MA3D9: Geometry of curves and surfaces

Exercises 3.

(1) Compute the first fundamental forms of the following parameterised surfaces:
ellipsoid: $\mathbf{r}(u,v) = (a\cos u\cos v, b\sin u\cos v, c\sin v)$
elliptic paraboloid: $\mathbf{r}(u,v) = (au\cos v, bu\sin v, u^2)$.
hyperbolic paraboloid: $\mathbf{r}(u,v) = (au\cosh v, bu\sinh v, u^2)$
2-sheeted hyperboloid: $\mathbf{r}(u,v) = (a\sinh u\cos v, b\sinh u\sin v, c\cosh u)$.

(2) Given $\theta \in \mathbf{R}$, let $\mathbf{r}(\theta)$ be the parametrised surface, $\mathbf{r}(\theta) : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$ defined by $\mathbf{r}(\theta)(u,v) = (x(\theta)(u,v), y(\theta)(u,v), z(\theta)(u,v))$ where

$$\begin{aligned} x(\theta)(u,v) &= \cos\theta\cosh u\cos v + \sin\theta\sinh u\sin v \\ y(\theta)(u,v) &= \cos\theta\cosh u\sin v - \sin\theta\sinh u\cos v \\ z(\theta)(u,v) &= u\cos\theta + v\sin\theta. \end{aligned}$$

Calculate the first fundamental form of $\mathbf{r}(\theta)$, and show that $\mathbf{r}(\theta)$ is isometric to $\mathbf{r}(\phi)$ for all θ and ϕ.
What are these surfaces when $\theta = 0$ and $\theta = \pi/2$?

(3) Let $\mathbf{r}$ be a regular surface. The "coordinate curves" are curves of the form $u \mapsto \mathbf{r}(u,v)$ for constant v, or $v \mapsto \mathbf{r}(u,v)$ for constant u. Show that coordinate curves are regular.
Show that $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0$ if and only if the any quadrilateral formed out of coordinate curves has opposite sides of equal length.
In such a case, show that we can find a reparametisation so that the first fundamental form has the form $E = 1$, $F = \cos\theta$, $G = 1$, where θ is the angle between the (new) coordinate curves.

(4) Show that the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ for $a, b, c > 0$ is an embedded regular surface, by describing explicit charts.
Show that for any $c \neq 0$, the set $\{(x,y,z) \in \mathbf{R}^2 \mid x^2 + y^2 - z^2 = c\}$ is a regular surface, by describing explicit charts.

(5) Show that any two ellipsoids are diffeomorphic.

(6) Show that the image of a surface of revolution of a Jordan curve is an embedded surface in $\mathbf{R}^3$. Show that any two such images are diffeomorphic.

(7) A *loxodrome* on the sphere is a path that makes a constant non-zero angle $\alpha \in (-\pi, \pi)$. θ with every latitude (i.e. constant compass bearing).
Show that the image of a loxodrome under steriographic projection is a logarithmic spiral.
Calculate the total length of a loxodrome from pole to pole.
Show that the interior angles of a triangle made from three loxodrome segments add up to π.