APPENDIX TO CHAPTER II

EXAMPLES

We give here some worked examples of the methods described in the preceding chapter, to illustrate and clarify the different situations which arise. The first example is $N = 11$, which is the first non-trivial level; here we give most detail. Then we consider $N = 33$, where we encounter oldforms and more complicated M-symbols, and $N = 37$, where there are two newforms, one of which has $L(f,1) = 0$, necessitating a different method of computing Hecke eigenvalues. Finally we look at a square level, $N = 49$, to illustrate the direct method of computing periods.

Example 1: $N = 11$

For simplicity we will only work in $H(11)$, rather than the smaller quotient space $H^+(11)$. The M-symbols for $N = 11$ are $(c:1)$ for c modulo 11 and $(1:0)$, which we abbreviate as (c) and (∞) respectively, with $|c| \le 5$. (Similarly with other prime levels). The 2-term and 3-term relations (2.2.6) and (2.2.7) are as follows.

$$\begin{aligned} (0) + (\infty) &= 0 \\ (1) + (-1) &= 0 \\ (2) + (5) &= 0 \\ (-2) + (-5) &= 0 \\ (3) + (-4) &= 0 \\ (-3) + (4) &= 0 \end{aligned} \qquad \begin{aligned} (0) + (\infty) + (-1) &= 0 \\ (1) + (-2) + (5) &= 0 \\ (2) + (4) + (-4) &= 0 \\ (3) + (-5) + (-3) &= 0 \end{aligned}$$

Solving these equations we can express all 12 symbols in terms of $A = (2)$, $B = (3)$ and $C = (0)$:

$$\begin{aligned} (0) &= C \\ (\infty) &= -C \\ (1) = (-1) &= 0 \\ (2) = (-2) &= A \\ (5) = (-5) &= -A \end{aligned} \qquad \begin{aligned} (3) &= B \\ (-3) &= A - B \\ (4) &= B - A \\ (-4) &= -B \end{aligned}$$

There are two classes of cusps, $[0]$ and $[\infty]$, with $[a/b] = [0]$ if $11 \nmid b$ and $[a/b] = [\infty]$ if $11 \mid b$. Hence $\delta((c)) = \delta(\{0, 1/c\}) = [1/c] - [0] = 0$ for $c \not\equiv 0$. It follows that

$$H(11) = \ker(\delta) = \langle A, B \rangle ,$$

with $2g = \dim H(11) = 2$, so that the genus is 1. There is therefore one newform f. This makes the rest of the calculation simpler, as we do not have to find and split off eigenspaces.

The conjugation $*$ involution maps $(c) \mapsto (-c)$, so $A^* = A$ and $B^* = A - B$. This has matrix $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ with respect to the basis A, B. The $+1$- and -1-eigenspaces are generated by A and $A - 2B$ respectively, and we have left eigenvectors $v^+ = (2, 1)$ and $v^- = (0, 1)$. Thus the period lattice is Type 1 (non-rectangular), and $\Omega(f) = \Omega_0(f) = \langle A, f\rangle$.

If we had worked in $H^+(11)$, viewed as the quotient $H(11)/H^-(11)$, by including relations $(c) = (-c)$, the effect would be to identify (c) and $(-c)$. This gives a 1-dimensional space generated by $\overline{B}$ with $\overline{A} = 2\overline{B}$, where the bars denote the projections to the quotient. Notice that although $\overline{B}$ is a generator here, the integral of f over B is not a real period; its real part is half the real period. However we do still have $\Omega(f) = \langle B + B^*, f\rangle = 2\mathrm{Re}\,\langle B, f\rangle$, so we could compute $\Omega(f)$ in this context without actually knowing whether it was 1 or 2 times the smallest real period.

