$\boxed{\text{Exercises 16}}$

(1) let X, Y be tangent vector fields to S. Show $D_X Y - D_Y X$ is also tangential, and deduce this is equal to $\nabla_X Y - \nabla_Y X$.

This defines the <u>Lie bracket</u> $[X, Y]$

Show if $(u,v) \mapsto r(u,v)$ then $[r_u, r_v] = 0$

Suppose $F : S \to \Sigma$ is a diffeo, and F_* is the derivative of F at p. Show $F_*[X,Y] = [F_* X, F_* Y]$.

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let $r : U \to S$ be the parametrisation.
$T_p S$ is spanned by r_u, r_v.

Let $X = \lambda_1 r_u + \mu_1 r_v$, $\lambda_1, \lambda_2, \mu_1, \mu_2 \in C^\infty(S)$
$\quad\; Y = \lambda_2 r_u + \mu_2 r_v$

$D_X Y - D_Y X$

$= \lambda_1 D_u Y + \mu_1 D_v Y - \lambda_2 D_u X - \mu_2 D_v Y$

$= \underbrace{\cancel{\lambda_1 \lambda_2 D_u r_u}}_{A} + \lambda_1 r_u D_u \lambda_2 + \underbrace{\cancel{\lambda_1 \mu_2 D_u r_u}}_{D} + \lambda_1 r_v D_u \mu_2$

$+ \underbrace{\cancel{\mu_1 \lambda_2 D_v r_u}}_{C} + \mu_1 r_u D_v \mu_2 + \underbrace{\cancel{\mu_1 \mu_2 D_v r_v}}_{B} + \mu_1 r_v D_v \mu_2$

$- \underbrace{\cancel{\lambda_2 \lambda_1 D_u r_u}}_{A} - \lambda_2 r_u D_u \lambda_1 - \underbrace{\cancel{\lambda_2 \mu_1 D_u r_v}}_{C} - \lambda_1 r_v D_u \mu_1$

$- \underbrace{\cancel{\mu_2 \lambda_1 D_v r_u}}_{D} - \mu_2 r_u D_v \lambda_1 - \underbrace{\cancel{\mu_2 \mu_1 D_v r_v}}_{B} - \mu_2 r_v D_v \mu_1$

$$= r_u \left(\lambda_1 D_u \lambda_2 + \mu_1 D_v \lambda_2 - \lambda_2 D_u \lambda_1 - \mu_2 D_v \lambda_1 \right)$$

$$+ r_v \left(\lambda_1 D_u \mu_2 + \mu_1 D_v \mu_2 - \lambda_1 D_u \mu_1 - \mu_2 D_v \mu_1 \right)$$

$$= r_u \left(D_X \lambda_2 - D_Y \lambda_1 \right) + r_v \left(D_X \mu_2 - D_Y \mu_1 \right)$$

Thus $D_X Y - D_Y X$ can be written as $A r_u + B r_v$ for $A, B \in C^\infty(S)$. $\therefore$ $D_X Y - D_Y X$ is tangential.

$$\nabla_X Y - \nabla_Y X = D_X Y - D_X Y \cdot N - D_Y X + D_Y X \cdot N \qquad \left[N = \text{normal vector field.} \right]$$

$$= D_X Y - D_Y X \underbrace{- N \cdot (D_X Y - D_Y X)}_{= 0 \text{ since tangential.}}$$

$$f_*[X, Y] = f_* \left[r_u \left(D_X \lambda_2 - D_Y \lambda_1 \right) + r_v \left(D_X \mu_2 - D_Y \mu_1 \right) \right]$$

f a diffeo so define $\tilde{r} = r \circ f^{-1}$. This gives a chart on Σ, and moreover each tangent space has basis $\tilde{r}_u, \tilde{r}_v$. $\left[= f^*(r_u), \; f^*(r_v) \right]$

$$(\text{Chain rule}) \quad = \tilde{r}_u \left(D_{f^*(X)}(f \circ \lambda_2) - D_{f^*(Y)}(f \circ \lambda_1) \right) \qquad (*)$$

$$+ \tilde{r}_v \left(D_{f^*(X)}(f \circ \mu_2) - D_{f^*(Y)}(f \circ \mu_1) \right)$$

