GEOMETRY OF CURVES AND SURFACES

Exercises 1

1) Suppose $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is linear. Show the following are equivalent:
   (a) ϕ orthogonal
   (b) $\forall x, y \in \mathbb{R}^n$, $\phi(x) \cdot \phi(y) = x \cdot y$
   (c) $\forall x \in \mathbb{R}^n$, $\|\phi(x)\| = \|x\|$

Fix a basis of $\mathbb{R}^n$ and let $A \in M_n(\mathbb{R})$ represent ϕ.

$$\phi \text{ orthogonal} \iff A^T A = I$$

Now we may write
$$\phi(x) \cdot \phi(y) = (Ax) \cdot (Ay) = (Ax)^T Ay = x^T A^T A y.$$

Similarly,
$$x \cdot y = x^T y$$

It is clear we have equality $\iff A^T A = I$ i.e. (a) $\iff$ (b).

Note (b) $\iff$ (c) is also clear:

$$\|x\| = \sqrt{x \cdot x} \quad \text{and} \quad x \cdot y = \tfrac{1}{2}\left(\|x\|^2 + \|y\|^2 - \|x-y\|^2\right)$$

so norms are preserved $\iff$ associated quadratic forms are too.

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(2) (a) Show a rigid motion of $\mathbb{R}^n$ preserves distances
   (b) Show an affine map of $\mathbb{R}^n$ preserving distances is a rigid motion
   (c) Show any map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ preserving distances is a rigid motion

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(a) First note that any rigid motion is a composition of first an orthogonal map, and then a translation. From Q1, orthogonal maps preserve distances, and it is clear that translations preserve distances.

(b) Write the affine map as $\varphi : x \mapsto Ax + b$, where $A \in M_n(\mathbb{R})$ and $b \in \mathbb{R}^n$.

let $x, y \in \mathbb{R}^n$.

$$d(x,y) = \|x-y\|$$
$$d(\varphi(x), \varphi(y)) = \|Ax+b - Ay - b\|$$
$$= \|A(x-y)\|$$

From Q1 we conclude A is orthogonal. Hence φ a rigid motion.

(c) It suffices to show that such an f is affine.

First note we may assume f fixes the origin. If not we may postcompose with a translation of $-f(0)$ - and since translations preserve distances, the composition does too.

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$.

We claim $f(e_1), \ldots, f(e_n)$ is an orthonormal basis of $\mathbb{R}^n$.

First note $1 = \|e_i\| = \|f(e_i)\|$ $(1 \leq i \leq n)$, since the distance between 0 and e_i is the same as the distance between 0 and $f(e_i)$. [We assume as above $f(0) = 0$]

Now consider $f(e_i) \cdot f(e_j)$.

As before, this is equal to: $\frac{1}{2}\left(\|f(e_i)\|^2 + \|f(e_j)\|^2 - \|f(e_i) - f(e_j)\|^2\right)$

$$= 1 - \frac{1}{2}\|e_i - e_j\|^2$$

$$= \begin{cases} 0 & e_i \neq e_j \\ 1 & e_i = e_j \end{cases}$$

$$\boxed{\begin{aligned} &\|f(e_i) - f(e_j)\|^2 \\ &= d(f(e_i), f(e_j))^2 \\ &= d(e_i, e_j)^2 \\ &= \|e_i - e_j\|^2 \end{aligned}}$$

Hence the $f(e_i)$ are orthonormal.

$\therefore \exists$ an orthogonal map ϕ s.t. $\phi \circ f(e_i) = e_i$. (each i)

Now observe the action of $\phi \circ f$ on a point $x \in \mathbb{R}^n$.

$x = \sum_{i=1}^{n} x_i e_i \qquad (x_i \in \mathbb{R})$.

$\phi \circ f(x) = \sum_{i=1}^{n} y_i e_i \qquad (y_i \in \mathbb{R})$

Moreover $x_i = x \cdot e_i$ and
$$y_i = \phi \circ f(x) \cdot e_i$$
$$= \tfrac{1}{2}\left(\|\phi \circ f(x)\|^2 + \|e_i\|^2 - \|\phi \circ f(x) - e_i\|^2 \right)$$
$$= \tfrac{1}{2}\left(\|x\|^2 + \|e_i\|^2 - \|x - e_i\|^2 \right)$$
$$= x \cdot e_i = x_i.$$

That is $x \equiv \phi \circ f(x)$. So $\phi \circ f$ is the identity map, and thus (since the orthogonal maps of $\mathbb{R}^n$ form a group) f is orthogonal. In particular, it's affine!

(3) [Assuming the Inverse Function Theorem] deduce:

Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is smooth, and the derivative (at each point) f_* has rank n everywhere. Then f is locally injective.

We implicitly have $m \geq n$...

We first show an $n \times m$ matrix, A, of rank n can be extended to an $m \times m$ matrix of rank m

This is equivalent to the statement that n linearly independent vectors of $\mathbb{R}^m$ (i.e. the columns of A) can be extende

Call them $c_1, \ldots, c_{m-n}$

to a basis of $\mathbb{R}^m$. Adding these $(m-n)$ vectors to the columns of the $\times$ matrix ensures the resulting matrix has rank m; that is it's invertible.

Call this extended matrix B.

