the two ending laminations must be different. This case calls for some special attention. (We remark that, in most of the above, end invariants are described in terms of the end of the manifold, rather than its non-cuspidal part, and so some re-interpretation is necessary to give a clean statement of the classification.)

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1. The Ending Lamination Conjecture.

In this section, we give a more precise statement of the Ending Lamination Conjecture, and outline the basic strategy for its proof.

Let Σ be a compact orientable surface. As in [Bow3], we write $\kappa(\Sigma) = 3g + p - 3$, where g is the genus and p is the number of boundary components. The only significance of this quantity here is that it measures the “complexity” of the topological type of Σ . Note that if $\kappa(\Sigma) = 1$, then Σ is either a one-holed torus or a four-holed sphere, abbreviated respectively to 1HT and 4HS. These cases require some special attention.

We write $\mathcal{T}(\Sigma)$ for the Teichmüller space of Σ — the space of marked finite-type conformal structures on $\text{int}(\Sigma)$.

Let $X(\Sigma)$ be the set of simple non-trivial non-peripheral closed curves in Σ , defined up to homotopy. We shall frequently refer to elements of $X(\Sigma)$ simply as “curves”. The curve graph, $\mathcal{G} = \mathcal{G}(\Sigma)$ has vertex set $X(\Sigma)$, and two curves are deemed to be adjacent if they can be realised disjointly in Σ [Har]. We write $d_{\mathcal{G}}$ for the combinatorial metric on $\mathcal{G}(\Sigma)$. It was shown in [MasM1] that is hyperbolic in the sense of Gromov [Gr] (see also [Bow1], [Ham2]), provided $\kappa(\Sigma) \geq 2$. In [Kl], the Gromov boundary $\partial\mathcal{G}(\Sigma)$ is identified with the space of arational laminations on Σ (see also [Ham1]). In this paper, we will describe end invariants in terms of $\partial\mathcal{G}(\Sigma)$, rather than the more traditional laminations, so we shall bypass this particular discussion. The equivalence of these formulations is not hard to see given the above.

In the special case where $\kappa(\Sigma) = 1$ we replace $\mathcal{G}(\Sigma)$ by the Farey graph, $\mathcal{G}'(\Sigma)$, with vertex set $X(\Sigma)$ and two curves deemed adjacent if they have minimal intersection. In this case arational laminations correspond to points of $\partial\mathcal{G}'(\Sigma)$ (irrational points on the circle). This identification has long been well understood.

We need to clarify how we understand the “marking” of end invariants. Suppose that Ψ is a 3-manifold with a topologically finite end, e . This means that there is a compact surface, Σ , and a proper injective map $\theta : \Sigma \rightarrow \Psi$ so that $\theta(\Sigma \times [0, \infty))$ is a neighbourhood of the end (and hence a homeomorphism to its range). If $\theta' : \Sigma' \times [0, \infty) \rightarrow \Psi$ is another such map, then there is a canonically defined homotopy equivalence from Σ to Σ' — take any $t \geq 0$ large enough so that $\theta(\Sigma \times \{t\}) \subseteq \theta'(\Sigma' \times [0, \infty))$, and postcompose $(\theta')^{-1} \circ (\theta|_{\Sigma \times \{t\}})$ with projection to Σ' . This homotopy equivalence respects the peripheral structure of these surfaces. We note, in particular, that Σ and Σ' are homeomorphic. This gives us a basis for using e (thought of formally as a directed set of subsets of Ψ) as a topological model for marking structures associated to Σ . In particular, we can define the Teichmüller space, $\mathcal{T}(e)$ associated to e by canonically identifying it with $\mathcal{T}(\Sigma)$ via θ . Similarly, we define the curve graph $\mathcal{G}(e)$ by identifying it with $\mathcal{G}(\Sigma)$. Note that any element of $V(\mathcal{G}(e))$ can be realised as a curve in any neighbourhood of e . In the case where $\kappa(\Sigma) = 1$, we can similarly define $\mathcal{G}'(e)$ to be the Farey graph, $\mathcal{G}'(\Sigma)$. We refer to $\Sigma = \Sigma(e)$ as the *base surface* of the end (which implies an implicit choice of map θ). In what follows, we reinterpret $\mathcal{G}(\Sigma)$ to mean the Farey graph, $\mathcal{G}'(\Sigma)$ if Σ is a 1HT or a 4HS.

Suppose we have a homeomorphism $f : \Psi \rightarrow \Psi'$ between two such manifolds. This will associate to each end, e , of Ψ , an end of Ψ' , which we denote by $f(e)$. Moreover, there is a canonical homotopy equivalence between the base surfaces, respecting their peripheral structure. Thus we get induced isomorphisms $f_* : \mathcal{T}(e) \rightarrow \mathcal{T}(f(e))$ and $f_* : \mathcal{G}(e) \rightarrow \mathcal{G}(f(e))$.

Now suppose that M is a complete hyperbolic 3-manifold, and that $\Psi(M)$ is the non-cuspidal part with respect to some Margulis constant $\eta > 0$. We write $\mathcal{E}(M)$ for the set

of ends of $\Psi(M)$. Thus $\mathcal{E}(M)$ is finite and is partitioned into geometrically finite and simply degenerate ends: $\mathcal{E}(M) = \mathcal{E}_F(M) \sqcup \mathcal{E}_D(M)$. We note that these sets and the associated constructions are canonically defined irrespective of the choice of (sufficiently small) Margulis constant $\eta > 0$. For the purpose of stating the Ending Lamination Conjecture, we could therefore fix some number $\eta > 0$ independently of Ψ . For the purposes of the proof however it is convenient allow ourselves the freedom to choose η depending on the topological type of Ψ , or more precicely, on the maximal complexity of base surfaces of ends. For the remainder of the discussion we will assume that η is fixed.

To each end $e \in \mathcal{E}_F(M)$ we have an end invariant $a(e) \in \mathcal{T}(e)$, namely a component of the quotient of the discontinuity domain in $\partial\mathbf{H}^3$, appropriately marked.

Suppose that $e \in \mathcal{E}_D(M)$. We fix a homeomorphism with a neighbourhood of this end with $\Sigma \times [0, \infty)$, and abuse notation slightly by identifying e with this neighbourhood. We take, as our starting point, the fact that there are an infinite number of curves in $X(\Sigma)$ that can be realised by curves of bounded length in Σ . (By our definition, two realisations of the same curve will be homotopic in e .) Here we could interpret the bound to be dependent on the geometry, though (as we shall see) it can always be chosen to depend only on $\kappa(\Sigma)$.

In what follows, we reinterpret $\mathcal{G}(\Sigma)$ to mean the Farey graph, $\Gamma'(\Sigma)$ if Σ is a 1HT or a 4HS.

The following is a standard fact about the geometry of such an end.

Proposition 1.1 : *There is some $a(e) \in \partial\mathcal{G}(e)$ such that of $(\gamma_i)_{i \in \mathbf{N}}$ is any sequence of distinct elements of $X(\Sigma)$ which have representatives of bounded length in e , then γ_i tends to $a(e)$ in $\mathcal{G}(e) \cup \partial\mathcal{G}(e)$.* \diamond

Here we take the usual topology of $\mathcal{G}(e) \cup \partial\mathcal{G}(e)$, in the sense of Gromov hyperbolic spaces (see, for example [GhH]). Again, this holds true for any length bound, though it would be enough to fix a certain bound depending only on $\kappa(\Sigma)$. Clearly, $a(e) = a(M, e)$ is uniquely defined, and is referred to as the *end invariant*.

Proposition 1.1 arises from reintrepreting the more traditional description of an ending lamination, see [K]. At least modulo this reinterpretation, Proposition 1.1 has been known from the work of Thurston and Bonahon [Bon]. Some refinements that we require here, will be described in Section 2.

We can now give a more formal statement of the Ending Lamination Conjecture.

Theorem 1.2 : *Suppose that M and M' are complete hyperbolic 3-manifolds with finitely generated fundamental groups. Suppose that $f : \Psi(M) \rightarrow \Psi(M')$ is a homeomorphism between the non-cuspidal parts such that for each $e \in \mathcal{E}(M)$, we have $f_*(a(M, e)) = a(M', f(e))$. Then there is an isometry $g : M \rightarrow M'$ such that $g|_{\Psi(M)} : \Psi(M) \rightarrow \Psi(M')$ is properly homotopic to f .* \diamond

This can, of course, be rephrased in terms of “marking” of the non-cuspidal parts by a topological model, as in Section 0.

The strategy for proving Theorem 2.1 is as follows. Starting from our topological manifold, Ψ , a partition of its ends into two sets deemed “geometrically finite” or “simply degenerate”, and an assignment of an element $a(e) \in \partial\mathcal{G}(e)$ to each simply degenerate end, e , we construct a “geometric model”, P . This a riemannian manifold, with a preferred

submanifold, $\Psi(P)$, called the “non-cuspidal part” of P . This model has the following property:

Proposition 1.3 : *Suppose that M is a complete hyperbolic 3-manifold, and that $g' : \Psi \rightarrow \Psi(M)$ is a homeomorphism. Suppose that g respects the partition of ends into geometrically finite and simply degenerate, and suppose that $g_*(a(e)) = a(M, g(e))$ for each simply degenerate end. Then there is a proper lipschitz homotopy equivalence $f : P \rightarrow M$ such that $f^{-1}(\Psi(M)) = \Psi(P)$, and $f|_{\Psi(P)} : \Psi(P) \rightarrow \Psi(M)$ is properly homotopic to g , and such a lift of f to the universal covers, $\tilde{f} : \tilde{P} \rightarrow \tilde{M} \equiv \mathbf{H}^3$ is a quasi-isometry.*

Now if M' is another manifold with the same topological type and degenerate end invariants, we have another map, $f' : \Psi \rightarrow \Psi(M')$ as above. We thus get an equivariant quasi-isometry between $\tilde{M} \equiv \mathbf{H}^3$ and $\tilde{M}' \equiv \mathbf{H}^3$.

If it happens that the geometrically finite end invariants are also equal, then we can go on to deduce, by established arguments, that there is an equivariant isometry between the universal covers. In other words, M and M' are isometric by an isometry in the appropriate homotopy class, thereby proving Theorem 1.2. This will be discussed in Section 8.

2. Degenerate ends.

Most of the work of this paper is to explain how the analysis of the geometry of simply degenerate ends can be effectively reduced to the incompressible case. We begin here by elaborating further on their geometry. Since, by tameness, all non-geometrically finite ends are known to be simply degenerate, we will omit the adverb “simply” from subsequent discussion.

Let M be a hyperbolic 3-manifold with end $e \in \mathcal{E}_D(M)$. Let $\mathcal{G}(e) \equiv \mathcal{G}(\Sigma)$ be the curve graph, where $\Sigma = \Sigma(e)$ is the base surface. As usual, we take this to mean the Farey graph, $\mathcal{G}'(\Sigma)$, when Σ is a 1HT or a 4HS. Recall that $\kappa(\Sigma)$ is the complexity of Σ , which here serves only as convenient notation to control the topological type of Σ .

Note that we will be using the term “geodesic” in two different senses. A geodesic in $\mathcal{G}(e)$ will be a globally distance minimising path, in the usual combinatorial sense. A geodesic in M will, unless otherwise specified, will only be assumed to be locally distance minimising, i.e. geodesic in the riemannian sense. We will often abuse notation and regard such a geodesic as a subset of M , even if it is not injective.

Proposition 2.1 : *There is some constant, $L = L(\kappa(\Sigma))$ depending only on $\kappa(\Sigma)$ such that there is a geodesic ray $(\gamma_i)_{i \in \mathbf{N}}$ in $\mathcal{G}(e)$ such that for all i , γ_i is represented by a closed curve in e of length at most L .*

In fact, as we shall see, these curves can be taken to be geodesic in M .

In [MasM2] Masur and Minsky introduced a preferred class of “tight” geodesics in $\mathcal{G}(\Sigma)$. A slightly modified version is used in [Bow2]. In any case, we shall only be quoting

results about tight geodesics from elsewhere, so we not worry about the definition here. A key fact is that any two vertices of $\mathcal{G}(\Sigma)$ are connected by a non-empty finite set of tight geodesics. The ray in Proposition 2.1 can be taken to be tight. In fact:

Proposition 2.2 : *There is some constant, $L = L(\kappa(\Sigma))$ depending only on $\kappa(\Sigma)$ such that if $(\gamma_i)_{i \in \mathbf{N}}$ is any tight geodesic ray in $\mathcal{G}(e)$ tending to $a(M, e) \in \partial\mathcal{G}(e)$, then, for all sufficiently large i , γ_i is represented by a curve of length at most L in e (which can be taken to be geodesic in M).*

These results will be proven in Section 7. In the incompressible case, Proposition 2.2 is a variant of the ‘‘a-priori bounds’’ theorem of Minsky [Mi2], reproven in this form, in [Bow2]. We shall explain how the arguments of [Bow2] can be adapted to this situation. This result, or rather its refinement, described at the end of Section 7, are key to establishing that the map from the model space to the hyperbolic 3-manifold is lipschitz. From that point on, only a few minor modifications of the proof given on [Bow3] are required. These are discussed in Section 8.

In order to reduce to the incompressible case, we will need a purely topological observation, which we can give at this point.

Lemma 2.3 : *Suppose that N is a 3-manifold with boundary, ∂N , and that $F \subseteq N$ is a compact subsurface such that no boundary component of F is homotopically trivial in N . Suppose that F can be homotoped in N into some closed subset $K \subseteq N \setminus F$. Then F is π_1 -injective in $N \setminus K$.*

Proof : We begin with a preliminary observation. Suppose that $S \subseteq \partial N$ is a compact subsurface homotopic to a point in N . Then S is planar (i.e. genus 0). This can be shown by using Dehn’s Lemma [He] to cut S into 3-holed spheres glued along boundary curves which bound disjoint embedded discs in N . These must be connected in a treelike fashion, since any closed cycle of adjacencies would give rise to curve in S that is non-trivial in N . In other words, S is planar.

