Exercises 1.

(1) Suppose that $\Phi : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is a linear map. Show that the following are equivalent.
(a) Φ is orthogonal
(b) $(\forall \mathbf{x}, \mathbf{y} \in \mathbf{R}^n)(\Phi(\mathbf{x}).\Phi(\mathbf{y}) = \mathbf{x}.\mathbf{y})$.
(c) $(\forall \mathbf{x} \in \mathbf{R}^n)(||\Phi(\mathbf{x})|| = ||\mathbf{x}||)$.

(2) (a) Show that a rigid motion of $\mathbf{R}^n$ preserves distances.
(b) Show that an affine map of $\mathbf{R}^n$ that preserves distances is a rigid motion.
(c) $(*)$ Show that any map, f, of $\mathbf{R}^n$ to itself that preserves distances is a rigid motion.
[First note that we can assume that it fixes the origin. Let $e_1, \dots, e_n$ be the standard basis for $\mathbf{R}^n$. Show that $f(e_1), \dots, f(e_n)$ is an orthonormal basis of $\mathbf{R}^n$. Composing f with an orthogonal linear map, we can now assume that $f(e_i) = e_i$ for all i. Now show that any point of $\mathbf{R}^n$ is determined by its distances from the origin and each of the basis vectors. Deduce that f must be the identity map on $\mathbf{R}^n$.]

(3) $(*)$ Assuming the inverse function theorem deduce Theorem 0.1, namely:
Suppose $f : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ is smooth and that the derivative f_* has rank n everywhere. Then f is locally injective.
[Hint: First show that any $n \times m$ matrix A of rank m can be extended to an invertible $m \times m$ matrix. Now let A be the jacobian of f at the point p. Extend f to a map g from $\mathbf{R}^m = \mathbf{R}^n \times \mathbf{R}^{m-n}$ to $\mathbf{R}^n$. with $g|\mathbf{R}^n = f$, and with jacobian B at $(p, \mathbf{0})$. Apply the implicit function theorem to g.]

(4) Let $\gamma : [a, b] \longrightarrow \mathbf{R}^n$ be a smooth curve.
Show that $||\gamma(a) - \gamma(b)|| \leq \text{length}(\gamma)$.
[For example, compare the derivatives of $||\gamma(t) - \gamma(a)||$ and $\text{length}(\gamma|[a, t])$.]
Show that we have equality if γ is a straight line.

(5) The "cycloid" is the curve traced out by a point on the rim of a wheel as it rolls along a straight line, starting and finishing when the point is level with the ground.
Write a down an equation for the cycloid, with a suitable parameter. Calculate the total length of the cycloid.

(6) Suppose that $r : \mathbf{R} \longrightarrow (0, \infty)$ be smooth. Let γ be the curve given by $\gamma(\theta) = (r(\theta), \theta)$ in polar coordinates in $\mathbf{R}^2$. Derive a formula for the curvature of γ at θ in terms of r, r' and r''.
Suppose that γ is the logarithmic spiral of pitch a, i.e. $r(\theta) = e^{a\theta}$. Show that the curvature of a point of γ at distance r from the origin is $1/r\sqrt{1 + a^2}$. What is the interpretation as $a \to 0$ and as $a \to \infty$?