To compute Hecke eigenvalues we may work in the subspace $\langle A\rangle$; since this subspace is conjugation invariant (being the +1-eigenspace) we will have $T_p(A) = a_pA$ for all $p \neq 11$. We first compute T_2 explicitly. The first method, converting the M-symbol $A = (2 : 1)$ to the modular symbol $\{0, 1/2\}$, gives:

$$\begin{aligned} T_2(A) = T_2\left(\left\{0, \frac{1}{2}\right\}\right) &= \{0, 1\} + \left\{0, \frac{1}{4}\right\} + \left\{\frac{1}{2}, \frac{3}{4}\right\} \\ &= \{0, 1\} + \left\{0, \frac{1}{4}\right\} + \left\{\frac{1}{2}, 1\right\} + \left\{1, \frac{3}{4}\right\} \\ &= (1 : 1) + (4 : 1) + (1 : 2) + (-4 : 1) \\ &= (1) + (4) + (-5) + (-4) \\ &= 0 + (B - A) + (-A) + (-B) = -2A, \end{aligned}$$

so that $a_2 = -2$. Alternatively, using the Heilbronn matrices from Section 2.4, we compute:

$$\begin{aligned} T_2(A) = T_2((2 : 1)) &= (2 : 1)R_2 \\ &= (2:1)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + (2:1)\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + (2:1)\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + (2:1)\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \\ &= (2 : 2) + (4 : 1) + (4 : 3) + (3 : 2) \\ &= (1) + (4) + (5) + (-4) \\ &= 0 + (B - A) + (-A) + (-B) \\ &= -2A. \end{aligned}$$

Now $(1 + 2 - a_2)L(f, 1) = \langle\{0, 1/2\}, f\rangle = \langle A, f\rangle = \Omega(f)$, giving

$$\frac{L(f, 1)}{\Omega(f)} = \frac{1}{5}.$$

For all primes $p \neq 11$ we will evaluate $\mu_p = \sum_{a=0}^{p-1} \{0, a/p\} = n_pA$ for a certain integer n_p, since then also $1/5 = n_p/(1 + p - a_p)$, giving

$$a_p = 1 + p - 5n_p.$$

At this stage we already know that the corresponding elliptic curve has rank 0, and that $1 + p - a_p \equiv 0 \pmod 5$ for all $p \neq 11$, so that it will possess a rational 5-isogeny.

To save time, we can use the fact that $\{0, a/p\}^* = \{0, -a/p\}$; thus for odd p we need only evaluate half the sum, say

$$\mu'_p = \sum_{a=1}^{(p-1)/2} \left\{0, \frac{a}{p}\right\},$$

and then set $\mu_p = \mu_p' + (\mu_p')^*$.

For $p = 3$, we have $\mu_3' = \{0, 1/3\} = (3:1) = (3) = B$, so $\mu_3 = B + B^* = A$, giving $n_3 = 1$ and $a_3 = 1 + 3 - 5n_3 = -1$.

For $p = 5$ we compute:

$$\begin{aligned}
\left\{0, \frac{1}{5}\right\} &= (5:1) = (5) = -A; \\
\left\{0, \frac{2}{5}\right\} &= \left\{0, \frac{1}{2}\right\} + \left\{\frac{1}{2}, \frac{2}{5}\right\} \\
&= (2:1) + (-5:2) = (2) + (3) = A + B; \\
\mu_5' &= (-A) + (A+B) = B; \\
\mu_5 &= B + B^* = A, \qquad \text{so that } n_5 = 1; \\
a_5 &= 1 + 5 - 5n_5 = 1.
\end{aligned}$$

Similarly, with $p = 7$ we have $n_7 = 2$, so that $a_7 = 1 + 7 - 5n_7 = -2$, and with $p = 13$ we have $n_{13} = 2$ so that $a_{13} = 4$.

These computations can also be carried out using Heilbronn matrices, by applying the Hecke operators directly to $C = (0:1) = \{0, \infty\}$, as follows. Once we know that $a_2 = -2$, we have

$$-2C = T_2(C) = T_2((0:1)) = (0:1)R_2 = (0:2) + (0:1) + (0:1) + (1:2) = 3C - A,$$

giving $C = \frac{1}{5}A$ in agreement with the ratio $L(f,1)/\Omega(f)$ found earlier. Similarly, using the Heilbronn matrices R_3 listed in Section 2.4, we find

$$\begin{aligned}
a_3 C = T_3(C) &= (0:1)R_3 \\
&= (0:3) + (0:1) + (1:3) + (0:1) + (0:1) + (1:-3) \\
&= C + C + (B - A) + C + C + (-B) \\
&= 4C - A = -C,
\end{aligned}$$

giving $a_3 = -1$ again.

