Exercises 2

(1) Calculate the Frenet frame, curvature, torsion of the curve $\gamma : t \longmapsto (t, at, bt^2)$. Verify Frenet-Serret.

$$\frac{d\gamma}{dt} = (1, a, 2bt) \qquad \|\gamma'(t)\| = \frac{ds}{dt} = \sqrt{1+a^2+4b^2t^2}$$

$$T = \frac{d\gamma}{dt}\frac{dt}{ds} = \frac{(1,a,2bt)}{\sqrt{1+a^2+4b^2t^2}}$$

$$\kappa N = \frac{dT}{ds} = \frac{dT}{dt}\frac{dt}{ds}$$

$$= \frac{1}{\sqrt{1+a^2+4b^2t^2}}\left[\frac{(0,0,2b)}{\sqrt{1+a^2+4b^2t^2}} - \frac{4b^2t\,(1,a,2bt)}{\sqrt{1+a^2+4b^2t^2}^{\,3}}\right]$$

$$= \frac{1}{(1+a^2+4b^2t^2)^2}\left(-4b^2t, -4b^2at, 2b(1+a^2)\right)$$

$$\kappa = \frac{2b}{(1+a^2+4b^2t^2)^2}\sqrt{4b^2t^2(1+a^2) + (1+a^2)^2}$$

$$= \frac{2b\sqrt{1+a^2}}{\sqrt{1+a^2+4b^2t^2}^{\,3}} \Rightarrow N = \frac{(-2bt, -2bat, 1+a^2)}{\sqrt{1+a^2}\sqrt{1+a^2+4b^2t^2}}$$

$$B = T \times N = \frac{\begin{pmatrix} (1+a^2)a \\ +4b^2at^2 \end{pmatrix}, \begin{pmatrix} -4b^2t^2 \\ -(1+a^2) \end{pmatrix}, \begin{pmatrix} -2bat \\ +2abt \end{pmatrix}}{\sqrt{1+a^2}\,(1+a^2+4b^2t^2)}$$

$$= \frac{(a,-1,0)}{\sqrt{1+a^2}}$$

$\Rightarrow B' \equiv 0$ so $\tau \equiv 0$.

At $t=0$ $\quad T = \frac{(1,a,0)}{\sqrt{1+a^2}} \quad N = (0,0,1), \quad B = \frac{(a,-1,0)}{\sqrt{1+a^2}}$

$\kappa = \frac{2b}{1+a^2} \quad \tau = 0.$

Frenet-Serret:

$$\frac{dN}{ds} = \frac{dN}{dt}\frac{dt}{ds} = \frac{1}{\sqrt{1+a^2}\sqrt{1+a^2+4b^2t^2}^{\,2}}\left[\frac{(-2b,-2ab,0)}{\sqrt{1+a^2+4b^2t^2}} - \frac{4b^2t(-2bt,-2bat,1+a^2)}{\sqrt{1+a^2+4b^2t^2}^{\,3}}\right]$$

$$= \frac{\begin{pmatrix} -2b(1+a^2+4b^2t^2) \\ +8b^3t^2 \end{pmatrix}, \begin{pmatrix} -2ab(1+a^2+4b^2t^2) \\ +8ab^3t^2 \end{pmatrix}, -(1+a^2)(4b^2t^2)}{\sqrt{1+a^2}\,(1+a^2+4b^2t^2)^2}$$

$$= \frac{\sqrt{1+a^2}\,(-2b, -2ab, -4b^2t^2)}{(1+a^2+4b^2t^2)^2}$$

$$-\kappa T + \tau B = \frac{-2b\sqrt{1+a^2}\,(1,a,2bt)}{(1+a^2+4b^2t^2)^2} + 0$$

(2) Consider the curve $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3$ given by

$$\gamma(s) = (a \cos ws,\ a \sin ws,\ bws) \qquad \text{(The Helix)}$$

where $a > 0$, b, w constant.

Calculate w, if s is an arc-length parameter. Calculate the Frenet frame, curvature, torsion of γ at an arbitrary point. Verify Frenet-Serret in this case.

Show that for suitable a, b we can find such a curve with any given constant torsion and constant positive curvature.