N.B. $f_*(X) = f_*(\lambda_1 \cdot r_u + \mu_1 \cdot r_v) = (f \circ \lambda_1)\, \tilde{r}_u + (f \circ \mu_1)\, \tilde{r}_v$

So $f_*[X, Y] = [f_* X, f_* Y]$

$\left[= (*) \text{ by calculation above, and chain rule} \right]$

(2) Suppose X is a tangential vector field on S. Show that if $\nabla_X f = 0$ for each smooth $f : S \to \mathbb{R}$, then $X = 0$ everywhere.

Deduce X is determined by the operation of ∇_X on smooth functions.

Show that if X, Y are smooth vector fields, and f a smooth function then:

$$\nabla_{[X,Y]} f = \nabla_X \nabla_Y f - \nabla_Y \nabla_X f$$

Show that $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

(JACOBI IDENTITY)

$\nabla_X f = \lambda f_u + \mu f_v$ $\qquad X = \lambda r_u + \mu r_v$

$\lambda, \mu \in C^\infty(S)$

(Abuse of notation - e.g. $f_u = \frac{\partial}{\partial u}(f \circ r)$)

Now, if $(\lambda, \mu) \neq (0,0)$ (as smooth functions), then we can always choose some f s.t. f_u, f_v are linearly independent.

So $\nabla_X f = 0 \iff \lambda, \mu \equiv 0 \iff X = 0 r_u + 0 r_v = 0$

Now suppose $\nabla_X f = \nabla_Y f$ $\quad \forall f \in C^\infty(S)$.

Then $\nabla_{X-Y} f \equiv 0 \iff X - Y = 0 \iff X = Y$.

So we can determine X by its action on smooth function

Now:

$$\nabla_{[X,Y]} f = \nabla_{\nabla_X Y - \nabla_Y X} f$$

$$= f_u(\nabla_X \lambda_2 - \nabla_Y \lambda_1) + f_v(\nabla_X \mu_2 - \nabla_Y \mu_1)$$

(product rule)
$$= \nabla_X(f_u \lambda_2) - \lambda_2 \nabla_X f_u - \nabla_Y(f_u \lambda_1) + \lambda_1 \nabla_Y f_u$$
$$+ \underbrace{\nabla_X(f_v \mu_2)} - \mu_2 \nabla_X f_v - \underbrace{\nabla_Y(f_v \mu_1)} + \mu_1 \nabla_Y f_v$$

$$= \nabla_X(\nabla_Y f) - \nabla_Y(\nabla_X f)$$
$$+ \Big(-\overset{A}{\cancel{\lambda_1 \lambda_2 \nabla_u}} f_u - \overset{B}{\cancel{\mu_1 \lambda_2 \nabla_v}} f_u + \overset{A}{\cancel{\lambda_1 \lambda_2}} \nabla_u f_u + \overset{C}{\cancel{\lambda_1 \mu_2 \nabla_v}} f_u$$
$$- \underset{C}{\cancel{\mu_2 \lambda_1 \nabla_u}} f_v - \underset{D}{\cancel{\mu_2 \mu_1 \nabla_v}} f_v + \underset{B}{\cancel{\mu_1 \lambda_2 \nabla_u}} f_v + \underset{D}{\cancel{\mu_1 \mu_2 \nabla_v}} f_v \Big)$$

(Noting that $\nabla_v f_u = f_{vu} = f_{uv} = \nabla_u f_v$)

<u>JACOBI IDENTITY</u> $[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0$

Note that $\nabla_{[[X,Y],Z]} f = \nabla_{[X,Y]} \nabla_Z f - \nabla_Z \nabla_{[X,Y]} f$

$$= \nabla_X \nabla_Y \nabla_Z f - \nabla_Y \nabla_X \nabla_Z f$$
$$- \nabla_Z \nabla_X \nabla_Y f - \nabla_Z \nabla_Y \nabla_X f.$$

The 3 terms produce 6 positive terms $\nabla_i \nabla_j \nabla_k f$
~~and~~ 6 negative terms $-\nabla_i \nabla_j \nabla_k f$.

i,j,k are determined by permutations of X, Y, Z - of which there are exactly 6. Each possibility for i,j,k arises once as + and once as - since the 3 terms themselves are cyclic permutations.
$\therefore$ By a symmetry argument, everything cancels when summed.