For the prime $q = 11$ we compute the involution W_{11} induced by the action of the matrix $\begin{pmatrix} 0 & -1 \\ 11 & 0 \end{pmatrix}$:

$$\begin{aligned}
W_{11}(A) = \begin{pmatrix} 0 & -1 \\ 11 & 0 \end{pmatrix} \left\{0, \frac{1}{2}\right\} &= \left\{\infty, \frac{-2}{11}\right\} \\
&= \{\infty, 0\} + \left\{0, \frac{-1}{5}\right\} + \left\{\frac{-1}{5}, \frac{-2}{11}\right\} \\
&= (1:0) + (-5:1) + (11:5) \\
&= (\infty) + (-5) + (0) \\
&= -A,
\end{aligned}$$

so that the eigenvalue ε_{11} of W_{11} is -1. In fact, this was implicit earlier, since $L(f,1) \neq 0$ implies that the sign of the functional equation is $+1$, which is minus the eigenvalue of the Fricke involution W_{11}.

The Fourier coefficients $a(n) = a(n,f)$ for $1 \le n \le 16$ are now given by the following table.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$a(n)$	1	−2	−1	2	1	2	−2	0	−2	−2	1	−2	4	4	−1	−4

Here we have used multiplicativity, and:

$$\begin{aligned} a(11) &= -\varepsilon_{11} = +1; \\ a(4) &= a(2)^2 - 2a(1) = 2; \\ a(8) &= a(2)a(4) - 2a(2) = 0; \\ a(16) &= a(2)a(8) - 2a(4) = -4; \\ a(9) &= a(3)^2 - 3a(1) = -2. \end{aligned}$$

We know that the period lattice Λ_f has a $\mathbb{Z}$-basis of the form $[\omega_1, \omega_2] = [2x, x + iy]$, where $\omega_1 = P_f(A)$ and $\omega_2 = P_f(B)$. We can compute the real period $\omega_1 = \Omega(f) = 5L(f, 1)$ by computing $L(f, 1)$:

$$L(f,1) = 2\sum_{n=1}^{\infty} \frac{a(n)}{n} t^n$$

where $t = \exp(-2\pi/\sqrt{11}) = 0.15\ldots$. Using the first 16 terms which we have, already gives this to 13 decimal places:

$$L(f,1) = 0.2538418608559\ldots;$$

thus

$$\omega_1 = \Omega(f) = 1.269209304279\ldots.$$

For the imaginary period y we twist with a prime $l \equiv 3 \pmod 4$. Here $l = 3$ will do, since

$$\gamma_3 = \left\{0, \frac{1}{3}\right\} - \left\{0, \frac{-1}{3}\right\} = (3) - (-3) = -A + 2B \neq 0.$$

To project onto the minus eigenspace we take the dot product of this cycle (expressed as a row vector $(-1, 2)$) with $v^- = (0, 1)$ to get $m^-(3) = 2$. Hence

$$y = \frac{1}{2i} P(3, f) = \frac{\sqrt{3}}{2} L(f \otimes 3, 1).$$

Summing the series for $L(f \otimes 3, 1)$ to 16 terms gives only 4 decimals:

$$L(f \otimes 3, 1) = 1.6845\ldots.$$

This is less accurate than $L(f, 1)$ since this series is a power series in $\exp(-2\pi/3\sqrt{11}) = 0.53\ldots$, compared with $0.15\ldots$. Hence $y = 1.4588\ldots$, so that

$$\omega_2 = 0.634604652139\ldots + 1.4588\ldots i.$$

So far we have only used the Hecke eigenvalues a_p for $p \leq 13$, and only 16 terms of each series. If we use these approximate values for the period lattice generators ω_1 and ω_2 we already find the approximate values $c_4 = 495.99$ and $c_6 = 20008.09$ which round to the integer values $c_4 = 496$ and $c_6 = 20008$. Taking the first 25 a_p and the first 100 terms of the series gives

$$c_4 = 495.9999999999954\ldots \qquad \text{and} \qquad c_6 = 20008.0000000085.$$

The exact values $c_4 = 496$ and $c_6 = 20008$ are the invariants of an elliptic curve of conductor 11, which is in fact the modular curve E_f:

$$y^2 + y = x^3 - x^2 - 10x - 20.$$

This is the first curve in the tables, with code 11A1 (or Antwerp code 11B). The value $L(f,1)/\Omega(f) = 1/5$ agrees with the value predicted by the Birch–Swinnerton-Dyer conjecture for $L(E_f,1)/\Omega(E_f)$, provided that E_f has trivial Tate–Shafarevich group.