Now suppose, for contradiction, that $\pi_1(F)$ does not inject into $\pi_1(N \setminus K)$. After lifting to the cover of N corresponding to F , we can assume that F carries all of $\pi_1(N)$. (Note that the homotopy of F into K also lifts, as does any disc spanning a non-trivial curve in F .)

By Dehn’s Lemma, there is an embedded disc $D \subseteq N \setminus K$ with $\partial D = D \cap \partial N = D \cap F$ a non-trivial curve in F . Let U be a small open product neighbourhood of D in $N \setminus K$ so that $U \cap \partial N = U \cap F$ is a small annular neighbourhood of ∂D in F . Let $P = N \setminus U$.

If P is connected then it determines a splitting of $\pi_1(N)$ as a free product $\pi_1(N) \cong \pi_1(P) * \mathbf{Z}$. But F is homotopic into $K \subseteq P$, and so $\pi_1(N)$ is conjugate into the $\pi_1(P)$ factor giving a contradiction.

Thus P has two components, P_0 and P_1 . We can suppose that $K \subseteq P_1$. Now $\pi_1(N) \cong \pi_1(P_0) * \pi_1(P_1)$ and as above, $\pi_1(N)$ is conjugate into $\pi_1(P_1)$. Thus $\pi_1(P_0)$ is trivial, and so in particular $P_0 \cap F$ is homotopic to a point in N . It follows by the observation of the first paragraph, that $P_0 \cap F$ is planar. Since ∂D is non-trivial in F , $P_0 \cap F$ cannot

be a disc. It must therefore have another boundary component, say $\beta \subseteq P_0 \cap \partial F$. But, by hypothesis, β is non-trivial in N , giving a contradiction. \diamond

We shall apply Lemma 2.3 in the following form. Suppose that $e \cong \Sigma \times [0, \infty)$ is an end of $\Psi(M)$. We shall write $\partial_H e = \Sigma \times \{0\}$ for the relative or *horizontal* boundary of e in $\Psi(M)$. This determines a proper homotopy class of proper maps of $\text{int } \Sigma$ into M , called the *ambient fibre class* — “ambient” referring to homotopies that take place in M . Suppose that $\phi : \text{int } \Sigma \rightarrow M$ is a proper map in this class, and suppose moreover that $\phi^{-1}\Psi(M)$ is compact. We can shrink e (i.e. replace it by $\Sigma \times [t, \infty)$ for some $t \geq 0$) so that $e \cap \phi(\text{int}(\Sigma)) = \emptyset$. Let U be the component of $M \setminus \phi(\text{int}(\Sigma))$ containing e . Then e is π_1 -injective in U .

This can be seen, for example, by taking $N = M \setminus \text{int}(e)$, so that $\partial N = \partial e = \partial_H e \cup \partial_V e$, and setting $F = \partial_H e$. Note that M retracts onto N , and so F is homotopic into $K = \phi(\text{int } \Sigma)$ in N . We now apply Lemma 2.3, to see that F and hence e is π_1 -injective in U as claimed.

3. Pleating surfaces.

The original notion of a “pleated surface” surface goes back to Thurston [T] (see [CanEG] for a more detailed discussion). The notion is quite robust in that many of the basic properties survive some tinkering with the definition. We shall find it convenient to use more than one formulation in this paper. In this section, we restrict attention to maps into a hyperbolic 3-manifold, though much of it can be interpreted more generally as we explain in Section 4. Since our definition does not quite coincide with the standard one, we use the term “pleating surface” instead.

Let (M, d) be a complete hyperbolic 3-manifold. We fix a Margulis constant and let $\Psi(M)$ be the non-cuspidal part. Let $\Theta(M) \subseteq \Psi(M)$ be the *thick part* of M , i.e. $\Psi(M)$ with open Margulis tubes removed. (For some applications, it is convenient to allow the Margulis constants defining different tubes to vary between two suitably chosen positive constants, as in [Bow3] for example. This makes no essential difference to what we have to say, and would only confuse the exposition here.)

Given $x, y \in \Psi(M)$, let $\rho(x, y)$ be the infimum (in fact minimum) of $\text{length}(\beta \cap \Theta(M))$ as β varies over all paths from x to y in $\Psi(M)$. Thus, $(\Psi(M), \rho)$ is a pseudometric space. We refer to this process as “collapsing” the Margulis tubes (even though we consider the topology on $\Psi(M)$ to remain unchanged).

Let Σ be a compact surface and let $S = \text{int } \Sigma$. We say that a homotopy class of maps from Σ into M (or equivalently S) into M is *type-preserving* if it sends each boundary component of Σ (or equivalently each end of S) homotopically to a generator of a \mathbf{Z} -cusp of M . We refer to this as the *associated cusp*. We remark that if two proper type preserving maps of S are homotopic then they are, in fact, properly homotopic.

Definition : A *pleating surface* is a uniformly lipschitz type preserving map $\phi : (S, \sigma) \rightarrow (M, d)$, where σ is a finite area hyperbolic metric on S .

By “uniformly lipschitz” we mean μ -lipschitz for some $\mu \geq 0$. In many situations, as in the traditional notion of pleated surface, one can take $\mu = 1$, though we might want to allow for a larger constant depending only on $\kappa(\Sigma)$. The metric $\sigma = \sigma_\phi$ is regarded as part of the data of the pleating surface. Note that ϕ is necessarily proper.

In practice, the pleating surfaces we deal with will all have the property that any ray going out a cusp of S will, from a certain point on, get sent to a ray going out the associated cusp of M .

Given some $\alpha \in X(\Sigma)$, we write α_S for its geodesic realisation in (S, σ) . Similarly, we write β_M for the geodesic realisation of a homotopy class, β , of closed curves in M — that is assuming β is non-trivial and non-parabolic. If we fix a homotopy class of type-preserving maps $\phi : S \rightarrow M$, we write $\bar{\alpha} = (\phi(\alpha))_M$. The notation is taken to imply that this exists.

Definition : A pleating surface, $\phi : S \rightarrow M$ is said to *realise* $\alpha \in X(\Sigma)$ if $\phi|_{\alpha_S}$ maps α_S locally isometrically and with degree ± 1 to $\bar{\alpha}$.

By a *multicurve* in Σ we mean a non-empty (finite) set of elements of $X(\Sigma)$ which can be realised disjointly in Σ . We say that a pleating surface realises a multicurve if it realises every component thereof.

Lemma 3.1 : *Suppose we are given a type-preserving homotopy class, ϕ , from S into M , and a multicurve, γ , in Σ , such that the ϕ -image of component of γ is non-trivial and non-parabolic in M . Then there is a pleating surface in this class realising γ . \diamond*

This is a standard construction, due to Thurston, which we shall discuss shortly. First, we shall consider some consequences, and related results.

The following are now fairly routine observations. Suppose that $\phi : S \rightarrow M$ is a π_1 -injective pleating surface. Then S with the preimages of the associated cusps removed has a *principal component*, F , which carries all of $\pi_1(\Sigma)$. In fact, by choosing the Margulis constant of M sufficiently small in relation to $\kappa(\Sigma)$, we can assume that F contains each horocycle of length 1. Now it is well known that any simple geodesic in S cannot cross any such horocycle, and so must lie inside F . As an immediate consequence, applying Lemma 3.1, we have:

Lemma 3.2 : *Suppose we have a π_1 -injective homotopy class, $\phi : S \rightarrow M$, and $\alpha \in X(\Sigma)$ such that $\phi(\alpha)$ is non-parabolic in M . Then the closed geodesic in M in the class of $\phi(\alpha)$ cannot enter any of the cusps of M associated to ϕ . \diamond*

We will be applying this principle in a slightly different form — see Lemma 5.4.

Returning to our π_1 -injective pleating surface ϕ , another observation is that each component of $\phi^{-1}\Theta(M)$ has bounded diameter. It follows easily that the ρ -diameter of $\phi(F)$ is bounded. In fact:

Lemma 3.3 : *There is some $r = r(\kappa(\Sigma))$ such that if $\phi : S \rightarrow M$ is a π_1 -injective pleating surface whose image, $\phi(S)$, only meets the associated cusps. Then the ρ -diameter of $\phi(S) \cap \Psi(M)$ is at most r .*

Proof : In this case, the principle component, F , is a component of $\phi^{-1}\Psi(M)$, and we have seen that $\phi(F)$ has bounded ρ -diameter. Any other components will be homotopic into an associated cusp. These are easily dealt with, but in practice we won't need to worry about them. \diamond

Pleated surfaces in this form are somewhat clumsy to deal with. We can perform a “tidying up” operation. At the cost of increasing the lipschitz constant by a bounded amount, we can assume that the principle component, F , is the same as $\phi^{-1}\Psi(M)$ and that its boundary consists entirely of horocycles of length 1. All that is now relevant is the metric restricted to F , which we can identify with the original surface Σ .

We will also require a somewhat deeper result concerning pleating surfaces, namely a version of the Uniform Injectivity Theorem, the original being due to Thurston.

Let $E \rightarrow M$ be the projectivised tangent bundle of M . (We can think of it as the unit tangent bundle factored by an involution.) If $\phi : S \rightarrow M$ is a pleating surface realising a multicurve γ , we can lift $\phi|_{\gamma_S}$ to a simple curve $\gamma_E \subseteq E$. We write $\psi = \psi_\phi : \gamma_S \rightarrow \gamma_E$ for the lift of $\phi|_{\gamma_S}$. Thus, ψ is a homeomorphism and a local isometry with respect to the metrics σ and d_E .

Lemma 3.4 : *Given $\kappa, \mu, \eta, \epsilon > 0$, there is some $\delta > 0$ with the following property. Suppose that $S = \text{int } \Sigma$ is a surface with $\kappa(\Sigma) = \kappa$. Suppose that $\phi : S \rightarrow M$ is a μ -lipschitz pleating surface and let $\psi : \gamma_S \rightarrow \gamma_E \subseteq E$ is the lift described above. Suppose that there is some $\eta > 0$ such that the injectivity radius of M at each point of $\gamma_M = \phi(\gamma_S)$ is at least η . Suppose moreover that there is a map $\theta : N(\gamma_M, \eta) \rightarrow S$ such that $\theta \circ \phi : N(\gamma, \eta/\mu) \rightarrow S$ is homotopic to the inclusion of $N(\gamma, \eta/\mu)$ into S . If $x, y \in \gamma_S$ with $d_E(\psi(x), \psi(y)) \leq \delta$, then $\sigma(x, y) \leq \epsilon$.*

Here $N(., r)$ denotes the open r -neighbourhood. The map θ need only be defined up to homotopy.

Lemma 3.4 is an immediate consequence of the statement for laminations given as Proposition 9.1, and we postpone the discussion until then.

We now go back to discuss some constructions of pleating surfaces that will be needed later.

Suppose that $\phi : S \rightarrow M$ is a type-preserving homotopy class (not necessarily π_1 -injective) and $\alpha \in X(\Sigma)$ so that $\phi(\alpha)$ is non-trivial and non-parabolic. We realise α as some smooth curve, and choose any $x \in \alpha$, and extend this to an ideal triangulation of S , whose edges are loops based at x (including α) as well as properly embedded rays going out the cusps. We now chose any $y \in \bar{\alpha} = \phi(\alpha_M)$ and realise all these edges as geodesic loops or rays based at y (so that x gets sent to y , and α to $\bar{\alpha}$). We then extend to S by sending 2-simplices homeomorphically to totally geodesic triangles in M . We pull back the metric to S to give a pseudometric σ on S . The realisation $\phi : S \rightarrow M$ is then 1-lipschitz. This is the construction used by Bonahon in [Bon]. To obtain a hyperbolic metric on the domain, we need to adjust this somehow. One way is to use the “spinning” construction of Thurston. Note that there is a real line's worth of possibilities for y (that is, taking account of the based homotopy class of our realisation $(S, x) \rightarrow (M, y)$). By sending y off to infinity we converge on a pleated surface in the traditional sense. We note that, in

fact, the same argument can be applied to any multicurve as in the hypotheses of Lemma 3.1.

Returning to Bonahon's construction, we shall refer to a surface arising in this way as a *folding surface*. The essential point is that it is piecewise totally geodesic. This is sufficient for many purposes, but in order to ensure we get a metric on the domain, we need another assumption.

Definition : We say that a folding surface is *non-degenerate* if no 2-simplex in S gets collapsed to a geodesic (or point) in M .

In this case, the pull-back pseudometric is a metric. In fact it is hyperbolic apart from a singularity of angle at least 2π at the vertex x .

To ensure that we can arrange that our folding surface is non-degenerate, we allow ourselves to move y . It is not hard to see that any given triangle can be degenerate for at most a discrete set of y , and so these points can all be avoided by a small perturbation.

In summary, we have shown the following variation on Lemma 3.1 (which we only need and state for a single curve).

Lemma 3.5 : *Suppose we have a type-preserving homotopy class of maps from S into M and that $\alpha \in X(\Sigma)$ whose image in M is non-trivial and non-parabolic. Then there is an non-degenerate folding surface realising α .* \diamond

Let $\Delta_y(M)$ be the unit tangent space to M at y , which we can identify with the unit sphere. Suppose we have a non-degenerate piecewise totally geodesic map $\phi : S \rightarrow M$. Then any point $z \in S$ determines a closed polygonal path, $\zeta(z)$, in $\Delta_{\phi(z)}(M)$.

Definition : We say that ϕ is *balanced* if for all $x \in S$, $\zeta(x)$ is not contained in any open hemisphere.