(3) Given three tangential vector fields X, Y, Z, write:

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

("Curvature Tensor")

Show that if $\lambda, \mu, \nu \in C^\infty(S)$ then:

$$R(\lambda X, \mu Y, \nu Z) = \lambda \mu \nu \, R(X,Y,Z)$$

$$R(\lambda X, \mu Y, \nu Z) = \nabla_{\lambda X} \nabla_{\mu Y} (\nu Z) - \nabla_{\mu Y} \nabla_{\lambda X} \nu Z - \nabla_{[\lambda X, \mu Y]} \nu Z$$

$$= \left[\nu \nabla_{\lambda X} \nabla_{\mu Y} Z + \cancel{\nabla_{\mu Y} Z \, \underbrace{\nabla_{\lambda X} \nu}_{3}} + \cancel{\underbrace{\nabla_{\mu Y} \nu}_{2}} \nabla_{\lambda X} Z + Z \cancel{\underbrace{\nabla_{\lambda X} \nabla_{\mu Y} \nu}_{1}} \right]$$

$$- \left[\nu \nabla_{\mu Y} \nabla_{\lambda X} Z + \cancel{\nabla_{\lambda X} Z \, \overset{2}{\nabla_{\mu Y} \nu}} + \cancel{\overset{3}{\nabla_{\lambda X} \nu}} \nabla_{\mu Y} Z + Z \cancel{\overset{1}{\nabla_{\mu Y} \nabla_{\lambda X} \nu}} \right]$$

$$- \Big[\nu \nabla_{[\lambda X, \mu Y]} Z + \cancel{Z \overset{1}{\nabla_{[\lambda X, \mu Y]} \nu}} \Big]$$

these terms cancel

Now, (for example) $\nu \nabla_{\lambda X} \nabla_{\mu Y} Z$

$$= \nu \nabla_{\lambda X} (\mu \nabla_Y Z)$$

$$= \nu \lambda \mu \, \nabla_X \nabla_Y Z + \nabla_Y Z \cdot \nu \lambda \nabla_X \mu$$

$$\nu \nabla_{\mu Y} \nabla_{\lambda X} Z = \nu \lambda \mu \, \nabla_Y \nabla_X Z + \nabla_X Z \cdot \mu \nu \nabla_X \lambda$$

$$\nu \nabla_{[\lambda X, \mu Y]} Z = \nu \nabla_{\lambda \mu \nabla_X Y + \lambda \nabla_X \mu Y - \mu \lambda \nabla_Y X - \mu \nabla_Y \lambda X} Z$$

$$= \nu \lambda \mu \, \nabla_{\nabla_X Y - \nabla_Y X} Z + \nu \lambda \nabla_{\nabla_X \mu Y} Z - \nu \mu \nabla_{\nabla_Y \lambda X} Z$$

$$= \nu \lambda \mu \, \nabla_{[X,Y]} Z + \nu \lambda \cdot \nabla_X \mu \lambda \cdot \nabla_Y Z - \nu \mu \nabla_Y \lambda \nabla_X Z$$

Thus we are left with

$$R(\lambda X, \mu Y, \nu Z) = \lambda\mu\nu \, \nabla_X \nabla_Y Z - \lambda\mu\nu \, \nabla_Y \nabla_X Z - \lambda\mu\nu \, \nabla_{[X,Y]} Z$$

$$= \lambda\mu\nu \, R(X,Y,Z)$$

NOTES:

It is easy to confuse vector fields with smooth functions. The covariant derivative generalises to differentiate both these objects - make sure you understand how.