We now illustrate the shortcut method presented in Section 2.11, where we guess the imaginary period and lattice type without computing $H(11)$. Having computed $P(3,f) = 2.9176\ldots i$ which is certainly non-zero, we consider the lattices $\langle x, yi\rangle$ and $\langle 2x, x+yi\rangle$, where $2x = 1.2692\ldots$ (from above) and $yi = P(3,f)/m^-$, for $m^- = 1,2,3,\ldots$. With $m^- = 1$ we do not find integral invariants, but for $m^- = 2$ and lattice type 1 we find the curve $E_f = [0,-1,1,-10,-20]$ given above[1].

Using the first variant of the method, where we do not even know x, we can take $l^+ = 5$ since $P(5,f) = 6.346\ldots \neq 0$. The correct value of m^+ here is 10; if we do not know this, but try $m^+ = 1,2,3\ldots$ in a double loop with m^-, the first valid lattice we come across is with $(m^+,m^-) = (2,2)$ and type 1, which leads to the curve $E' = [0,-1,1,0,0]$, also of conductor 11; this is 5-isogenous to the "correct" curve E_f, which comes from $(m^+,m^-) = (10,2)$ and type 1.

We may also consider the ratios $P(l,f)/P(3,f)$ for other primes $l \equiv 3 \pmod 4$; we restrict to those l satisfying $\left(\frac{-11}{l}\right) = \left(\frac{l}{11}\right) = +1$, since otherwise $P(l,f)$ is trivially 0 (since the sign of the functional equation for the corresponding $L(f\otimes\chi,s)$ is then -1). We find the following table of values (rounded: they are only computed approximately):

l	3	23	31	47	59	67	71	103	163	179	191	199	223	251
$\dfrac{P(l,f)}{P(3,f)}$	1	1	1	0	1	9	1	0	4	25	1	4	1	1

The zero values for $l = 47$ and $l = 103$ indicate that the corresponding twists of the newform f have positive even analytic rank (one can check that the corresponding twists of the curve E_f do indeed have rank 2). As all these values are integral here (*a priori* they are only known to be rational) we do not find any nontrivial divisor of m^- (which we know in fact equals 2). The fact that all the integers are perfect squares is an amusing observation, but has a simple explanation in terms of the numbers appearing in the Birch–Swinnerton-Dyer conjecture for the twists of E_f.

There is one other curve E'' isogenous to E_f in addition to E' (found above). If the period lattice of $E_f = [0,-1,1,-10,-20]$ is $\langle 2x, yi\rangle$ with $x = 0.6346...$ and $y = 1.4588...$, then $E' = [0,-1,1,0,0]$ has period lattice $\langle 10x, 5x+yi\rangle$, and $E'' = [0,-1,1,-7820,263580]$ has lattice $\langle x/5, 2x/5+yi\rangle$. These curves are linked by 5-isogenies $E_f \leftrightarrow E'$ and $E_f \leftrightarrow E''$.

Finally, we compute the degree of the modular parametrization $\varphi\colon X_0(11) \to E_f$. Of course, this is obviously 1, since the modular curve $X_0(11)$ has genus 1, so that φ is the identity map in this case; but this example will serve to illustrate the general method.

The twelve M-symbols form 4 triangles which we choose as follows:

$$(1,0),(-1,1),(0,1); \qquad (1,1),(-2,1),(-1,2);$$
$$(1,2),(-3,1),(-2,3); \qquad (1,3),(-4,1),(-3,4).$$

[1] Here, $[a_1,a_2,a_3,a_4,a_6]$ denote the Weierstrass coefficients of the curve; see Chapter 3.