Note that this is only an issue at the vertices of the triangulation of Σ . At the vertex x of a folding surface, $\zeta(x)$ contains two antipodal points coming from $\bar{\alpha}$. Therefore:

Lemma 3.6 : *Any non-degenerate folding surface (of the type featuring in Lemma 3.5) is balanced.* \diamond

4. Negatively curved spaces.

We will need to consider a notion of pleating surface in a broader context than that discussed in Section 3, in particular when the range is “negatively curved”, with upper curvature bound -1 . Morally, having concentrated negative curvature can only work in our favour, though it introduces a number of technical issues that need to be addressed. Most of what need can be phrased in terms of locally $\text{CAT}(-1)$ metrics, though in practice, all our metrics will be at least piecewise riemannian. In particular, we have a natural notion of area. We remark that $\text{CAT}(-1)$ geometry has been used in a related context in [So].

Let (R, d) be a metric space. A (*global*) *geodesic* in R can be defined as a path whose rectifiable length equals the distance between its endpoints. We say that (R, d) is a *geodesic space* if every pair of points can be connected by a geodesic. We can also define a *local geodesic* in the obvious manner, which for a riemannian metric coincides (up to parameterisation) with the riemannian notion. We shall abbreviate “closed local geodesic” to “closed geodesic” since there can be no confusion in that case.

For any $k \in \mathbf{R}$, we have the notion of a “CAT(k)” (or “locally CAT(k)”) space which satisfies the CAT(k) comparison axiom globally (or locally). We refer to [BriH] for a detailed account of such spaces. The Cartan-Hadamard Theorem in this context says that if $k \leq 0$, then a locally CAT(k) space is globally CAT(k) if and only if it is simply connected. For a proper (complete locally compact) CAT(-1) space we have the usual classification of isometries into elliptic, parabolic and loxodromic. We are only interested here in discrete torsion-free groups, so there are no elliptics. This gives rise to the following “thick-thin” decomposition.

Let (R, d) be a proper locally CAT(-1) space and $\eta > 0$. Let $\tau(R)$ be the set of x such that x lies in a homotopically non-trivial curve γ of length less than η . We write $\tau_0(R) \subseteq \tau(R)$ for the set of x such that some such γ can be homotoped in R to be arbitrarily short. If α is a closed geodesic, we write $\tau(R, \alpha) \subseteq \tau(R)$ for the set of x such that some such γ can be homotoped to some multiple of α . We write $\tau_+(R)$ for the union of all $\tau(R, \alpha)$ as α varies over all closed geodesics in R . One can show that all of these sets are open, and that $\tau(R) = \tau_0(R) \cup \tau_+(R)$. (Without a lower curvature bound, these sets need not be disjoint.) We write $\Psi(R) = \Psi_\eta(R) = R \setminus \tau_+(R)$ and $\Theta(R) = \Theta_\eta(R) = R \setminus \tau(R)$. (If R is a complete hyperbolic 3-manifolds, these sets agree with those already defined.)

Suppose that $S = \text{int } \Sigma$ is a finite type surface with a complete locally CAT(-1) metric. We say that S has *finite area* if $\Psi_\eta(S)$ is compact for some (hence any) $\eta > 0$.

An example of such a surface would be a riemannian metric of curvature at most -1 , possibly with a discrete set of cone points of angles at least 2π . More generally we can allow a piecewise riemannian metric which is riemannian outside an embedded 1-complex which has cone angles at the vertices and where the edges are smooth and outwardly curved with respect to the metric on each side. Note that in this situation, the notion of “finite area” coincides with the usual one.

We remark that we can also allow for S to have boundary components. In this case, there need be no constraint on the geometry at the boundary.

We will use the following means of constructing a locally CAT(-1) by cutting up a hyperbolic manifold.

Let M be a complete hyperbolic 3-manifold. We say that an open subset $U \subseteq M$ is *polyhedral* if ∂U is an embedded locally finite 2-complex, where all the simplices are totally geodesic, and each edge is contained in at least one 2-simplex. Let Π be the metric completion of U , and set $\partial\Pi = \Pi \setminus U$. There is a natural map, $\pi : \Pi \rightarrow M$ which is injective on U . Note that Π is locally compact. We write d_Π for the induced path metric on Π (which is the same as the completion of the induced path metric on U). Thus, (Π, d_Π) is a geodesic space.

If $x \in \Pi$, we can define the unit tangent space $\Delta_x(\Pi)$ of Π at x in the obvious sense. Removing those tangents in the boundary, we obtain $\Delta_x(\Pi, U)$. There is a natural map

$\pi : \Delta_x(\Pi) \longrightarrow \Delta_{\pi(x)}(M)$. This is injective on $\Delta_x(\Pi, U)$.

Lemma 4.1 : *Suppose that Π is a polyhedral set of the type described above. Then Π is locally CAT(-1) if and only if $\Delta_x(\Pi)$ is globally CAT(1) for all $x \in \partial\Pi$.*

Proof : This is a standard fact about a polyhedral complex built out of hyperbolic polyhedra (see [BriH]). One can triangulate Π so that it has such a structure (though this is a somewhat artificial given that the relevant arguments apply directly to this situation). \diamond

Now, $\Delta_x(\Pi)$ is necessarily locally CAT(1), and the statement that it is globally CAT(1) is equivalent to asserting that there is no (intrinsic) closed geodesic of length less than 2π .

Given $y \in M$, write $\Delta_y(M, \partial U) \subseteq \Delta_y(M)$ for those tangent vectors lying in ∂U . This is a 1-complex in $\Delta_y(M)$ with geodesic edges.

Lemma 4.2 : *Suppose that Π is a polyhedral complex as constructed above. If for all $y \in M$, no component of $\Delta_y(M, \partial U)$ is contained in an open hemisphere of $\Delta_y(M)$, then Π is locally CAT(-1).*

Proof : Note that if $x \in \partial\Pi$, then $\Delta_x(\Pi, U)$ maps injectively onto a component of $\Delta_{\pi(x)}(M) \setminus \Delta_{\pi(x)}(M, \partial U)$. Now any intrinsic geodesic of length less than 2π in $\Delta_x(\Pi)$ would map to a closed path in $\Delta_{\pi(x)}(M)$ enclosing a component of $\Delta_{\pi(x)}(M, \partial U)$ contained in an open hemisphere, contrary to the hypotheses. \diamond

We also note that if for all y , each component of $\Delta_y(M) \setminus \Delta_y(M, \partial U)$ is a topological disc, then U is π_1 -injective in Π .

Now all the above criteria hold for the image of a non-degenerate balanced piecewise geodesic surface, as defined in Section 3. Thus:

Lemma 4.3 : *Suppose that M is a complete hyperbolic 3-manifold, and $\phi : S \longrightarrow M$ is a proper non-degenerate balanced piecewise geodesic map, and let U be a component of $M \setminus \phi(S)$, and let Π be the metric completion of U . Then Π is locally CAT(-1). Moreover, U is π_1 -injective in Π .* \diamond

In particular, this applies to a folding map, as in Lemma 3.6.

Now suppose that (Π, d_Π) is a proper locally CAT(-1) space. Much of the discussion of pleating surfaces in Section 3 applies with M replaced by Π . We can define a pleating surface as a type preserving uniformly lipschitz map $\phi : (S, \sigma) \longrightarrow (\Pi, d_\Pi)$ where σ is a finite area locally CAT(-1) metric on the surface S . In practice, we only need to consider a polyhedral space Π of the type constructed above, and piecewise riemannian metrics (or pseudometrics) on S . Moreover, we can assume that the cusps of S are isometric to standard hyperbolic cusps in some neighbourhood of the end. In this setting we only consider π_1 -injective pleating surfaces.

We have an exact analogue of Lemma 3.1. The map can be constructed via the folding construction. The only difficulty is that the triangles can no longer be assumed to lie in totally geodesic subspaces. We therefore need to use ruled surfaces: coning an edge over the

opposite vertex. If one vertex is ideal, we cone over this vertex. This construction makes sense in any locally CAT(-1) space. It is a general fact that the pull-back (pseudo)metric on S is locally CAT(-1).

There are some technical issues involved here, but we need not worry about the details of the general case. In practice, our surface is built out of triangles obtained by coning a $d_{\mathbb{H}}$ -geodesic segment over a (possibly ideal) point. It is thus foliated by lines, the “generators” meeting at the vertex. It would be tedious to enumerate all kinds of local structure such a surface may have. However, it is not hard to see that such a triangle can be subdivided into cells. Each 1-cell of the cellulation either lies in a generator or is transverse to the generators. In the latter case, it maps to a (geodesic) 1-cell in the polyhedral complex $\partial\Pi$, or perhaps collapses to a point. The 2-cells are either totally geodesic, or have the form of a ruled surface joining two skew lines in hyperbolic 3-space. It is possible, in the former case, that the cell may collapse onto a geodesic segment. In all other cases, the pull-back metric to the cell is (non-singular) riemannian. Any edge that is not a generator will have non-negative outward curvature. If the metric does not degenerate at a particular vertex, then the angle sum around that vertex will be at least 2π . This needs to be appropriately interpreted when edges of the cellulation collapse to points — the adjacent 2-cells do not contribute to the angle sum. In any case, the Gauss-Bonnet Theorem can be seen to apply to such a metric. (One can simply sum the contributions of the cells, making use of the angle sum inequality at the vertices, and the Euler formula.) In particular, this allows us to bound area in terms of the topological type, and hence diameters of the components of the thick part.

5. Isolating simply degenerate ends.

Suppose that $e \in \mathcal{E}_D(M)$. We want to focus our attention on the intrinsic geometry of e , so that the arguments of the incompressible case will apply equally well. We will do this by looking at the complement of a bending surface. (A different trick of passing to branched covers was used by Canary [Can], and could probably be used here also. It has the advantage of producing a riemannian metric, though it does raise some other issues that would need to be checked.)

We will write $e \cong \Sigma \times [0, \infty)$. Recall that $\partial_H e = \Sigma \times \{0\}$ is the relative boundary of e in $\Psi(M)$. Let $S = \text{int } \Sigma$. Note that e determines a homotopy class of type-preserving maps from S into M , which we refer to as the *ambient fibre class*.

To get started we note:

Lemma 5.1 : *There is some $\alpha \in X(e)$ which is homotopic in M to a closed geodesic $\bar{\alpha}$ in M .*

In other words, we need one curve that is neither trivial nor parabolic. In fact, a basic property of a simply degenerate end is that there is a sequence of closed geodesics going out the end. This can be deduced from our description of a simply degenerate end by an argument similar to that of Lemma 5.6.

Now, by Lemma 3.5, there is a non-degenerate folding map $\phi_0 : S \rightarrow M$, in the ambient fibre class in M , realising α . As in the discussion at the end of Section 2, we can assume that $e \cap \phi_0(S) = \emptyset$. Let U be the component of $M \setminus \phi_0(S)$ containing e , and let Π be its metric completion. By Lemma 2.3, and the subsequent remarks, we know that e is π_1 -injective in U . By Lemma 4.3, it is also π_1 -injective in Π . By Lemmas 3.6 and 4.3, we see that Π is locally CAT(-1) in the induced path metric d_Π .

There is a canonical map, $\pi : \Pi \rightarrow M$, that is injective on U . For notational convenience in what follows, we will pretend that π is injective on $\partial\Pi$. (This is what one would expect generically.) We can thus identify Π as a subset of M . All our constructions have an obvious interpretation in the case where π might not be injective on $\partial\Pi$.

Recall, from Section 4, the definition of the thin part, $\tau(\Pi) = \tau_0(\Pi) \cup \tau_+(\Pi)$ of Π . The set $\tau_+(\Pi)$ is a union of $\tau(\Pi, \alpha)$ as α ranges over all closed d_Π -geodesics. If $\alpha \cap \partial\Pi = \emptyset$, then α is also a closed geodesic in M (in particular non trivial in M). If $\tau(\Pi, \alpha) \cap \partial\Pi = \emptyset$, then $\tau(\Pi, \alpha)$ is a Margulis tube in M with respect to the hyperbolic metric. Suppose that α, β are closed d_Π -geodesics with $\tau(\Pi, \alpha) \cap \tau(\Pi, \beta) \cap U \neq \emptyset$. Let x be a point of this intersection. By definition of the sets $\tau(\Pi, \alpha)$ and $\tau(\Pi, \beta)$, x lies in curves, α' and β' , of length at most η , homotopic in Π respectively to powers of α and β . Then the Margulis lemma (applied in M) tells us that α' and β' generate an abelian subgroup of $\pi_1(M)$. Such a group must be trivial, loxodromic or parabolic in M . Thus, if $\alpha, \beta \subseteq U$, then there are both hyperbolic geodesics and hence equal. If $\alpha \neq \beta$, we conclude that at least one of α or β must meet $\partial\Pi$.

By similar reasoning, we see that if α is a closed d_Π -geodesic and $\tau(\Pi, \alpha) \cap \tau_0(\Pi) \cap U \neq \emptyset$, then $\alpha \cap \partial\Pi \neq \emptyset$.

Now, by discreteness, only finitely many $\tau(\Pi, \alpha)$ can meet $\partial_H e$, and these include all those meeting both e and $\partial\Pi$. Thus, after shrinking e , we can assume that if $\tau(\Pi, \alpha) \cap e \neq \emptyset$, then $\tau(\Pi, \alpha) \subseteq U$, and so it is a Margulis tube in M . We also assume that e only meets the associated cusps of M . From this, it follows easily that $\partial_V e = e \cap \Psi(M) = e \cap \Psi(\Pi)$. Thus $e \cap \Theta(M) = e \cap \Theta(\Pi)$.

Recall that, in Section 3, we defined a pseudometric, $\rho = \rho_M$, on M by collapsing the Margulis tubes. It will be more convenient, for the moment, to work with another metric, ρ_e , where we also collapse the complement of $e \in \Psi(M)$. In other words, $\rho_e(x, y)$, is the minimum length of $\beta \cap e \cap \Theta(M)$ as β ranges over all paths from x to y in $\Psi(M)$. We can also view this as a pseudometric on $\Psi(\Pi)$. Clearly $\rho_e \leq \rho_M$.