Now extend f to a map g from $\mathbb{R}^n \times \mathbb{R}^{m-n}$ to $\mathbb{R}^m$, defined by: $g(\underline{x}, y) = (f_1(x), \ldots, f_m(x)) + (h_1(y), \ldots, h_m(y))$ $(h : \mathbb{R}^{m-n} \to \mathbb{R}^m)$

s.t. $\left.\frac{\partial h}{\partial x_i}\right|_{y=0} = c_{i-n}$ $\quad n+1 \leq i \leq m$

and $h(0) = 0$. So $g|_{\mathbb{R}^n} = f$ and $\mathrm{Jac}\, g = B$ at $(p, 0)$.

Now the IFT applies: We can solve for $m-n$ variables of f in terms of n.

That is f is locally injective. (bijective onto its image)

(4) Let $\gamma : [a,b] \to \mathbb{R}^n$ be a smooth curve. Show
$\| \gamma(a) - \gamma(b) \| \leq \text{length}(\gamma)$.

Define $f(t) = \| \gamma(a) - \gamma(t) \|$
$g(t) = \int_a^t \| \gamma'(\tilde{t}) \| d\tilde{t}$ for $t \in [a,b]$

Consider $f'(t)$ and $g'(t)$.

Clearly, $g'(t) = \| \gamma'(t) \|$

and by the chain rule,

$$f'(t) = \frac{\gamma'(t) \cdot (\gamma(t) - \gamma(a))}{\| \gamma(t) - \gamma(a) \|}$$

Cauchy-Schwarz $\Rightarrow$ $$\leq \frac{\| \gamma'(t) \| \, \| \gamma(t) - \gamma(a) \|}{\| \gamma(t) - \gamma(a) \|}$$

$$= g'(t)$$

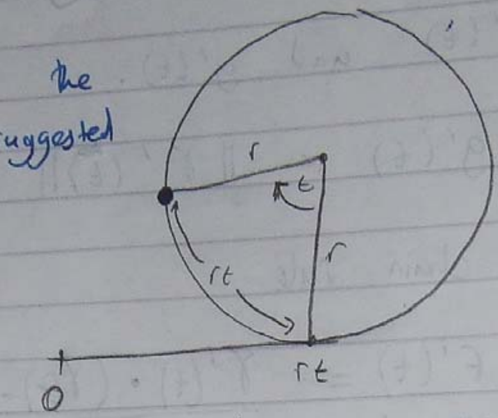
Now, it is clear f is increasing in t - so together with the condition $f'(t) \leq g'(t)$ $\forall t$ we have:

$$0 \leq f'(t) \leq g'(t) \Rightarrow 0 \leq f \leq g.$$

(5) The "Cycloid" is the curve traced out by a point on the rim of a wheel as it rolls along a straight line, starting and finishing ~~with~~ when the point is level with the ground.

Write down an equation for the cycloid, with a suitable parameter - and calculate its length.

Let t measure the angle of the point from the vertical, as suggested in the diagram.



It's clear to see $x(t) = rt - r\sin t$

$y(t) = r - r\cos t$

so $(\dot{x}, \dot{y}) = (r - r\cos t, \; r\sin t)$

and $\|(\dot{x}, \dot{y})\| = r\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}$

$= r\sqrt{2 - 2\cos t}$

$= r\sqrt{4\sin^2 t/2}$

$= 2r\sin t/2$

$\text{length (cycloid)} = \int_0^{2\pi} 2r\sin \frac{t}{2} \, dt$

$= 8r.$

(6) Suppose $r : \mathbb{R} \to (0, \infty)$ is smooth. Let γ be the curve given by $\gamma(\theta) = (r(\theta), \theta)$ in polar coordinates.

Show the curvature of γ at θ is in terms of r, r', r''.

Now let $r(\theta) = e^{a\theta}$. Show the curvature is $\frac{1}{r\sqrt{1+a^2}}$.

Interpret as $a \to 0$

$a \to \infty$.

$$k = \frac{|\ddot{x}\dot{y} - \ddot{y}\dot{x}|}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$ where $\gamma = (x(t), y(t))$

[Should be able to derive this].

In cartesian coordinates:

$$\gamma(\theta) = (r\cos\theta, r\sin\theta)$$

$$\gamma'(\theta) = (-r\sin\theta + r'\cos\theta, r\cos\theta + r'\sin\theta)$$

$$\gamma''(\theta) = (\cos\theta(r'' - r) - 2r'\sin\theta, \sin\theta(r'' - r) + 2r'\cos\theta)$$

$$\therefore \quad k = \frac{|2r'^2 - rr'' + r^2|}{\sqrt{r^2 + r'^2}^{\,3}}$$

If $r = e^{a\theta}$, $r' = ae^{a\theta}$, $r'' = a^2 e^{a\theta}$

and
$$k = \frac{|e^{2a\theta}[2a^2 - a^2 + 1]|}{\sqrt{e^{2a\theta}(1 + a^2)}^{\,3}} = e^{-a\theta} \cdot (1 + a^2)^{-1/2} = \frac{1}{r\sqrt{1 + a^2}}$$

If $a \to 0$, $K \to \frac{1}{r}$ $\Rightarrow$ curve approximates a circle.

If $a \to \infty$, $k \to 0$ $\Rightarrow$ curve approximates a line.