There are two τ-orbits, corresponding to the two cusps at ∞ (of width 1) and at 0 (of width 11). The first contributes nothing. The second is as follows:

$$\begin{aligned}(1,0) &\mapsto (1,1) \mapsto (1,2) \mapsto (1,3) \mapsto (1,4) \equiv (-2,3) \mapsto (-2,1) \mapsto (-2,-1)\\ &\equiv (-3,4) \mapsto (-3,1) \mapsto (-3,-2) \equiv (-4,1) \mapsto (-4,-3) \equiv (-1,2) \mapsto (-1,1) \mapsto (1,0).\end{aligned}$$

There are four jump matrices coming from the above sequence. From $(1,4) \equiv (-2,3)$ we obtain

$$\delta_1 = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & -1 \\ 11 & 5 \end{pmatrix};$$

the others are $\delta_2 = \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}$, $\delta_3 = \begin{pmatrix} -5 & -1 \\ 11 & 2 \end{pmatrix}$ and $\delta_4 = \begin{pmatrix} -3 & 1 \\ 11 & -4 \end{pmatrix}$. Using modular symbols, we can compute the coefficients of $P_f(\delta_i)$ with respect to the period basis ω_1, ω_2, to obtain $P_f(\delta_1) = -\omega_1$, $P_f(\delta_2) = \omega_2$, $P_f(\delta_3) = \omega_1$, and $P_f(\delta_4) = -\omega_2$. Hence

$$\begin{aligned}\deg(\varphi) &= \frac{1}{2}\left(\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \right)\\ &= \frac{1}{2}(1 + 0 - 1 + 1 + 0 + 1) = 1,\end{aligned}$$

as expected.

Example 2: $N = 33$

Since $33 = 3{\cdot}11$, the number of M-symbols is $48 = 4{\cdot}12$, consisting of 33 symbols $(c) = (c : 1)$, 13 symbols $(1 : d)$ with $\gcd(d, 33) > 1$, and the symbols $(3 : 11)$ and $(11 : 3)$. (In fact, whenever N is a product pq of 2 distinct primes, the M-symbols have this form, with exactly two symbols, $(p : q)$ and $(q : p)$ not of the form $(c : 1)$ or $(1 : d)$).

There are four cusp classes represented by 0, 1/3, 1/11 and ∞, with the class of a cusp a/b being determined by $\gcd(b, 33)$. (Similarly, whenever N is square-free, the cusp classes are in one-one correspondence with the divisors of N).

Using the two-term and three-term relations, and including the relations $(c : d) = (-c : d)$, we can express all the M-symbols in terms of six of them, and $\ker(\delta^+) = \langle (7), (2), (15) - (9) \rangle$. Hence $H^+(33)$ is three-dimensional. We know there will be a two-dimensional oldclass coming from the newform at level 11, so there will also be a single newform f at this level.

If we compute the images of the basis modular symbols $\{0, 1/7\}$, $\{0, 1/2\}$ and $\{1/9, 1/15\}$ under T_2 and W_{33}, we find that they have matrices

$$T_2 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{pmatrix} \qquad \text{and} \qquad W_{33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

T_2 has a double eigenvalue of -2, coming from the oldforms, which we ignore, and also the new eigenvalue $a_2 = 1$ with left eigenvector $v = (0, 1, 0)$. The corresponding eigenvalue for W_{33} is $\varepsilon_{33} = -1$. Hence the sign of the functional equation is +, and the analytic rank is even. Moreover since the eigencycle for a_2 is the second basis element, which is $\{0, 1/2\} = \mu_2$, we have $2(1 + 2 - a_2)L(f, 1) = \Omega(f)$, so that

$$\frac{L(f,1)}{\Omega(f)} = \frac{1}{4}.$$

In particular, $L(f,1) \neq 0$, so that the analytic rank is 0. Note that because we have factored out the pure imaginary component, we do not usually know at this stage whether the least real period $\Omega_0(f)$ is equal to $\Omega(f)$ or half this; all we can say is that $\Omega(f)/2 = \mathrm{Re}\,\langle\{0,1/2\},f\rangle$ is the least real part of a period (up to sign). But in this case, $\{0,1/2\}$ is certainly an integral cycle, and since $\langle\{0,1/2\},f\rangle$ is real, we can in fact deduce already that the period lattice is of type 2 (rectangular) with $\Omega(f) = 2\Omega_0(f)$.

To compute more a_p we express each cycle μ_p as a linear combination of the basis and project to the eigenspace by taking the dot product with the left eigenvector v, which just amounts in this case to taking the second component. In this way we find $a_5 = -2$, $a_7 = 4$, $a_{13} = -2$, and so on. For the involutions W_3 and W_{11} we can either compute their 3×3 matrices or just apply them directly to the eigencycle $\{0,1/2\}$, and we find that $\varepsilon_3 = +1$ and $\varepsilon_{11} = -1$. In fact we already knew that the product of these was $\varepsilon_{33} = -1$, so we need not have computed ε_{11} directly, though doing so serves as a check.