Given $x \in e$, we write $D(x) = \rho_M(x, \partial_H e) = \rho_e(x, \partial_H e)$ for the *depth* of x in e . We note that if $x, y \in e$ have depth greater than the ρ_M -diameter of $\partial_H e$, then $\rho_M(x, y) = \rho_e(x, y)$. Given a subset $Q \subseteq e$, write $D(Q) = \rho_e(Q, \partial_H e) = \inf\{D(x) \mid x \in Q\}$. We write $\rho_e\text{-diam}(Q)$ for the ρ_e diameter of Q .

There is natural type-preserving π_1 -injective homotopy class of maps $S \rightarrow \Pi$, determined by e , which we refer to as the *fibre class*.

Lemma 5.2 : *There is some $h_0 = h_0(\kappa(\sigma))$ such that if $\phi : S \rightarrow \Pi$ is a pleating surface in the fibre class, then $\rho_e\text{-diam}(\phi(S) \cap \Psi(\Pi)) < h_0$.*

Proof : In applications, we only need this where the (S, σ_ϕ) is piecewise riemannian. In

this case we can apply the Gauss-Bonnet Theorem, so the argument follows as with Lemma 3.3. \diamond

In particular, we get:

Lemma 5.3 : *If $\phi : S \rightarrow \Pi$ is a pleating surface in the fibre class, and $\phi(S) \cap e$ contains some point of depth at least h_0 in e , then $\phi(S) \cap \Psi(\Pi) \subseteq e$.* \diamond

Given $\alpha \in X(e) \equiv X(\Sigma)$, we write α_Π for its realisation as a closed geodesic in Π . Thus if $\alpha_\Pi \cap \partial\Pi = \emptyset$, then $\alpha_\Pi = \alpha_M$ is also a closed geodesic in M .

Lemma 5.4 : *Suppose that $\alpha \in X(e)$ and $\alpha_\Pi \cap e$ contains some point of depth at least h_0 . Then $\alpha_\Pi \subseteq e$, $\alpha_\Pi = \alpha_M$, and $\rho_e - \text{diam}(\alpha_\Pi) < h_0$.*

Proof : Let $\phi : S \rightarrow \Pi$ be a pleating surface in the fibre class realising α_Π . By Lemma 5.3, $\phi(S) \cap \Psi(M) \subseteq e$. Let F be the principal component of $\phi^{-1}(\Psi(\Pi)) = \phi^{-1}(e) \subseteq S$. A similar argument to that of Lemma 3.2 now applies. Assuming that we have chosen the Margulis constant, η , sufficiently small (depending on $\kappa(\Sigma)$), any simple geodesic in S will lie in F . (One way to see this is to note that any point in the non-cuspidal part of S lies in a non-trivial non-peripheral curve of bounded length in S . This maps to a curve of bounded length in Π , which will lie outside all Margulis cusps for sufficiently small Margulis constant. We have only used the fact that the curvature on S is at most -1 .) In particular, $\alpha_S \subseteq F$, and so $\alpha_\Pi = \phi(\alpha_S) \subseteq e$. Since $e \cap \partial\Pi = \emptyset$, it follows that $\alpha_\Pi = \alpha_M$. Finally, $\rho_e - \text{diam}(\alpha_\Pi) \leq \rho_e - \text{diam}(\phi(S) \cap \Psi(\Pi)) < h_0$. \diamond

Lemma 5.5 : *Suppose that $\alpha, \beta \in X(e)$ are adjacent in $\mathcal{G}(e)$ and $\alpha_\Pi \cap e$ contains some point of depth at least h_0 . Then $\beta_\Pi = \beta_M \subseteq e$, and $\rho_e - \text{diam}(\alpha_\Pi \cup \beta_\Pi) < h_0$.*

Proof : The proof follows that of Lemma 5.4. This time we take ϕ to realise the multicurve $\{\alpha, \beta\}$. \diamond

We remark that, in retrospect, we see that the pleating surfaces arising in the proofs of Lemmas 5.3 and 5.4 do not meet $\partial\Pi$. If they were constructed by the folding procedure described in Section 3, then all the 2-simplices would be totally geodesic in M . We can then proceed to spin around the curves to give us pleated surfaces whose domains are hyperbolic surfaces.

Lemma 5.6 : *For all $l \geq 0$, there is some $h_1(l)$ such that if $\alpha \in X(e)$ is realised by a curve, α_0 , in e of length at most l and containing a point of depth at least $h_1(l)$, then $\alpha_\Pi = \alpha_M \subseteq e$ and $\rho_e(\alpha_0 \cup \alpha_\Pi) < h_1(l)$.* \diamond

Proof : This is now a fairly standard argument (cf. [Bon]). We can realise the homotopy between α_0 and α_Π in Π by a ruled surface, $\phi : A \rightarrow \Pi$, where A is an annulus whose boundary components get mapped to α_0 and α_Π . This can be constructed as for pleated surfaces. The pull-back (pseudo)metric on A is locally $\text{CAT}(-1)$, and can be assumed

piecewise riemannian. (In practice, we can approximate α_0 by a piecewise geodesic curve, which that the surface will have the form described at the end of Section 4.) As with Lemma 5.2, we see that the $\rho_e - \text{diam}(\phi(A) \cap \Psi(\Pi))$ is bounded in terms of the area of A . (Note that there must be a path in $\phi^{-1}(\Psi(\Pi))$ connecting the two boundary components. Applying Gauss-Bonnet, this is in turn bounded in terms of the length of ∂A which is at most $2l$. In other words, there is some constant $h' = h'(l)$ such that $\rho_e - \text{diam}(\alpha_0 \cup \alpha_\Pi) \leq \rho_e - \text{diam}(\phi(A)) \leq h'$. We can now set $h_1 = h_0 + h'(l)$. It then follows that $D(\alpha_\Pi) > h_0$, and so by Lemma 5.4, $\alpha_\Pi = \alpha_M \subseteq e$ and $\rho_e - \text{diam}(\alpha_\Pi) < h_0$, and so the result follows. \diamond

We have shown the essential properties we need. In order to restrict our attention to the end, e , we can perform a few “tidying up” operations on pleating surfaces. We will only be interested in pleating surfaces realising multicurves of depth at least h_0 . As observed after Lemma 5.5, such a surface can be assumed to have domain a hyperbolic surface. Then, as discussed in Section 4, at the cost of increasing the lipschitz constant, we can assume that the preimage of the non-cuspidal part is a core bounded by horocycles of some fixed length. We can assume that these horocycles get mapped to euclidean geodesics in $\partial_V e$. From this point on, the remainder of the manifold is of little interest to us. We can reinterpret a “pleating surface” accordingly, as we set out at the beginning of the next section.

6. Quasiprojections.

In this section, we continue with the analysis of our end $e \in \mathcal{E}_D(M)$. We show how the notion of projection defined in [Bow2] adapts to this context. For the moment, we assume that $\kappa(\Sigma) \geq 2$. We will explain at the end what happens in the special case of a 1HT or 4HS.

Following on from the last section, we shall define a *pleating surface* in e to mean a uniformly lipschitz map $\phi : (\Sigma, \sigma) \rightarrow (e, d)$, where σ is some hyperbolic structure on Σ with each boundary component horocyclic of fixed length. (Geodesic of fixed length would do just as well.) We assume that each boundary component gets mapped to a euclidean geodesic in $\partial_V e$. All our pleated surfaces will be in the fibre class, i.e. identifying e topologically with $\Sigma \times [0, \infty)$, it is homotopic to the inclusion of Σ as the first co-ordinate. We note that Lemmas 5.2 and 5.3 apply equally well for a pleating surface interpreted in this way. We also note that Lemma 3.4 (uniform injectivity) applies in this situation: if an η -neighbourhood of the curve lies in e , then projection of e to Σ to this neighbourhood gives us our map θ . For this we should insist that the curve has depth at least η , but after shrinking e slightly, we can forget this detail.

Let $J \subseteq X(\Sigma)$ be the set of curves that are realised by a closed geodesic in the interior of e , which we shall henceforth denote by $\bar{\alpha}$. This is also a closed geodesic in M . We write $l_M(\alpha)$ for the length of $\bar{\alpha}$, and write $D(\alpha) = D(\bar{\alpha}) = \rho_e(\alpha)$ for the depth of $\bar{\alpha}$ in e . We write $J(h, r) = \{\alpha \in J \mid D(\alpha) > h, l_M(\alpha) \leq r\}$.

Let h_0 be the constant of Lemma 5.5. An immediate consequence of this lemma applied inductively is:

Lemma 6.1 : *If $k \in \mathbf{N}$ and $h \geq kh_0$, then $N(J(h, \infty), k) \subseteq J(h - kh_0, \infty)$.* \diamond

Here $N(Q, k)$ denotes the k -neighbourhood of $Q \subseteq X(\Sigma)$ with respect to the combinatorial metric, $d_{\mathcal{G}}$, on $\mathcal{G}(\Sigma)$.

Now suppose that $\alpha \in J(h_0, \infty)$, and let $(\Sigma, \sigma) \rightarrow e$ be a pleating surface in e realising α , as given by Lemma 5.3. Let $J_\sigma(r) \subseteq X(\Sigma)$ be the set of curves whose geodesic realisations in (Σ, σ) have length at most r . It is a standard and relatively straightforward fact that the diameter of $J_\sigma(r)$ in $X(\Sigma)$ is bounded in terms of r . Moreover, if $r \geq r_0 = r_0(\kappa(\Sigma))$, then it is non-empty. We fix r_0 and abbreviate $J_\sigma = J_\sigma(r_0)$. Suppose that $\phi' : (\Sigma, \sigma') \rightarrow e$ is another such pleating surface realising α . A consequence of uniform injectivity (Lemma 3.4) is that the diameter of $J_\sigma \cup J_{\sigma'}$ is bounded in terms of $\kappa(\Sigma)$. We can therefore choose some $\gamma \in J_\sigma$ and set $\text{proj}(\alpha) = \gamma$. This is then well-defined up to bounded distance in $X(\Sigma)$. If $\alpha \in J(h_0, r_0)$ we can set $\text{proj}(\alpha) = \alpha$. Note that if α and β are adjacent in $X(\Sigma)$, then we can choose the same pleating surface for both, showing that $d_{\mathcal{G}}(\text{proj}(\alpha), \text{proj}(\beta))$ is bounded.

Suppose $h \geq h_0 + h_1(r_0)$ where h_1 is the function of Lemma 5.6. It follows that if $\alpha \in J(h, \infty)$ then $\text{proj}(\alpha) \in J(h - (h_0 + h_1(r_0)), r_0)$.

Let us summarise what we have shown. For notational simplicity, we increase our original choice of constant h_0 to $h_0 + h_1(r_0)$. In this way, it will serve for all the above purposes.

Lemma 6.2 : *There is some $h_0 = h_0(\kappa(\Sigma))$, $r_0 = r_0(\kappa(\Sigma))$, $k_0 = k_0(\kappa(\Sigma)) \in \mathbf{N}$, and a map $\text{proj} : J(h_0, \infty) \rightarrow J(0, r_0)$ with the following properties:*

(P1) *If $\alpha \in J(h, \infty)$ for some $h \geq h_0$, then $\text{proj}(\alpha) \in J(h - h_0, r_0)$.*

(P2) *If $\alpha \in J(h_0, r_0)$, then $\text{proj}(\alpha) = \alpha$.*

(P3) *If $\alpha, \beta \in J(h_0, \infty)$ are adjacent in $X(\Sigma)$, then $d_{\mathcal{G}}(\text{proj}(\alpha), \text{proj}(\beta)) \leq k_0$.* \diamond

To deduce more about the map proj , we bring the hyperbolicity of the $\mathcal{G}(\Sigma)$ into play. We are specifically aiming at Lemma 6.4. This depends on the following general observation about quasiprojections in a hyperbolic graph.

Lemma 6.3 : *For all $k, s, t \geq 0$ there is some $u \geq 0$ with the following property. Suppose that (\mathcal{G}, d) is a k -hyperbolic graph, $A, B \subseteq V(\mathcal{G})$ and $\omega : A \rightarrow B$ is a map satisfying $\omega(x) = x$ for all $x \in A \cap B$ and $d(\omega(x), \omega(y)) \leq t$ whenever $x, y \in A$ are adjacent. Suppose that $x_0 x_1 \cdots x_n$ is a geodesic in \mathcal{G} with $d(x_0, B) \leq s$, $d(x_n, B) \leq s$ and $N(x_i, u) \subseteq A$ for all i . Then for all i , $d(x_i, \omega(x_i)) \leq tu$.*

In the case where $A = X$, this follows from the fact that the image of a quasiprojection (here $\omega(X)$) is quasiconvex. (This is discussed in [Bow2] for example.) Here we have to take account of the fact the quasiprojection is only defined on a certain subset.

Proof : We first make the observation that if $x \in A$ and $d(x, A \cap B) \leq v \leq u$ (however u is defined) then $d(x, \omega(x)) \leq tv$. This follows by straightforward induction.