I have underlined in red the smooth functions in the calculation.

Finally, although in general $\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \neq \nabla_{[X,Y]} Z$,

we do have $\nabla_X \nabla_Y \lambda - \nabla_Y \nabla_X \lambda = \nabla_{[X,Y]} \lambda \quad \forall \lambda \in C^\infty(S)$.

(Why?)

Really finally, Note that $\nabla_{\lambda X} Y = \lambda \nabla_X Y$.

(again convince yourself of this - $\nabla_{\lambda X} Y$ differentiates Y in the direction of the tangent field λX. At a given point $p \in S$ λX is simply a scalar multiple of X.)

(4) Let $r(u,v) = (\lambda(u)\cos v, \lambda(u)\sin v, \mu(u))$ be a surface of revolution

Show the first and second fundamental forms are given respectively by:

$$(\lambda'^2 + \mu'^2)\,du^2 + \lambda^2 dv^2$$

$$\frac{1}{\sqrt{\lambda'^2+\mu'^2}}\left([\lambda'\mu'' - \lambda''\mu']\,du^2 + \lambda\mu'\,dv^2\right)$$

Calculate the matrix for the shape operator. Show the principal curvatures are given by:

$$K_1 = \frac{\lambda''\mu' - \lambda'\mu''}{\sqrt{\lambda'^2 + \mu'^2}^{\,3}} \qquad K_2 = \frac{-\mu'}{\lambda\sqrt{\lambda'^2+\mu'^2}}$$

and that the Gauss curvature is given by:

$$K = \frac{\lambda'\mu'\mu'' - \lambda''(\mu')^2}{\lambda[\lambda'^2 + \mu'^2]}$$

$$r_u = (\lambda'\cos v, \lambda'\sin v, \mu')$$
$$r_v = (-\lambda\sin v, \lambda\cos v, 0)$$

$$\Rightarrow E = \lambda'^2 + \mu'^2 \qquad F = 0 \qquad G = \lambda^2$$

$$n = \frac{r_u \times r_v}{\|r_u \times r_v\|} = \frac{1}{\lambda\sqrt{\mu'^2 + \lambda'^2}}(-\lambda\mu'\cos v, -\lambda\mu'\sin v, \lambda\lambda')$$

$$r_{uu} = (\lambda''\cos v, \lambda''\sin v, \mu'') \qquad r_{uv} = (-\lambda'\sin v, \lambda'\cos v, 0)$$
$$r_{vv} = (-\lambda\cos v, -\lambda\sin v, 0)$$

So:

$$e = \frac{1}{\sqrt{\mu'^2+\lambda'^2}} \left(-\mu'\lambda'' + \mu''\lambda'\right)$$

$$f = 0$$

$$g = \frac{1}{\sqrt{\mu'^2+\lambda'^2}} \left(\lambda\mu'\right)$$

The Shape Operator matrix is given by

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

$$= \frac{1}{\sqrt{\mu'^2+\lambda'^2}} \cdot \frac{1}{\lambda^2(\mu'^2+\lambda'^2)} \begin{pmatrix} [(\lambda')^2+(\mu')^2]\lambda\mu' & 0 \\ 0 & [\mu''\lambda'-\mu'\lambda'']\lambda^2 \end{pmatrix}$$

so $$k_1 = \frac{\lambda\mu'(\lambda'^2+\mu'^2)}{\lambda^2\sqrt{\mu'^2+\lambda'^2}^{\,3}} = \frac{\mu'}{\lambda\sqrt{\mu'^2+\lambda'^2}}$$

$$k_2 = \frac{\mu''\lambda'-\mu'\lambda''}{\sqrt{\mu'^2+\lambda'^2}^{\,3}}$$

and $$k = k_1 k_2 = \frac{\mu'(\mu''\lambda'-\mu'\lambda'')}{(\mu'^2+\lambda'^2)^2}.$$