Now we go back and compute the full space $H(33)$, which is six-dimensional, with basis

$$\left\{0,\frac{1}{7}\right\},\left\{0,\frac{1}{4}\right\},\left\{0,\frac{-1}{4}\right\},\left\{\frac{1}{12},\frac{-1}{6}\right\},\left\{\frac{1}{12},\frac{-1}{3}\right\},\left\{0,\frac{1}{10}\right\}.$$

By computing the 6×6 matrices of conjugation and T_2, we may pick out the left eigenvectors

$$v^+ = (0,1,-1,1,2,0) \qquad \text{and} \qquad v^- = (-1,0,0,2,1,1).$$

Since these vectors are independent modulo 2, it follows (as expected) that the period lattice is type 2, with a $\mathbb{Z}$-basis of the form $[\omega_1,\omega_2] = [x, yi]$.

Firstly, $x = \Omega_0(f) = \Omega(f)/2 = 2L(f,1)$. Summing the series for $L(f,1)$ we obtain $L(f,1) = 0.74734\ldots$, so that $\omega_1 = x = 1.49468\ldots$ and $\Omega(f) = 2x = 2.98936\ldots$. Then we use the twisting prime $l = 7$: the twisting cycle

$$\gamma_7 = \sum_{a=1}^{6}\left(\frac{a}{7}\right)\left\{0,\frac{a}{7}\right\}$$

is evaluated in terms of our basis to be $(2,2,0,-2,0,0)$, whose dot product with v^- is -6. Hence $y = \sqrt{7}L(f\otimes 7,1)/6$. The value of $L(f\otimes 7,1)$ is determined by summing the series to be $3.11212\ldots$, so that $y = 1.37232\ldots$ and $\omega_2 = 1.37232\ldots i$. If we evaluate these from the first 100 terms of the series, using a_p for $p < 100$, we find the approximate values $c_4 = 552.99999\ldots$ and $c_6 = -4084.99947\ldots$. These round to $c_4 = 553$ and $c_6 = -4085$, which are the invariants of the curve 33A1: $y^2 + xy = x^3 + x^2 - 11x$. Notice that this curve has four rational points, which we could have predicted since the ratio $L(f,1)/\Omega(f) = 1/4$ implies that $1 + p - a_p \equiv 0 \pmod{4}$ for all $p \neq 2, 3, 11$.

Example 3: $N = 37$

Since 37 is prime the M-symbols are simple here, as for $N = 11$. We find that $H^+(37)$ is two-dimensional, generated by $A = (8)$ and $B = (13)$. With this basis the matrices of T_2 and W_{37} are

$$T_2 = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad W_{37} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus we have two one-dimensional eigenspaces, generated by A and B respectively, with eigenvalues $(a_2 = -2, \varepsilon_{37} = +1)$ for A and $(a_2 = 0, \varepsilon_{37} = -1)$ for B. The left eigenvectors are simply

$v_1 = (1, 0)$ and $v_2 = (0, 1)$. Let us denote the corresponding newforms by f and g respectively. Now $\{0, 1/2\} = 2B$, so

$$\frac{L(f,1)}{\Omega(f)} = 0 \qquad \text{and} \qquad \frac{L(g,1)}{\Omega(g)} = \frac{1}{3}.$$

The fact that $\varepsilon_{37}(f) = +1$ implies that f has odd analytic rank, while the previous line shows that g has analytic rank 0..

To compute Hecke eigenvalues, the method we used previously would only work for g, so instead we use the variation discussed in Section 2.9. The cycle $\{1/5, \infty\}$ projects non-trivially onto both eigenspaces. In fact $(1 + 2 - T_2)\{1/5, \infty\} = -5A - B$, so the components in the two eigenspaces are $(-5)/(1 + 2 - (-2)) = -1$ and $(-1)/(1 + 2 - 0) = -1/3$. Hence by computing $(1 + p - T_p)\{1/5, \infty\} = n_1(p)A + n_2(p)B$ for other primes $p \neq 37$, we may deduce that

$$a(p, f) = 1 + p + n_1(p) \qquad \text{and} \qquad a(p, g) = 1 + p + 3n_2(p).$$

In this way we find that the first few Hecke eigenvalues are as follows:

p	2	3	5	7	11	13	17	19	...
$a(p, f)$	-2	-3	-2	-1	-5	-2	0	0	...
$a(p, g)$	0	1	0	-1	3	-4	6	2	...