Now fix $v \geq r$ to be determined shortly. Let $\{p+1, p+2, \dots, q-1\}$ be a maximal set of consecutive indices such that $d(x_i, B) > v$ for all $p+1 \leq i \leq q-1$. In other words, $d(x_p, B) \leq v$, $d(x_q, B) \leq v$ and $d(x_i, B) > v$ whenever $p < i < q$. We can assume that $u \geq v$ so that in particular, $d(x_p, A \cap B) \leq v$ and $d(x_q, A \cap B) \leq v$. Let $y_i = \pi(x_i)$. Thus, $d(x_p, y_p) \leq tv$ and $d(x_q, y_q) \leq tv$ and $d(y_i, y_{i+1}) \leq t$ for all i . Now the piecewise geodesic path $\zeta = [y_p, y_{p+1}] \cup \dots \cup [y_{q-1}, y_q]$ has length at most $t(q-p)$ and lies outside a $(v-t/2)$ -neighbourhood of the geodesic segment $x_p x_{p+1} \dots x_q$. Now, given any $a \geq 0$, we can choose v sufficiently large in relation to a , t and k so that the length of ζ is at least $ad(y_p, y_q) - c$, where c depends on a , t , k and v . This is a standard fact about distances outside a uniform neighbourhood of a geodesic segment in a hyperbolic space (see, for example [GhH]: it is used, for example, to show that quasigeodesics remain close to geodesics). Now set $a = 2t$, and fix some such v . Thus c depends only on t and k . Now $2td(y_p, y_q) - c \leq \text{length}(\zeta) \leq t(q-p)$ and so $2d(y_p, y_q) \leq (q-p) + c/t$. Also $d(y_p, y_q) \geq d(x_p, x_q) - 2tv = (q-p) - 2tv$ and so $2((q-p) - 2tv) \leq (q-p) + c/t$, and so $q-p \leq 5tv + c/t$, which is bounded in terms of k and t . We now set $u = (4tv + c/t)$. It follows that for each i , $d(x_i, y_p) \leq d(x_i, x_p) + d(x_p, y_p) \leq (4tv + c/t) + tv = u$. Since this applies to any such segment, $x_p x_{p+1} \dots x_q$, we conclude that $d(x_i, B) \leq u$ for all i .

Now by the first paragraph again, we see that for all i , $d(x_i, \pi(x_i)) \leq tu$ as required.

◇

We apply this as follows:

Lemma 6.4 : *Given $s \geq 0$, there exist $h_2, k_2 \geq 0$, depending only on s and $\kappa(\Sigma)$ such that if $\alpha_0, \alpha_1, \dots, \alpha_n$ is a geodesic in $\mathcal{G}(\Sigma)$ with $d_{\mathcal{G}}(\alpha_0, J(0, r_0)) \leq s$, $d_{\mathcal{G}}(\alpha_n, J(0, r_0)) \leq s$ and with $\alpha_i \in J(h_2, \infty)$ for all i , then for all i , $d_{\mathcal{G}}(\alpha_i, \text{proj}(\alpha_i)) \leq k_2$.*

Here r_0 is the constant involved in the definition of $\text{proj} : J(h_0, \infty) \rightarrow J(0, r_0)$. We can certainly choose $h_2 \geq h_0$ so that $\text{proj}(\alpha_i)$ is defined.

Proof : Let $A = J(h_0, \infty)$ and $B = J(0, r_0)$, so that $A \cap B = J(h_0, r_0)$. We now apply Lemma 6.3, with $\mathcal{G} = \mathcal{G}(\Sigma)$ and $\omega = \text{proj} : A \rightarrow B$. Let $k = k(\kappa(\Sigma))$ be the hyperbolicity constant of $\mathcal{G}(\Sigma)$, and let $t = k_0$ be the constant of Lemma 6.2. Let u be the constant of Lemma 6.3 and set $k_2 = tu$ and $h_2 = (k_2 + 1)h_0$. Then, by Lemma 6.1, $N(J(h_2, \infty), k_2) \subseteq J(h_0, \infty)$. In particular, by hypothesis, $N(\alpha_i, k_2) \subseteq J(h_0, \infty) = A$ for all i . Thus, applying Lemma 6.3, we see that $d_{\mathcal{G}}(\alpha_i, \text{proj}(\alpha_i)) \leq tu = k_2$ for all i , as required. ◇

We now have all the ingredients to apply the arguments of [Bow2] to deduce:

Lemma 6.5 :

- (1) $(\exists h)(\forall r)(\exists r')$ such that if $(\gamma_i)_{i=0}^n$ is a tight geodesic in $\mathcal{G}(\Sigma)$ with $\gamma_0, \gamma_n \in J(h, r)$ and $\gamma_i \in J(h, \infty)$ for all i , then $\gamma_i \in J(0, r')$ for all i .
- (2) $(\exists h)(\forall k, r)(\exists k', r')$ such that if $(\gamma_i)_{i=0}^n$ is a tight geodesic in $\mathcal{G}(\Sigma)$ with $\gamma_0, \gamma_n \in N(J(h, r), k)$ and $\gamma_i \in J(h, \infty)$ for all i , then $\gamma_i \in J(h, r')$ for all i with $k' \leq i \leq n + k'$.

(Here r' just depends on r and $\kappa(\Sigma)$ and in part (2), k' just depends on k and $\kappa(\Sigma)$.)

This is a version of the ‘‘a-priori bounds’’ result (cf. [Mi2], [Bow2]).

The proof of Lemma 6.5 follows exactly that of [Bow2], where the lower bound, h , on the depths of curves was redundant. Here it is needed to ensure that none of our constructions take us outside e . The argument is by contradiction. If the conclusion fails, we can find sequences, $(\gamma_i^n)_i$ of tight geodesics in ends, e^n , of thick parts, $\Psi(M^n)$, of complete hyperbolic 3-manifolds, M^n , that satisfy the hypotheses of the lemma, but for which certain γ_i^n get arbitrarily long as geodesics in M^n . To derive the same contradiction as in [Bow2] it is sufficient to have a homotopy equivalence, $e^n \rightarrow \Sigma$, defined on each end e^n . The remainder of the manifold $\Psi(M^n)$ is then irrelevant.

Lemma 6.5 is all we need to prove Propositions 2.1 and 2.2, as we do in the next section. However, for their application we will also need a relative version of this, or at least of part (1) of the statement. Let Φ be a connected proper π_1 -injective subsurface of Σ . We shall assume here that any boundary component of Φ that is homotopic to a boundary component of Σ equals that boundary component. We write $\partial_\Sigma \Phi$ for the relative boundary of Φ in Σ , and write $X(\partial_\Sigma \Phi) \subseteq X(\Sigma)$ for the set of components of $\partial_\Sigma \Phi$. (It is possible that two such components might get identified in $X(\Sigma)$.) We can also identify $X(\Phi)$ as a subset of $X(\Sigma)$. Note that, by Lemma 6.1, if for some $h \geq h_0$, $X(\partial_\Sigma \Phi) \cap J(h, \infty) \neq \emptyset$, then $X(\Phi) \subseteq J(h - h_0, \infty)$.

Lemma 6.6 : *($\exists h$)($\forall r$)($\exists r'$) such that if $\Phi \subseteq \Sigma$ is a proper subsurface with $X(\partial_\Sigma \Phi) \subseteq J(h, r)$ and $(\gamma_i)_{i=0}^n$ is a tight geodesic in $\mathcal{G}(\Phi)$ with $\gamma_0, \gamma_n \in J(0, r)$, then $\gamma_i \in J(0, r')$ for all i .*

Again this follows exactly as in [Bow2] (modulo restricting the homotopy equivalence to the ends). This time, the control on depth is automatic, given the assumption on $\partial_\Sigma \Psi$, and the observation preceding the lemma.

As we stated at the beginning we have assumed that $\kappa(\Sigma) \geq 2$, and for Lemma 6.6, that $\kappa(\Phi) \geq 2$. One can reinterpret these statements for the case of a 1HT or 4HS, where we replace the curve graph by the Farey graph. In this case, all geodesics are deemed ‘‘tight’’. The same results are valid, though the argument, in this case is rather different.

We can first note that any end with base surface a 1HT or a 4HS is necessarily incompressible, and so it is covered by the discussion given in [Bow3]. In Lemma 6.6, we note, for the same reason given there, each curve of our (tight) geodesic lies in $J(0, \infty)$ and so, in particular, is non-trivial in M . One can use trace identities, exactly as in [Bow2], for example, to bound the length of these geodesics. These trace identities are valid regardless of whether or not the representation of the surface group is injective (or indeed discrete).

7. A-priori bounds.

In this section, we give proofs of Propositions 2.2 and 2.3, and describe a variant of the ‘‘a-priori bounds’’ result for our version of hierarchies.

For the moment, we restrict attention to the case where $\kappa(\Sigma) \geq 2$.

Let $e \in \mathcal{E}_D(M)$, and let $D : e \rightarrow [0, \infty)$ be the depth function as defined in Section 5. Let $\text{proj} : J(h_0, \infty) \rightarrow J(0, r_0)$ be the quasiprojection defined in Section 6. We note that for all $\alpha \in J(h_0, \infty)$, $|D(\alpha) - D(\text{proj}(\alpha))| \leq h_0$, and if $\alpha, \beta \in J(h_0, \infty)$ are adjacent in $\mathcal{G}(\Sigma)$, then $|D(\alpha) - D(\beta)| \leq h_0$, $|D(\text{proj}(\alpha)) - D(\text{proj}(\beta))| \leq h_0$ and $d_{\mathcal{G}}(\text{proj}(\alpha), \text{proj}(\beta)) \leq k_0$.

By discreteness, we note that $J(0, r) \setminus J(h, r)$ is finite for all h and r . In particular, if $(\gamma_i)_{i \in \mathbf{N}}$ is a sequence of distinct elements of $J(0, r)$ then $D(\gamma_i) \rightarrow \infty$. In other words, the geodesics $\bar{\gamma}_i$ go out the end e .

Lemma 7.1 : *There exist $k_0 = k_0(\kappa(\Sigma))$ and $r_0 = r_0(\kappa(\Sigma))$ such that there is a sequence $\alpha \in J(0, r_0)$ with $d_{\mathcal{G}}(\alpha_i, \alpha_{i+1}) \leq k_0$ for all i with $D(\alpha_i) \rightarrow \infty$.*

We can take r_0 and h_0 to be the same as those above.

Proof : First note that from the description of a simply degenerate end, we have curves $\gamma_i \in J(0, \infty)$ with $D(\gamma_i) \rightarrow \infty$, and so $D(\text{proj}(\gamma_i)) \rightarrow \infty$. Thus $J(0, r_0)$ is infinite.

Now choose any $\beta_0 \in J(0, r_0)$ with $D_0 = D(\beta_0) \geq h_0$. Let \mathcal{H} be the graph with vertex set $V(\mathcal{H}) = J(D_0, r_0)$ and with $\alpha, \beta \in V(\mathcal{H})$ deemed adjacent if $d_{\mathcal{G}}(\alpha, \beta) \leq k_0$. Note that \mathcal{H} is locally finite.

Let $B = \{\beta \in V(\mathcal{H}) \mid D(\beta) \leq D_0 + 3h_0\}$. Thus, $B \subseteq V(\mathcal{H})$ is finite. Now each point of $V(\mathcal{H})$ can be connected to B by some path in \mathcal{H} . To see this, suppose $\beta \in V(\mathcal{H})$ and let $\beta_0, \beta_1, \dots, \beta_n = \beta$ be any path in $X(\Sigma)$ from β_0 to β . Thus $|D(\beta_i) - D(\beta_{i+1})| \leq h_0$ for all i . If $\beta \notin B$, there is some p so that $D(\beta_i) \geq D_0 + h_0$ for all $i \geq p$ and with $D(\beta_p) \leq D_0 + 2h_0$. Let $\delta_i = \text{proj}(\beta_i)$ for all $i \geq p$. Thus $\delta_p \in B$ and $\delta_n = \beta$. We see that $\delta_p, \delta_{p+1}, \dots, \delta_n$ connects B to β in \mathcal{H} as claimed.

In summary, \mathcal{H} is infinite, locally finite, and has finitely many components. It thus has a component of infinite diameter, which then contains an infinite arc $(\alpha_i)_{i \in \mathbf{N}}$. It follows that $D(\alpha_i) \rightarrow \infty$ as required. \diamond

Proof of Proposition 2.1 : We begin with the sequence, $(\alpha_i)_{i \in \mathbf{N}}$, given by Lemma 7.1. We want to replace this by a geodesic $(\gamma_i)_{i \in \mathbf{N}}$, perhaps at the cost of increasing the length bound.

For each pair, $i, j \in \mathbf{N}$, let $\pi(i, j) \subseteq X(\Sigma)$ be a tight geodesic in \mathcal{G} from α_i to α_j . Let $m(i, j) = \min\{D(\delta) \mid \delta \in \pi(i, j)\}$ (where $m(i, j) = 0$ if some curve of $\pi(i, j)$ lies outside $J(0, \infty)$). Note that $\pi(i, i) = \{\alpha_i\}$, so $m(i, i) \rightarrow \infty$ as $i \rightarrow \infty$. By Lemma 6.5(1), there is some $h \geq 0$ such that if $m(i, j) \geq h$, then $\pi(i, j) \subseteq J(0, L)$. Here h and L depend only on $\kappa(\Sigma)$.

By construction, $d_{\mathcal{G}}(\gamma_j, \gamma_{j+1}) \leq k_0$, and so, by the hyperbolicity of $\mathcal{G}(\Sigma)$, the geodesics $\pi(i, j)$ and $\pi(i, j+1)$ lie a bounded distance apart (depending only on $\kappa(\Sigma)$) for each i and j . From this it follows that $|m(i, j) - m(i, j+1)|$ is bounded by some constant $m_0 = m_0(\kappa(\Sigma))$ for all i, j .

We distinguish two cases:

Case (1): $(\exists i)(\forall j \geq i)(m(i, j) \geq h)$.

In this case, $\pi(i, j) \subseteq J(0, L)$ for all $j \geq i$. Let γ_k^j be the k th curve in $\pi(i, j)$ so that

$\gamma_0^j = \alpha_i$. Now $D(\gamma_k^j)$ is bounded above in terms of $D(\alpha_i)$ and k , and so there are only finitely many possibilities for any given k . After passing to a diagonal subsequence, we can therefore suppose that for each k , γ_k^j eventually stabilises on some curve $\gamma_k \in J(0, L)$ as $j \rightarrow \infty$. Thus, $(\gamma_k)_k$ gives us our required geodesic.

