Exercises 5

(1) Suppose $A \in Sym_2(\mathbb{R})$ is positive semi-definite (i.e. $x^T A x \geq 0 \; \forall x \in \mathbb{R}^2$)
show $\det A \geq 0$

Suppose that $B \in Sym_2(\mathbb{R})$ is s.t. $B - A$ is positive semidefinite. Show B is positive semi-definite and that $\det B \geq \det A \geq 0$

Suppose $f, g : \mathbb{R}^2 \to \mathbb{R}$ are smooth with $g(u,v) \geq f(u,v) \geq 0$ for each (u,v), and with $f(0,0) = g(0,0) = 0$.
If K_f, K_g are the respective Gauss curvatures of the graphs of f and g at the origin, that $K_g \geq K_f \geq 0$

$$x^T A x = x^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} x = a x_1^2 + 2b x_1 x_2 + c x_2^2$$

The discriminant of this polynomial is $ac - b^2 = \det A$, thus $x^T A x \geq 0$ iff $\det A \geq 0$.

If $x^T (B - A) x \geq 0 \quad \forall x \in \mathbb{R}^2$, then

$$x^T B x \geq x^T A x \geq 0 \qquad \forall x \in \mathbb{R}^2.$$

This again (linear algebra) $\Rightarrow \det(B) \geq \det(A)$.

(2) Let S be a regular surface and $p \in S$.
Suppose that the normal at p has a non-zero z-component.
Show $\exists$ a chart $r : U \to S$ with $p \in r(U)$ and with
$r(u,v) = (u, v, f(u,v))$ $\forall (u,v) \in U$ for some smooth funct
f. Suppose S lies on one side of its tangent plane at p.
Show that the Gauss curvature at p is non-negative

Existence of the chart follows from the implicit function theorem:

If $\varphi : \underset{\substack{\cup \\ \varphi^{-1}(p)}}{V} \to S$ is any chart, then

$$n = \frac{\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}}{\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|}$$

The normal at p has a non-zero component iff $\frac{\partial \varphi_1}{\partial v} \frac{\partial \varphi_2}{\partial u} \neq \frac{\partial \varphi_2}{\partial v} \frac{\partial \varphi_1}{\partial u}$

i.e. $\det \begin{pmatrix} \frac{\partial \varphi_1}{\partial u} & \frac{\partial \varphi_1}{\partial v} \\ \frac{\partial \varphi_2}{\partial u} & \frac{\partial \varphi_2}{\partial v} \end{pmatrix} \neq 0$

or the matrix is invertible, (evaluated at p).

So we may solve $\varphi_3(x,y)$ in terms of φ_1 and φ_2, in some nhd. of p.

Calculate Gauss curvature in the new coordinates.

$r_u = (1, 0, f_u)$
$r_v = (0, 1, f_v)$

$$n = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}$$

$r_{uu} = (0, 0, f_{uu})$
$r_{uv} = (0, 0, f_{uv})$
$r_{vv} = (0, 0, f_{vv})$

$E = 1 + f_u^2$ $\qquad e = \dfrac{f_{uu}}{\sqrt{1+f_u^2+f_v^2}}$

$F = f_u f_v$ $\qquad f = \dfrac{f_{uv}}{\sqrt{1+f_u^2+f_v^2}}$

$G = 1 + f_v^2$ $\qquad g = \dfrac{f_{vv}}{\sqrt{1+f_u^2+f_v^2}}$

$$eg - f^2 = \frac{f_{uu}f_{vv} - f_{uv}^2}{1+f_u^2+f_v^2}$$

$$EG - F^2 = 1 + f_u^2 + f_v^2$$

so $K = \dfrac{f_{uu}f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2}$

so $K \geq 0$ iff $f_{uu}f_{vv} - f_{uv}^2 \geq 0$

(determinant of)

This expression is the hessian of the function f.

S lies on one side of its tangent plane $\Rightarrow$ f attains either a local minimum or maximum at p.

$\Rightarrow$ the eigenvalues of the hessian are both positive or negative in either case $K > 0$.

If S is locally flat then $K = 0$.

$\downarrow$

i.e. agrees with its tangent plane to higher order derivatives.

(3) Consider the quadratic form on $\mathbb{R}^2$ induced by the matrix $P = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$

Show that $\max\limits_{\|x\|=1} |x^T \cdot P x| = \max\{|\lambda|, |\mu|\}$

Let S be a surface and $p \in S$. Deduce the maximal value of $|e \cdot \nabla_e n|$ for $e \in T_p S$ with $\|e\|=1$ is equal to $\max\{|\kappa_1|, |\kappa_2|\}$ where κ_1, κ_2 are the principal curvatures.

If $\kappa_1, \kappa_2 \neq 0$ show the minimal value is $\min\{|\kappa_1|, |\kappa_2|\}$

Suppose γ is a unit speed curve in S with $\gamma(t) = p$, and that the Gauss curvature of S at p is positive.
Show that $|\gamma''(t) \cdot n| \geq \min\{|\kappa_1|, |\kappa_2|\}$

Deduce that the curvature of γ at p is at least $\min\{|\kappa_1|, |\kappa_2|\}$

Firstly, $x^T P x = \lambda x_1^2 + \mu x_2^2$.
If $\|x\| = 1$, then $x = (\cos\theta, \sin\theta)$ some θ.

$\Rightarrow |x^T P x| = |\lambda|\cos^2\theta + |\mu|\sin^2\theta$

It is clear the maximal value attained is $\max\{|\lambda|, |\mu|\}$ as θ ranges from 0 to 2π.