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s arclength $\Rightarrow \|\gamma'(s)\| \equiv 1$.

$$\gamma'(s) = (-aw \sin ws,\ aw \cos ws,\ bw)$$

$$\|\gamma'(s)\| = \sqrt{a^2w^2 + b^2w^2} = 1$$

$$\Rightarrow \quad w = \frac{1}{\sqrt{a^2+b^2}}$$

$$T = (-aw \sin ws,\ aw \cos ws,\ bw)$$

$$\kappa N = (-aw^2 \cos ws,\ -aw^2 \sin ws,\ 0)$$

$$\kappa = aw^2 = \frac{a}{a^2+b^2}$$

$$N = (-\cos ws,\ -\sin ws,\ 0)$$

$$B = (+bw \sin ws,\ -bw \cos ws,\ aw)$$

$$-\tau N = \left(+bw^2 \cos ws,\ +bw^2 \cos ws,\ 0 \right)$$

$$\Rightarrow \tau = +bw^2 = \frac{+b}{a^2+b^2}$$

Frenet-Serret: $\quad \frac{dN}{ds} = -\kappa T + \tau B$

LHS: $(w \sin ws,\ -w \cos ws,\ 0)$

RHS: $\left(a^2 w^3 \sin ws,\ -a^2 w^3 \cos ws,\ -abw^3 \right)$

$\quad + \left(b^2 w^3 \sin ws,\ -b^2 w^3 \cos ws,\ +abw^3 \right)$

Note $(a^2+b^2) w^3 = \frac{1}{w^2} w^3 = w$ so easy to verify F-S.

Finally, if $\tau \equiv \lambda$, $\kappa \equiv \mu$ for $\lambda, \mu \in \mathbb{R}$, $\mu > 0$, then we solve $\lambda = \frac{b}{a^2+b^2}$, $\mu = \frac{a}{a^2+b^2}$

to choose $\quad a = \frac{\mu}{\lambda^2+\mu^2}$

$$b = \frac{\lambda}{\lambda^2+\mu^2}$$

(3) Suppose that $\gamma : I \to \mathbb{R}^3$ is a regular curve (not necessarily arclength parametrised. Show that:

$$\kappa = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3} \qquad \tau = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{\|\gamma' \times \gamma''\|^2}$$

Let s be the arclength parameter. $s = \int_{t_0}^{t} \|\gamma'(\tilde{t})\| \, d\tilde{t}$

$$\gamma' = \frac{d\gamma}{dt} = \frac{\partial \gamma}{ds}\frac{ds}{dt} = T\frac{ds}{dt}$$

$$\gamma'' = \frac{d^2 s}{dt^2}\frac{d\gamma}{ds} + \left(\frac{ds}{dt}\right)^2 \frac{d^2\gamma}{ds^2} = T\frac{d^2 s}{dt^2} + \left(\frac{ds}{dt}\right)^2 \kappa N$$

$$\gamma''' = \frac{d^3 s}{dt^3}\frac{d\gamma}{ds} + \frac{ds}{dt}\frac{d^2 s}{dt^2}\frac{d^2\gamma}{ds^2} = T\frac{d^3 s}{dt^3} + 3\frac{ds}{dt}\frac{d^2 s}{dt^2}\kappa N + \left(\frac{ds}{dt}\right)^3\left[\frac{d\kappa}{ds}N + \kappa(\kappa T + \tau B)\right]$$

$$+ 2\frac{d^2 s}{dt^2}\frac{ds}{dt}\frac{d^2\gamma}{ds^2} + \left(\frac{ds}{dt}\right)^3 \frac{d^3\gamma}{ds^3}$$

Further, $\frac{ds}{dt} = \|\gamma'(t)\|$, $\frac{d^2 s}{dt^2} = \frac{\gamma' \cdot \gamma''}{\|\gamma'\|}$ [don't need!]

and $\frac{d^3 s}{dt^3} =$ [don't need!]