Case (2): $(\forall i)(\exists j \geq i)(m(i, j) < h)$.

Now $m(i, i) \rightarrow \infty$, and we can assume that $m(i, i) \geq h + m_0$ for all i . Since $|m(i, j) - m(i, j+1)| \leq m_0$ for all j , there is some $j(i)$ such that $h \leq m(i, j(i)) \leq h + m_0$. Note that $\pi(i, j(i)) \subseteq J(0, L)$. Now let $\delta_0^i \in \pi(i, j(i))$ be such that $D(\delta_0^i) \leq h + m_0$. Let $\gamma_k^i|_{k=0}^{p_i}$ be the subpath of $\pi(i, j(i))$ going backwards from δ_0^i to $\delta_i^{p_i} = \alpha_i$. As in Case (1), we see that there are only finitely many possibilities for the curve δ_k^i as $i \rightarrow \infty$, so we can suppose that δ_k^i stabilises on some curve $\gamma_k \in J(0, L)$. Thus $(\gamma_k)_k$ is the required geodesics. \diamond

Proof of Proposition 2.2 : Let $(\gamma_i)_i$ be a tight geodesic converging on $a(M, e) \in \mathcal{G}(\Sigma)$. Let $(\gamma'_i)_i$ be the tight geodesic given by Proposition 2.2. By Proposition 1.1, $(\gamma'_i)_i$ also converges on $a(M, e)$. Thus, by hyperbolicity, $(\gamma_i)_i$ and $(\gamma'_i)_i$ eventually remain a uniformly bounded distance apart. In other words, up to shifting the indices, we can assume that $d_{\mathcal{G}}(\gamma_i, \gamma'_i) \leq k_1 = k_1(\kappa(\Sigma))$ for all i . Since $\gamma'_i \in J(0, L)$ for all i , we can apply Lemma 6.5(2) to deduce that $\gamma_i \in J(0, L')$ for all i , where L' depends only on L, k and $\kappa(\Sigma)$, and hence ultimately only on $\kappa(\Sigma)$. \diamond

This result extends to a version of “hierarchies”. A theory of hierarchies was developed in [MasM2]. In [Bow3] we made do with a somewhat less sophisticated variant. The simplified description below will be seen to include both the above insofar as it applies to the a-priori bounds result.

Given a subset, $Q \subseteq X(\Sigma)$, let $Y(Q)$ be Q itself, together the set of curves which lie on some tight geodesics in $X(\Phi)$ whose endpoints lie in $Q \cap X(\Phi)$ for some subsurface $X(\partial_\Sigma(\Phi)) \subseteq Q$. We allow $\Phi = \Sigma$. By a “tight geodesic” we are referring to $\mathcal{G}(\Phi)$, or $\mathcal{G}'(\Phi)$ in the case where Φ is a 1HT or 4HS. (In the latter case, all geodesics are deemed tight.)

A combination of Propositions 6.5 and 6.6 now gives:

Proposition 7.2 : *For all h there is some h' and for all r there is some r' such that if $Q \subseteq J(h', r)$, then $Y(Q) \subseteq J(0, r')$.* \diamond

We can, of course, iterate this procedure. If $(\gamma_i)_i$ is any geodesic in $X(\Sigma)$ tending to $a(M, e)$, then for any given $\nu \in \mathbf{N}$ there is some $j \in \mathbf{N}$ such that $Y^\nu(\{\gamma_i \mid i \geq j\}) \subseteq J(0, L)$, where L depends only on ν and $\kappa(\Sigma)$. By fixing a suitable ν depending only on $\kappa(\Sigma)$ ($\nu = 2\kappa(\Sigma)$ will do) this includes all the curves featuring in the “a-priori bounds” statement of [Bow3], which was the key point needed to construct our lipschitz from a degenerate end of $\Psi(P)$ to e .

We can now proceed to the final step, described in the next section.

8. The model space.

In this section, we explain how the model space is constructed and give a proof of

Proposition 1.3. We go on to explain how this implies the Ending Lamination Theorem.

We adopt the notation and terminology of [Bow3]. We say that a map between two geodesic spaces is *sesquilipschitz* if it is a lipschitz quasi-isometry. We say that it is *universally sesquilipschitz* if it is a homotopy equivalence and the lift to universal covers is sesquilipschitz.

Now it is not hard to see that a quasi-isometry between two geodesic spaces that is equivariant with respect to some discrete group actions descends to a quasi-isometry of the quotients. In particular:

Lemma 8.1 : *A universally sesquilipschitz map is sesquilipschitz.* \diamond

Now suppose Ψ is a topologically finite 3-manifold such that $\partial\Psi$ is a disjoint union of tori and cylinders. Suppose we have a decomposition of its set of ends as $\mathcal{E} = \mathcal{E}_F \sqcup \mathcal{E}_D$, so that no base surface of any end is a disc, annulus, sphere or torus, and no base surface of an end of \mathcal{E}_D is a three-holed sphere. Suppose that to each end $e \in \mathcal{E}_D$, we have associated some $a(e) \in \partial\mathcal{G}(e)$. From this data, we construct a riemannian manifold, P , without boundary, with a submanifold, $\Psi(P)$, which is homeomorphic to Ψ . We show that it satisfies the conclusion of Proposition 1.3.

The construction of P follows almost verbatim that described in [Bow3]. For each $e \in \mathcal{E}$ we construct a “model end”, $\Psi(P_e)$, which we glue to a topological core, Ψ_0 , and extend the metric over Ψ_0 . This gives us our “thick part”, $\Psi(P)$. We then glue in a standard cusp (a \mathbf{Z} -cusp or $\mathbf{Z} \oplus \mathbf{Z}$ -cusp) to each boundary component of $\Psi(P)$ to give us P .

Let M be a hyperbolic 3-manifold with a homeomorphism, $g : \Psi \rightarrow \Psi(M)$, sending \mathcal{E}_F and \mathcal{E}_D respectively to $\mathcal{E}_F(M)$ and $\mathcal{E}_D(M)$, and such that $a(M, g(e)) = g_*(a(e))$ for all $e \in \mathcal{E}_D$. We now construct another $f : P \rightarrow M$ in stages, as follows.

For each $e \in \mathcal{E}$, we define $f_e : \Psi(P_e) \rightarrow \Psi(M_e)$, where $\Psi(M_e) \cong \Sigma(e) \times [0, \infty)$ is a neighbourhood of the end $g(e)$ of M . In the incompressible case, it was shown in [Bow3] that f_e is universally sesquilipschitz. We can apply the same argument here. For degenerate ends, the key point is the “a-priori bounds” result explained the end of Section 7. Beyond establishing the corresponding fact for incompressible ends, no essential use was made of incompressibility — all the constructions being intrinsic to the end. In dealing with geometrically finite ends, we made use in [Bow3] of certain multiplicative bounds on distance distortion when projecting to a uniform neighbourhood of the convex core. But exactly the same bounds are valid in this setting — indeed for projecting to any convex set in hyperbolic space.

The extension to the core, Ψ_0 , and then to the cusps is essentially elementary. This gives us a lipschitz map $f : P \rightarrow M$, and one verifies that $f|_{\Psi(P)}$ is proper. We still need to check that each end, e , of $\Psi(P)$ goes to the corresponding end, $g(e)$, of $\Psi(M)$. Using Waldhausen’s cobordism theorem, we can see that only case where this may be an issue is where $\Psi(M) \cong \Sigma \times \mathbf{R}$ for a compact surface Σ , already dealt with in [Bow3]. Since $\Psi(M)$ is no longer a-priori irreducible, we use the following form of the cobordism theorem. Suppose that Ψ_0 is a compact 3-manifold with boundary $\partial\Psi_0$, and compact subsurfaces, Σ_1 and Σ_2 in $\partial\Psi_0$ such that the pair $(\Sigma_1, \partial\Sigma_1)$ is homotopic to $(\Sigma_2, \partial\Sigma_2)$ in $(\Psi_0, \partial\Psi_0)$, then the triple $(\Psi_0, \Sigma_1, \Sigma_2)$ is homeomorphic to $(\Sigma \times [0, 1], \Sigma \times \{0\}, \Sigma \times \{1\})$ for a compact

surface Σ (possibly with fake 3-balls surgered in, which cannot occur here).

Since for all $e \in \mathcal{E}$, the map $f_e : \Psi(P_e) \rightarrow \Psi(M_e)$ is universally sesquilipschitz, by Lemma 8.1, the map $\hat{f}_e : \hat{\Psi}(P_e) \rightarrow \hat{\Psi}(M_e)$ is sesquilipschitz, where, on both sides, $\hat{\Psi}$ denotes the cover of Ψ corresponding to the kernel of $\pi_1(\Sigma(e))$ in $\pi_1(\Psi)$. Thus, $\hat{\Psi}(P_e)$ and $\hat{\Psi}(M_e)$ are subsets of the universal covers of $\Psi(P)$ and $\Psi(M)$ respectively. This is what we use when gluing the pieces together, to see, as in [Bow3] that the lift between universal covers is a quasi-isometry.

This proves Proposition 1.3.

Now we can use the same model space for two homeomorphic hyperbolic 3-manifolds with the same degenerate end invariants to deduce:

Proposition 8.3 : *Suppose that M and M' are complete hyperbolic 3-manifolds and that there is a homeomorphism from M to M' that sends cusps of M into cusps of M' and conversely. Suppose that the induced map between the non-cuspidal parts sends each geometrically finite end to a geometrically finite end and each degenerate end to a degenerate end. Suppose that (under the induced homeomorphisms of base surfaces) the end invariants of corresponding pairs of degenerate ends are equal. Then there is an equivariant quasi-isometry between the universal covers of M and M' . \diamond*

This is identical to a statement in [Bow3], with the hypothesis of “indecomposable” omitted. The argument is the same, given Proposition 1.3.

In particular, we get an equivariant quasi-conformal extension, $f : \partial\mathbf{H}^3 \rightarrow \partial\mathbf{H}^3$. In the case where the geometrically finite end invariants are also equal, we can find another equivariant map, $g : \partial\mathbf{H}^3 \rightarrow \partial\mathbf{H}^3$, which agrees with f on the limit set, and is conformal on the discontinuity domain. We need to verify that g is also quasiconformal.

Let U be component of the domain of discontinuity of $\Gamma \equiv \pi_1(M)$ acting on $\partial\mathbf{H}^3$. As in [Bow3], we see that $g^{-1} \circ f|_U$ moves every point a bounded distance in the Poincaré metric. To show that $g^{-1} \circ f$, and hence g , is conformal everywhere, it is enough to give a suitable bound on the euclidean metric, d_e , in term of the Poincaré metric (using an identification of $\partial\mathbf{H}^3$ with $\mathbf{C} \cup \{\infty\}$). In [Bow3], the formula we used was: $d_e(z, w) \leq (e^{2k} - 1) \max\{d_e(z, \partial U), d_e(w, \partial U)\}$, under the assumption that U was simply connected. We want a variant of this when U might not be simply connected. Let Δ be the universal cover of U , which we identify with the Poincaré disc. We have a group, G , acting on Δ , and a normal subgroup, $H \triangleleft G$, such that $U = \Delta/H$ and $R = \Delta/G$ is Riemann surface of finite type — here the corresponding geometrically finite end invariant. Note that H has no parabolic elements.

Lemma 8.3 : *If $z, w \in U$ are a distance at most k apart in the Poincaré metric on U , then $d_e(z, w) \leq (e^{\mu k} - 1) \max\{d_e(z, \partial U), d_e(w, \partial U)\}$, where $\mu > 0$ is a constant depending only on the Riemann surface R .*

Proof : Choose $t > 0$ so that the shortest closed geodesic on R has length greater than $2t$ in the Poincaré metric. Since H has no parabolics, any t -disc in R lifts to an embedded disc in U . Put another way, if $z \in U$, then the t -disc, D , about z in the Poincaré metric is embedded in U . We can lift this to the Poincaré disc, Δ , centred at z . Here D will

have euclidean radius $\tanh t$. Now consider the $D \rightarrow U$ with respect the euclidean metric on both D and U . By the Koebe Quarter Theorem, the norm of the derivative at z is at most $\frac{r}{4 \tanh t}$ at the centre. But a euclidean unit vector at the origin has Poincaré norm 2, and so we deduce that $|ds| \geq \frac{1}{\mu r} |dz|$, where $\mu = 2/\tanh t$, where $|ds|$ and $|dz|$ are the infinitesimal Poincaré and euclidean metrics, and $r = d_e(z, \partial U)$. We can now integrate this to derive the required inequality. \diamond

Now since there are only finitely many geometrically finite ends we can take the same μ for all components of the discontinuity domain. Thus, the inequality of Lemma 8.3 applies equally well when U is replaced by the whole discontinuity domain. This is sufficient to bound the metric quasiconformal distortion of $g^{-1} \circ f$ on the limit set, showing that $g^{-1} \circ f$ and hence f is quasiconformal.

The remainder of the argument is now standard. We see using the result of Sullivan [Su] that f must be conformal. It therefore gives rise to an isometric conjugacy between the actions of $\pi_1(M)$ and $\pi_1(M')$ on \mathbf{H}^3 , showing that M and M' are isometric. The fact that this isometry satisfies the conditions laid out in Theorem 1.2 is now elementary.

9. The uniform injectivity theorem.

The original Uniform Injectivity Theorem for pleated surfaces goes back to Thurston [T], and there have been several variations since. Most have made some assumption of incompressibility, which is sufficient for the indecomposable case of the Ending Lamination Conjecture, see for example [Mi1]. A variation for the pleating loci of pleated surfaces in handlebodies, is given in [N]. The argument there would apply to more general situations, where the surface is deep in the end of a 3-manifold, without assuming the end is incompressible. However the quantification means that the required depth may be dependent on other constants, and it is not clear that this result can be adapted to the argument given in [Bow2].