Now let $e \in T_p S$. $e = a r_u + b r_v$

$e \cdot \nabla_e n = (a r_u + b r_v) \cdot (a n_u + b n_v)$

$= a^2 r_u \cdot n_u + ab(r_v \cdot n_u + r_u \cdot n_v) + b^2 r_v \cdot n_v$

$$\therefore \quad e \cdot \nabla_e n = -\left((a,b) \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right)$$

$$= \Pi(e,e)$$

<u>But also:</u> $e \cdot \nabla_e n = e \cdot (D_e n - (D_e n \cdot n) n)$
$= e \cdot D_e n$
$= -e \cdot \Theta(e)$ where Θ the shape operator

So then the maximal absolute value attained by this expression is the maximum of the eigenvalues of the shape operator. (This is because the matrix for a quadratic form agrees up to [orthogonal] conjugation).

i.e $\max_{\|e\|=1} |e \cdot \nabla_e n| = \max\{|k_1|, |k_2|\}$ as before.

N.B $P = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}$ and $\underline{\underline{\text{not}}}$ $\begin{pmatrix} e & f \\ f & g \end{pmatrix}$

because we initially we expressed $\nabla_e n$ in terms of the tangent space basis n_u, n_v and not r_u, r_v.

Now, $\gamma'' \cdot n = K_\gamma N_\gamma \cdot n$

If T is the tangent to γ, $T \in T_pS$

$K_\gamma N_\gamma = k_s N_s + (K_\gamma N_\gamma \cdot n) n$ $\quad \begin{bmatrix} \text{where } N_s \in T_pS \\ N_s \cdot T = 0 \end{bmatrix}$

Now, $T' \cdot n = -T \cdot n'$
$= -T \cdot D_T n$

$$
\begin{aligned}
\text{So } \gamma'' \cdot n &= -T \cdot D_T n \\
&= \mathrm{II}(T, T) \\
&= T \cdot \nabla_T n
\end{aligned}
$$

Now, Gauss curvature $\geq 0 \Rightarrow \kappa_1, \kappa_2$ have the same sign.

$$\Rightarrow \min\{|\kappa_1|, |\kappa_2|\} \leq |e \cdot \nabla_e n| \leq \max\{|\kappa_1|, |\kappa_2|\}$$

Thus $|\gamma'' \cdot n| \geq \min\{|\kappa_1|, |\kappa_2|\}$

$$
\begin{aligned}
\text{And } |\gamma'' \cdot n| &= |\kappa_\gamma| |N_\gamma \cdot n| \\
&= \kappa_\gamma |\cos\theta| \quad \text{where } \theta \text{ the angle between } N_\gamma \text{ and } n.
\end{aligned}
$$

So the lower bound on $|\gamma'' \cdot n|$ applies equally well to κ_γ.

(4) Let γ be a smooth unit-speed curve in a regular surface S.
Write T, N_S and n respectively for the tangent to γ, the normal to γ in S and the normal to γ in $\mathbb{R}^3$.
Let Π denote the second fundamental form on $T_{\gamma(t)}S$, and let κ_S be the geodesic curvature of γ in S.

Show that:
$$T' = \kappa_S N_S + \Pi(T,T)\, n$$
$$N_S' = -\kappa_S T + \Pi(T, N_S)\, n$$
$$n' = -\Pi(T,T)\,T - \Pi(T,N_S)\,N_S$$

First note (T, N_S, n) is an orthonormal basis:

Recall that $\frac{D}{ds}\gamma'(s)$ is the projection of T' to the tangent plane.

Recall $T' = \kappa_\gamma N_\gamma$, so $\frac{D}{ds}\gamma'(s)$ is orthogonal to T.

We define a unit vector N_S s.t. $\frac{D}{ds}\gamma'(s) =: \kappa_S N_S$.

Note that N_S is orthogonal to T. So T, N_S span the tangent plane $T_{\gamma(t)}S$.

<u>So by definition:</u>

$$\kappa_S N_S = T' - (T' \cdot n)\, n$$

$$\Rightarrow T' = \kappa_S N_S + (T' \cdot n)\, n$$

We differentiate $T \cdot n = 0$ $\quad [n' =$

$\Rightarrow \quad T' \cdot n + T \cdot n' = 0$

$\Rightarrow \quad -T \cdot n' = T' \cdot n$

$\Pi(T,T) =$

$[\text{Recall } \Pi(e,f) = -e \cdot D_f n \quad \text{for } e, f \in T_p S]$

So we get $\underline{T' = k_s N_s + \Pi(T,T) n.}$

Similarly if we differentiate $N_s \cdot n = 0$

$\Rightarrow \quad N_s' \cdot n = -n' \cdot N_s$

$= \Pi(T, N_s).$

So $N'_s = (N'_s \cdot T) T + \underbrace{\cancel{(N'_s \cdot N_s)}}_{0} N_s + (N'_s \cdot n) n$

and $N'_s \cdot T = -N_s \cdot T'$ $\quad$ (differentiating $N_s \cdot T = 0$)

giving: $N'_s \cdot T = -N_s \cdot (k_s N_s + \Pi(T,T) n)$

$= -k_s$

So $\underline{N'_s = -k_s T + \Pi(T, N_s) n}$

Finally, observe that

$n' = (n' \cdot T) T + (n' \cdot N_s) N_s + \underbrace{\cancel{(n' \cdot n)}}_{0} n$

$= \underline{\Pi(T,T) T + \Pi(T, N_s) N_s.}$