Now: $\gamma' \times \gamma'' = \left(\frac{ds}{dt}\right)^3 \kappa B \Rightarrow \|\gamma' \times \gamma''\| = \|\gamma'\|^3 \kappa$

and: $\gamma' \times \gamma'' \cdot \gamma''' = \left(\frac{ds}{dt}\right)^6 \kappa^2 \tau = \left[\left(\frac{ds}{dt}\right)^3 \kappa\right]^2 \tau = \|\gamma' \times \gamma''\|^2 \tau$

(4) Suppose that $\beta, \gamma : I \longrightarrow \mathbb{R}^3$ are two unit speed smooth curves. Suppose that the curvatures and torsions are everywhere +ve, and that $B_\beta \equiv B_\gamma$. Show $\exists\, p \in \mathbb{R}^3$ s.t. for each s, $\gamma(s) = \beta(s) + p$

$B_\beta \equiv B_\gamma$. We differentiate:

$\Rightarrow \tau_\beta N_\beta \equiv \tau_\gamma N_\gamma$,

Since τ_β, τ_γ are both positive, we can conclude that $\tau_\beta \equiv \tau_\gamma$ and $N_\beta \equiv N_\gamma$.

Now, $T_\beta = N_\beta \times B_\beta \equiv N_\gamma \times B_\gamma = T_\gamma$

so $T_\beta \equiv T_\gamma$.

Differentiating tells us that $k_\beta N_\beta \equiv k_\gamma N_\gamma$ and so $k_\beta \equiv k_\gamma$.

So by existence and uniqueness of curves with prescribed curvature and torsion, the curves γ and β agree (up to a translation).

(5) Let γ be a unit-speed curve with non-zero curvature and torsion.
Let $r = 1/\kappa$, $t = 1/\tau$.

Show γ is a spherical curve $\iff r^2 + (r't)^2 \equiv R^2$

If $\gamma \subset S^2$, $\|\gamma(s) - p\|^2 = R^2$. We differentiate:

(1) $0 = \cancel{2}(\gamma(s) - p)\cdot\gamma'(s)$

(2) $0 = \cancel{2}\|\gamma'(s)\|^2 + \cancel{2}(\gamma(s)-p)\cdot\gamma''(s)$

(3) $0 = 2\gamma'(s)\cdot\gamma''(s) + \cancel{2}\gamma'(s)\cdot\gamma''(s) + \cancel{2}(\gamma(s)-p)\cdot\gamma'''(s)$

Rewriting (2):

$0 = 2 + \kappa 2(\gamma(s)-p)\cdot N \implies -r = (\gamma(s)-p)\cdot N$

Rewriting (3):

$0 = \underbrace{\cancel{3T\cdot\kappa N}}_{0} + (\gamma(s)-p)\cdot(\kappa' N + \kappa(-\tau B - \kappa T))$

$= (\gamma(s)-p)\cdot\kappa' N - (\gamma(s)-p)\cdot\kappa\tau B$

$= -r\kappa' - \kappa\tau(\gamma(s)-p)\cdot B$

so $(\gamma(s)-p)\cdot B = \dfrac{-r\kappa'}{\kappa\tau} = -r^2\kappa' t$

$= -r't$

Finally from (1) we obtain $(\gamma(s)-p)\cdot T = 0$

Recalling (T, N, B) is an orthonormal frame, we have:

$\gamma(s) - p = [(\gamma(s)-p)\cdot T]T + [(\gamma(s)-p)\cdot N]N + [(\gamma(s)-p)\cdot B]B$

so $R^2 = \|\gamma(s) - p\|^2 = |(\gamma(s)-p)\cdot T|^2 + |(\gamma(s)-p)\cdot N|^2 + |(\gamma(s)-p)\cdot B|^2$

$= 0 + r^2 + (r't)^2$

Conversely, now suppose $r^2 + (r't)^2 \equiv R^2$. $(*)$

Let $\beta(s) = \gamma(s) + rN + r't\,B$. We will β is constant.

$$\beta' = \underset{\overset{\shortparallel}{T}}{\gamma'} + r'N + r(\tau B - \kappa T) + (r't' + r''t)B - \tau r't N$$

N coefficients: $r' - \tau r't = 0$.

T coefficients: $1 - r\kappa = 0$

B coefficients: $r\tau + r't' + r''t = 0$

Differentiating $(*)$ gives $0 = 2r\cdot r' + 2r't(r't' + r''t)$.

That is, $(r't' + r''t) = \dfrac{-2rr'}{2r't} = -r\tau$.

So we have $\beta' \equiv 0$. $\Rightarrow \gamma(s) - p = -rN - r'tB$

$\Rightarrow \|\gamma(s)-p\|^2 = r^2 + (r't)^2 = R^2$.