In this section, we give a version which depends just on a local incompressibility assumption. As with earlier versions, we argue by contradiction, passing to a limit. This means that the constants involved are not a-priori computable. We will make our statement for laminations, though we only apply it in this paper for multicurves. It is easily seen to imply Lemma 3.4. To simplify notation we only deal with 1-lipschitz pleating surfaces, though the argument will be seen to apply equally well to uniformly lipschitz maps. The basic idea of constructing a partial covering space to derive a contradiction can be found in Thurston's original [T], though since we have altered a number of definitions and hypotheses, we work things through from first principles.

Let (M, d) be a complete hyperbolic 3-manifold, with projectivised tangent bundle, $E \rightarrow M$. We write d_E for the metric on E . The map $(E, d_E) \rightarrow (M, d)$ is then 1-lipschitz. We shall define a *pleating surface* as a 1-lipschitz map $\phi : (\Sigma, \sigma) \rightarrow (M, d)$ where (Σ, σ) is a compact hyperbolic surface with (possibly empty) horocyclic boundary. By a *lamination* $\lambda \subseteq \Sigma$, we mean a geodesic lamination in the usual sense, see for example [CanEG]. We say that a pleating surface, $\phi : \Sigma \rightarrow M$ *realises* λ if it sends each leaf of λ locally isometrically to a geodesic in M . We write $\psi = \psi_\phi : \lambda \rightarrow E$ for the lift to E , and

let $\Lambda = \psi(\lambda)$. (It will follow from subsequent hypotheses that ψ will be a homeomorphism to Λ , in which case, the notion coincides with that already defined for multicurves in Section 3.) Note that we are not assuming λ to be connected.

Here is our formulation of uniform injectivity:

Proposition 9.1 : *Given positive κ, η, ϵ , there is some $\delta > 0$ with the following property. Suppose that Σ is a compact surface with $\kappa(\Sigma) = \kappa$, and suppose that $\phi : (\Sigma, \sigma) \rightarrow (M, d)$ is a pleating surface to a complete hyperbolic 3-manifold M , realising a geodesic lamination $\lambda \subseteq \Sigma$. Suppose:*

(U1) *For all $x \in \lambda$, the injectivity radius of M at $\phi(x)$ is at least η .*

(U2) *There is a map $\theta : N(\phi(\lambda), \eta) \rightarrow \Sigma$ such that the composition $\theta \circ \phi|_{N(\lambda, \eta)} \rightarrow \Sigma$ is homotopic to the inclusion of $N(\lambda, \eta)$ in Σ .*

Then for all $x, y \in \lambda$, if $d_E(\psi_\phi(x), \psi_\phi(y)) \leq \delta$ then $\sigma(x, y) \leq \epsilon$.

In (U1), we are demanding the $f(\lambda)$ lie in the thick part of M . This is equivalent to putting a bound on the diameter of each component of $f(\lambda)$ (and excluding cores of Margulis tubes, though such components could be easily dealt with explicitly).

In (U2), $N(Q, r)$ denotes the open r -neighbourhood of Q . We are assuming that ϕ is 1-lipschitz, and so $\phi(N(\lambda, \eta)) \subseteq N(\phi(\lambda), \eta)$. As remarked earlier, it is not hard to see that the hypotheses imply that $\psi|_\lambda$ is injective (cf. Lemma 9.9). Thus, ψ is a homeomorphism to Λ . Note that Λ admits a decomposition into geodesic leaves, being invariant under a local geodesic flow. The map θ need only be defined up to homotopy.

Although it is implicit in our earlier definitions that Σ is connected, this is not really required. Indeed for the proof, it is convenient to allow for a disconnected surface. This allows us to cut away the thin part of Σ , so that all of Σ maps into the thick part of M .

We can assume that the length of each boundary curve of Σ is bounded below by some positive constant. To see this, note that λ cannot cross any horocycle of length 1, we can simply cut away the remainder of Σ . Similarly, we can cut away the thin part of Σ along curves of length bounded below, and constant outward curvature bounded above. Thus, we can assume that the injectivity radius of Σ is bounded below, though at the possible cost of disconnecting the surface.

We finally note that it would be enough to assume that ϕ is μ -lipschitz for some μ , in which case, of course, δ will also be a function of μ . The argument is unchanged modulo introducing various factors of μ into the proceedings.

Before beginning with the proof, we recall some basic facts and introduce some notation relating to laminations.

Let $\lambda \subseteq \Sigma$ be a lamination, and write $\vec{\lambda}$ for the unit tangent bundle to λ . Thus λ is a quotient of $\vec{\lambda}$ under the involution, denoted $[\vec{a} \mapsto -\vec{a}]$, that reverses direction. We write $a \in \lambda$ for the ‘‘basepoint’’ of \vec{a} (or of $-\vec{a}$). Given $\vec{a} \in \vec{\lambda}$, write $\vec{a}_t \in \vec{\lambda}$ for the vector obtained by flowing a distance t in the direction of \vec{a} . If $\vec{a}, \vec{b} \in \vec{\lambda}$, we write $\vec{a} \approx \vec{b}$ if $\sigma(a_t, b_t) \rightarrow 0$. This is an equivalence relation on $\vec{\lambda}$. Each equivalence class has at most two elements. (It identifies pairs in a finite set of non-closed directed boundary leaves.) If $\vec{a}, \vec{b} \in \vec{\lambda}$ with $\vec{a} \approx \vec{b}$ and with $\sigma(a, b) \leq \eta$, then $\sigma(a_t, b_t)$ is monotonically decreasing for $t \geq 0$. If $a, b \in \lambda$

lie in the same non-closed leaf, l , of λ , we write $[a, b]$ for the interval of λ connecting them. Similarly, if $a, b \in \Sigma$ with $\sigma(a, b) < \eta$, we write $[a, b]$ for the unique shortest geodesic connecting them. These notations are consistent.

We write $\Upsilon(\lambda)$ for the union of λ and all those intervals $[a, b]$ for which $\sigma(a, b) \leq \eta/8$ and $\vec{a} \approx \vec{b}$ for some tangents, \vec{a} and \vec{b} . Thus, each component of $\Upsilon(\lambda)$ is either a closed leaf of λ , or else a closed non-annular subsurface of Σ . Each component of $\Upsilon(\lambda) \setminus \lambda$ is a “spike” between two asymptotic rays in λ .

We can now begin the proof of Proposition 9.1. Let us assume that it fails. In this case, we can find a sequence, σ_i , of hyperbolic metrics on Σ , geodesic laminations, $\lambda_i \subseteq (\Sigma, \sigma_i)$, pleating surfaces $\phi_i : (\Sigma, \sigma_i) \rightarrow (M_i, d_i)$, maps $\theta_i : N(\phi_i(\lambda_i), \eta) \rightarrow \Sigma$, and points, $x_i, y_i \in \lambda_i$ so that $\sigma_i(x_i, y_i) \geq \epsilon$, but $d_{E_i}(\psi_i(x_i), \psi_i(y_i)) \rightarrow 0$, where $\psi_i = \psi_{\phi_i}$. For each i , the hypotheses of Proposition 9.1 are satisfied for fixed $\eta, \epsilon > 0$.

As observed earlier, we can assume that the lengths of the boundary curves of (Σ, σ_i) are all bounded below, and so the structures (Σ, σ_i) all lie in a compact subset of moduli space. We can thus pass to a subsequence so that these structures converge on some hyperbolic structure (Σ, σ) . This may involve precomposing the ϕ_i and postcomposing the θ_i with suitable inverse mapping classes of Σ . Indeed, after applying precomposing ϕ_i suitable self homeomorphisms of Σ , we can suppose that the metrics σ_i converge to σ .

Passing to another subsequence we can assume that λ_i converges to a lamination $\lambda \subseteq \Sigma$ in the Hausdorff topology. Note that $N_i = N(\lambda_i, \eta/2)$ converges on $N = N(\lambda, \eta/2)$. Let $O_i = N(\phi_i(\lambda_i), \eta/2)$. Thus $\phi_i(N_i) \subseteq O_i$. We can again pass to a subsequence so that (O_i, d_i) converges on a space, (O, d) , in the Gromov-Hausdorff topology. The space (O, d) is an incomplete hyperbolic 3-manifold, in the sense of being locally isometric to \mathbf{H}^3 . (The metric d need not be a path metric on O . Indeed O need not be connected.) We can also observe that the maps $\phi_i : N_i \rightarrow O_i$ converge to a 1-lipschitz map $\phi : N_i \rightarrow O_i$ (in the sense that their graphs converge in the Gromov-Hausdorff topology). Now let $E_O \rightarrow O$ be the projectivised tangent bundle to O , and let $\psi : \lambda \rightarrow E_O$ be the lift of $\phi|_{\lambda}$ to E_O . We write $\Lambda = \psi(\lambda)$. As before, Λ is partitioned into leaves, which are images of leaves of $\vec{\lambda}$ and $\vec{\Lambda}$ for the tangent spaces of λ and Λ respectively. Note that if $p, q \in \Lambda$ with $d(p, q) \leq \eta/2$, then there is a geodesic, $[p, q]$, of length $d(p, q)$ connecting p and q in O .

Finally, we can pass to yet another subsequence so that $x_i \rightarrow x \in \lambda$ and $y_i \rightarrow y \in \lambda$. Thus $\sigma(x, y) > \eta$, but $\psi(x) = \psi(y)$. In particular, ψ is not injective. (From this point on, we could focus our attention on one component of O where the restriction of ψ is not injective, though this is not logically necessary.)

Lemma 9.2 : *There is some $k \geq 0$ such that if π is a path in N connecting two points, $a, b \in \lambda$ with $d(\phi(a), \phi(b)) \leq \eta/2$ such that $\phi(\pi) \cup [\phi(a), \phi(b)]$ is homotopically trivial in O , then π is homotopic relative to a, b in Σ to a path in Σ of length at most k .*

Proof : We first note that if τ is any closed curve in N such that $\phi(\tau)$ is homotopically trivial in O , then τ is homotopically trivial in Σ . To see this, note that for sufficiently large i , τ lies in N_i and $\phi_i(\tau)$ is trivial in O_i . Thus, $\theta_i \circ \phi_i(\tau)$ is trivial in Σ . But $\theta_i \circ \phi_i$ is homotopic to the inclusion of N_i in Σ , and so τ is trivial in Σ as claimed.

We now consider a lift $\tilde{\phi} : \tilde{N} \rightarrow \tilde{O}$ to the universal cover, \tilde{O} , of O (or more precisely

the appropriate connected component of O), where \tilde{N} is some cover of N . Let $\tilde{\pi}$ be a lift of π to \tilde{N} . The endpoints of $\tilde{\phi}(\tau(\Pi))$ are $(\eta/2)$ -close in \tilde{O} . It now follows (from the discreteness of the covering group on O , and its coboundedness \tilde{N}) that the endpoints of $\tilde{\pi}$ are connected by a path $\tilde{\pi}'$ of bounded length in \tilde{N} . This projects to a path π' in N , with endpoints a, b . Now $\phi(\pi \cup \pi')$ is homotopically trivial in O . (It lifts to the closed curve $\tilde{\phi}(\tilde{\pi}) \cup \tilde{\phi}(\tilde{\pi}')$). Thus, $\pi \cup \pi'$ is homotopically trivial in Σ . In other words, π' is homotopic to π relative to a, b , as claimed. \diamond

Suppose $\vec{a}, \vec{b} \in \vec{\lambda}$ with $\vec{a} \approx \vec{b}$. Then $\sigma(a_t, b_t) \rightarrow 0$ and so $d(\phi(a_t), \phi(b_t)) \rightarrow 0$. We have the following converse:

Lemma 9.3 : *Suppose $\vec{a}, \vec{b} \in \vec{\lambda}$ with $\sigma(a, b) \leq \eta/2$ and with $d(\phi(a_t), \phi(b_t)) \rightarrow 0$, then $\vec{a} \approx \vec{b}$.*

Proof : Since $d(\phi(a_t), \phi(b_t))$ is monotonically decreasing for $t \geq 0$, we have $d(\phi(a_t), \phi(b_t)) \leq \eta/2$ for all $t \geq 0$. Let π_t be the path $[a_t, a] \cup [a, b] \cup [b, b_t]$ from a to b in N . Now $\phi(\pi_t) \cup [\phi(a_t), \phi(b_t)]$ is homotopically trivial in O , since $\phi([a, b]) \cup [\phi(a), \phi(b)]$ has length less than η and $\phi([a_t, a]) \cup [\phi(a), \phi(b)] \cup \phi([b, b_t]) \cup [\phi(b_t), \phi(a_t)]$ is spanned by the disc $\bigcup_{u \in [0, t]} [\phi(a_u), \phi(b_u)]$. It follows by Lemma 9.2 that π_t is homotopic in Σ to a path of bounded length k . This means that the half-leaves of λ , based at \vec{a} and \vec{b} are asymptotic, taking account of homotopy class. In other words, there is some $s \in \mathbf{R}$ with $\vec{c} \approx \vec{b}$. Now $\sigma(a, c) \leq \sigma(a, b) \leq \eta/2$, so $|s| \leq \eta/2$. Note that $\sigma(b_t, c_t) \rightarrow 0$ and $d(\phi(a_t), \phi(c_t)) \leq d(\phi(a_t), \phi(b_t)) + d(\phi(b_t), \phi(c_t)) \rightarrow 0$. Now some subsequence of \vec{a}_t must converge on some $\vec{p} \in \vec{\lambda}$. The corresponding \vec{c}_t converge on \vec{p}_s , and we have $\phi(p) = \phi(p_s)$. The loop $\phi([p, p_s])$ in O has length at most $\eta/2$ and so must be homotopically trivial. From this it follows that $s = 0$, and so $\vec{b} \approx \vec{a}_s = \vec{a}$ as required. \diamond

In particular, we see that if $\vec{a}, \vec{b} \in \vec{\lambda}$ with $\sigma(a, b) \leq \eta/2$, then $\vec{a} \approx \vec{b}$ if and only if $d(\phi(a_t), \phi(b_t)) \rightarrow 0$ and if and only if $d_E(\psi(a_t), \psi(b_t)) \rightarrow 0$.

Lemma 9.4 : *If $a, b \in \lambda$ with $\psi(a) = \psi(b)$, then either $a = b$ or $\sigma(a, b) > \eta/2$.*

Proof : Suppose $\sigma(a, b) \leq \eta/2$. Let \vec{a}, \vec{b} be unit tangent vectors at a, b with $\psi(\vec{a}) = \psi(\vec{b})$. Then $\psi(a_t) = \psi(b_t)$ for all $t \in \mathbf{R}$. Applying Lemma 9.3 for $t \geq 0$ we get $\vec{a} \approx \vec{b}$, and for $t \leq 0$, we get $-\vec{a} \approx -\vec{b}$. It follows that $a = b$. \diamond

Now consider the map $\psi : \lambda \rightarrow \Lambda = \psi(\lambda)$. Given $n \in \mathbf{N}$, let $\Lambda(n) = \{c \in \Lambda \mid |\psi^{-1}(c)| \geq n\}$, and let $\lambda(n) = \psi^{-1}(\Lambda(n))$. In view of Lemma 9.4, $\Lambda(n)$ and hence $\lambda(n)$ is closed. Moreover, these sets are invariant under flow along leaves. In particular, $\lambda(n)$ is a sublamination of λ .

Now choose n maximal so that $\lambda(n) \neq \emptyset$. Since ψ is not injective, $n \geq 2$. Let $\xi = \lambda(n)$ and let $\Xi = \Lambda(n)$. The map $\psi : \xi \rightarrow \Xi$ is now everywhere n to 1. This is also the case for the lifted map $\psi : \vec{\xi} \rightarrow \vec{\Xi}$.

We define the closed equivalence relation, \sim , on ξ by writing $a \sim b$ if $\psi(a) = \psi(b)$. We can thus identify ξ with Ξ/\sim . We similarly define \sim on $\vec{\xi}$, so that $\vec{\Xi}$ gets identified

with $\vec{\xi}/\sim$. We write $[a]$ for the equivalence class of a .

Lemma 9.5 : *For all $h \leq \eta/4$, there is some h' such that if $a, b, x \in \lambda$ with $a \sim x$, $\sigma(a, b) \leq h'$ and $a \neq b$. Then there is some $y \in \lambda \setminus \{x\}$ with $y \sim b$ and $\sigma(x, y) < h$.*

Proof : If this fails, there is a sequence $b_i \rightarrow a \in \lambda$ with $b_i \neq a$ and such that no point of $N(x, h) \setminus \{x\}$ is equivalent to b_i . Now by Lemma 9.4, as z varies over $[a] = [x]$, the neighbourhoods $N(z, h)$ are disjoint, and each can contain at most one element of $[b_i]$. Since all these classes have n elements, the pigeon-hole principle tells us that there is some $c_i \sim b_i$ with $\sigma(c_i, [a]) \geq h$. Passing to a subsequence, we get $c_i \rightarrow c \neq [a]$. But $b_i \rightarrow a$, contradicting the fact that \sim is closed. \diamond

Lemma 9.6 : *Suppose $\vec{a}, \vec{b}, \vec{x}, \vec{y} \in \vec{\xi}$ with $\sigma(a, b) < \eta/2$, $\sigma(x, y) < \eta/2$, $\vec{a} \sim \vec{x}$ and $\vec{b} \sim \vec{y}$. If $\vec{a} \approx \vec{b}$, then $\vec{x} \approx \vec{y}$.*

Proof : As observed before Lemma 9.3, we have $d_E(\psi(a_t), \psi(b_t)) \rightarrow 0$. But $\psi(a_t) = \psi(x_t)$ and $\psi(b_t) = \psi(y_t)$ for all t , and so $d_E(\psi(x_t), \psi(y_t)) \rightarrow 0$. By Lemma 9.3, $\vec{x} \approx \vec{y}$ as claimed. \diamond

Lemma 9.7 : *Suppose $\vec{a}, \vec{b}, \vec{x}, \vec{y} \in \vec{\xi}$ with $\sigma(a, b) < \eta/2$, $\sigma(x, y) < \eta/2$, $\vec{a} \approx \vec{b}$ and $\vec{x} \approx \vec{y}$. If $\vec{a} \sim \vec{x}$, then $\vec{b} \sim \vec{y}$.*

Proof : Let $h' > 0$ be the constant of Lemma 9.5 given $h = \eta/4$. We can assume that $h' \leq \eta/2$. Choose $t \geq 0$ so that $\sigma(a_t, b_t) < h'$. By Lemma 9.5, applied to a_t, b_t, x_t , there is some $z \in \lambda \setminus \{x_t\}$ with $z \sim b_t$ and $\sigma(x_t, z) < \eta/2$. Let $\vec{z} \in \vec{\lambda}$ be the vector with $\vec{z} \sim \vec{b}_t$. Now $\sigma(a_t, b_t) < \eta/2$ and $\vec{a}_t \sim \vec{x}_t$. Thus, applying Lemma 9.6 with $\vec{a}_t, \vec{b}_t, \vec{x}_t, \vec{z}$, we get $\vec{x}_t \approx \vec{z}$. But $\vec{x}_t \approx \vec{y}_t$, and $\vec{z} \neq \vec{x}_t$. Thus, since any \approx -class has at most two elements, it follows that $\vec{z} = \vec{y}_t$. In other words, $\vec{b}_t \sim \vec{y}_t$, and it follows that $\vec{b} \sim \vec{y}$ as claimed. \diamond

Lemmas 9.5, 9.6 and 9.7 are effectively telling us that the map $\psi : \xi \rightarrow \Xi$ is a covering space. This can be made precise as follows. Let $\Phi = \Upsilon(\xi) \subseteq \Sigma$ be the space obtained by filling in the spikes of $\Sigma \setminus \xi$ as described earlier. Note that $\Phi \subseteq N$. We can extend \sim to a closed equivalence relation on Φ as follows.

Suppose $p \in \Phi \setminus \xi$. Then p lies on a geodesic $[a, b]$, with $a, b \in \xi$, $\vec{a} \approx \vec{b}$ and $\sigma(a, b) \leq \eta/8$. Suppose $q \in \Phi \setminus \xi$ similarly lies in $[x, y]$. Then $\vec{a} \sim \vec{x}$ if and only if $\vec{b} \sim \vec{y}$. In this case, we write $p \sim q$ if and only if p cuts $[a, b]$ in the same ratio that q cuts $[x, y]$. It is now readily checked that \sim is a closed equivalence relation with n points in each class. Let $\Omega = \Phi/\sim$. The quotient map $\omega : \Phi \rightarrow \Omega$ is an n -fold covering. Each component of Ω is a circle or a closed surface.

Suppose $p, q, a, b, x, y \in \Phi$ are as above. By assumption $\sigma(a, b) \leq \eta/8$ and $\sigma(x, y) \leq \eta/8$. Since $\phi : N \rightarrow O$ is 1-lipschitz and $\phi(a) = \phi(x)$ and $\phi(b) = \phi(y)$, we see that $d(\phi(a), \phi(b)) \leq \eta/8$. Put another way, if $z \in \Omega$, then $\text{diam } \phi(\omega^{-1}(z)) \leq \eta/8$. Note also that $\phi(\Phi) \subseteq N(\phi(\lambda), \eta/8)$.

Recall that $\phi : N \rightarrow O$ is a limit of the maps $\phi_i : N_i \rightarrow O_i$. Moreover, by construction, $\Phi \subseteq N_i$ for all sufficiently large i . We see that for all large i , $\text{diam } \phi(\omega^{-1}(z)) < \eta/4$, say, for all $z \in \Omega$, and that $\phi(\Phi) \subseteq N(\phi_i(\lambda_i), \eta/4)$. Note that, by the condition of injectivity radius of $\phi_i(\lambda_i)$ in O_i , we see that $\phi_i(\omega^{-1}(z))$ lies in a hyperbolic $(\eta/2)$ -ball embedded in O_i .

We now fix some such i , and set $\mu(z) \in O_i$ to be the centre of $\phi_i(\omega^{-1}(z))$, in other words, the point so that $\phi_i(\omega^{-1}(z))$ lies in the closed r -ball about $\mu(z)$ for r minimal. Here $r \leq \eta/8$ and the point is uniquely defined, given the fact that $\phi_i(\omega^{-1}(z))$ lies in a hyperbolic $(\eta/2)$ -ball embedded in O_i . Moreover, it gives us a continuous map, $\mu : \Omega \rightarrow O_i$. Also, for each $x \in \Phi$, $d(\phi_i(x), \mu(\omega(x))) < \eta/4$. So again by the condition of injectivity radius, we see that the maps $\phi_i : \Phi \rightarrow O_i$ and $\mu \circ \omega : \Phi \rightarrow O_i$ are homotopic (by linear homotopy along short geodesics).

By hypothesis, $\theta_i \circ \phi_i : \Phi \rightarrow \Sigma$ (being a restriction of $\theta_i \circ \phi_i : N_i \rightarrow \Sigma$) is homotopic to inclusion, and so therefore is $\theta_i \circ \mu \circ \omega : \Phi \rightarrow \Sigma$. Writing $f = \theta_i \circ \mu$, we can summarise this as follows:

Lemma 9.8 : *We have an n to 1 covering map $\omega : \Phi \rightarrow \Omega$, with $n \geq 2$, and a map $f : \Omega \rightarrow \Sigma$ such that $f \circ \omega : \Omega \rightarrow \Sigma$ is homotopic to inclusion. \diamond*

In order to get a contradiction, we make the following purely topological observation.

Lemma 9.9 : *Suppose Φ is a (not necessarily connected) subsurface of the compact surface Σ . Suppose that $\omega : \Phi \rightarrow \Omega$ is a n -fold covering map to a (not necessarily connected) surface Ω . Suppose that there is a map $f : \Omega \rightarrow \Sigma$ such that $f \circ \omega : \Phi \rightarrow \Sigma$ is homotopic to inclusion. Then either $n = 1$ or Φ is homotopic into a 1-dimensional submanifold of Σ (a multicurve union the boundary of Σ).*

Proof : It is clearly sufficient to prove the result when Σ is connected. We first note that we can also reduce to the case where Φ is connected. For if Φ_0 is a component of Φ , then $\omega|_{\Phi_0}$ is an n_0 -fold cover of a component, Ω_0 , of Ω . Together with $f|_{\Omega_0}$, this satisfies the hypotheses of the lemma. Suppose that Ω is not homotopic into a closed curve. The lemma then tells us that $n_0 = 1$. If $n > 1$, then some other component, Φ_1 , of Φ also gets mapped homeomorphically to Ω_0 , so the inclusions of Ω_0 and Ω_1 into Σ are homotopic. In other words, Ω_0 can be homotoped to be disjoint from itself in Σ . But this is impossible since we are assuming that it cannot be homotoped into a curve. We conclude that all components of Ω are homotopic into curves. Moreover, it is easily seen that we can take these curves to be disjoint in Σ giving the result.

Let us therefore suppose that Φ is connected. Suppose first that each (intrinsic) boundary component of Φ is homotopically trivial in Σ . If Φ is not homotopic to a point in Σ , then Σ is closed, and each component of $\Sigma \setminus \Phi$ is a disc. Let Ω' be the closed surface obtained by gluing a disc to each boundary component of Ω . We can now extend ω to an n -fold branched cover $\omega' : \Sigma \rightarrow \Omega'$, and extend f to a map $f' : \Omega \rightarrow \Sigma$. The composition $f' \circ \omega' : \Sigma \rightarrow \Sigma$ is homotopic to the identity, and it follows that $n = 1$.

Suppose that α is a boundary curve of Φ that is homotopically non-trivial in Σ . Its inclusion into Σ factors through the boundary curve, $\omega(\alpha)$, of Ω . It follows that

$\omega|_{\alpha}$ is injective. If $n > 1$, then some other boundary curve, β , of Φ also gets mapped homeomorphically to $\omega(\alpha)$. Now α and β are homotopic in Σ , and hence bound an annulus. Unless this annulus contains Φ , it must be a component of $\Sigma \setminus \Phi$. No other boundary component of Φ can be homotopic to this annulus. We see that $n = 2$, and that the homotopically non-trivial boundary components of Φ occur in pairs that bound annular components of $\Sigma \setminus \Phi$. It follows that Σ is closed and that each component of $\Sigma \setminus \Phi$ is either such an annulus or a disc. Let Ω'' be the surface obtained by gluing a disc to each boundary curve of Ω whose f -image is trivial in Σ . We now extend ω to a map $\omega'' : \Sigma \rightarrow \Omega''$ by collapsing the annular components of $\Sigma \setminus \Phi$ to boundary curves, without twisting, and extending (anyhow) over the disc components. Let $f'' : \Omega'' \rightarrow \Sigma$ be any extension of f . Thus, $f'' \circ \omega'' : \Sigma \rightarrow \Sigma$ is homotopic to the identity. But it factors through the non-closed surface Ω'' , giving a contradiction. \diamond

We can now apply this to the situation described by Lemma 9.8. Certain components of Φ may be circles, as we have defined it, but these can be thickened up to annuli, so that makes no essential difference. From the construction, no two distinct components of Ω can be homotoped into the same closed curve. Lemma 9.9 now gives the contradiction that $n = 1$, finally proving Proposition 9.1.

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