Two things can be noticed here: the preponderance of negative values amongst the first few $a(p, f)$ means that the curve E_f has many points modulo p for small p, which we might expect heuristically since we know that its analytic rank is odd, and hence positive. Secondly, since $1 + p - a(p, g) \equiv 0 \pmod 3$ for all $p \neq 37$, we know that E_g will have a rational 3-isogeny.

Turning to the full space $H(37)$, we find that it has basis $\langle (8), (16), (20), (28) \rangle$. Conjugation and W_{37} have matrices

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

respectively.

For the A eigenspace corresponding to f we find left eigenvectors $v_1^+ = (-1, 1, 0, 0)$ and $v_1^- = (-1, 0, -1, 1)$. These are independent modulo 2, so the period lattice is rectangular, say $[x, yi]$. To find x we must twist by a real quadratic character, using a prime $l \equiv 1 \pmod 4$. Here $l = 5$ will do: the twisting cycle is $\{0, 1/5\} - \{0, 2/5\} - \{0, 3/5\} + \{0, 4/5\} = (2, -2, 0, 2)$, whose dot product with v_1^+ is -4, so that $x = \sqrt{5}L(f \otimes 5, 1)/4$. For the imaginary period we use $l = 3$ with twisting cycle $(0, 0, -1, 1)$ and a dot product of 2 with v_1^-, so that $y = \sqrt{3}L(f \otimes 3, 1)/2$. Evaluating numerically, using 100 terms of the series and a_p for $p < 100$, we find the values

$$L(f \otimes 5, 1) = 5.35486\ldots, \qquad \text{so that} \qquad x = 2.99346\ldots;$$
$$L(f \otimes 3, 1) = 2.83062\ldots, \qquad \text{so that} \qquad y = 2.45139\ldots;$$

and finally,

$$c_4 = 47.9999999996\ldots,$$
$$c_6 = -216.000000004\ldots.$$

The rounded values $c_4 = 48$ and $c_6 = -216$ are those of the curve 37A1, with equation $y^2 + y = x^3 - x$. This curve does have rank 1. We may also check that the analytic rank is 1 by computing $L'(f,1)$ by summing the series given in Section 2.13: we find that $L'(f,1) = 0.306\ldots$, which is certainly non-zero.

The B eigenspace is handled similarly to the example at level 33. We find $v_2^+ = (0,1,1,1)$ and $v_2^- = (1,1,0,0)$. The period lattice is $[x, iy]$ with $x = 3L(g,1)/2$ and $y = \sqrt{19}L(g\otimes 19,1)/4$. The latter needs more terms to compute to sufficient accuracy, as 19 is larger than the twisting primes we have previously used. Using $p < 100$ as before we find $c_4 = 1119.878\ldots$, which rounds to the correct (with hindsight) value 1120, but for c_6 we get 36304.495, and neither 36304 nor 36305 is correct. Going back to compute a_p for $100 < p < 200$ we reevaluate the series to 200 terms, and find

$$\begin{aligned} L(g,1) &= 0.72568\ldots, &\quad \text{so that} \quad & x = 1.08852\ldots; \\ L(g\otimes 19,1) &= 1.62207\ldots, &\quad \text{so that} \quad & y = 1.76761\ldots; \end{aligned}$$

and hence

$$c_4 = 1120.000008\ldots, \qquad \text{and} \qquad c_6 = 36295.99943\ldots.$$

Now the rounded values $c_4 = 1120$ and $c_6 = 36296$ are the invariants of the curve 37B1 with equation $y^2 + y = x^3 + x^2 - 23x - 50$. As expected, this curve does admit a rational 3-isogeny.

Example 4: $N = 49$

$H(49)$ is two-dimensional, with a basis consisting of the M-symbols (11), (2). Hence there is a unique newform f at this level, which must be its own -7-twist, or in other words have complex multiplication by -7. The conjugation matrix with respect to this basis is $\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$, so we take $v^+ = (1,-2)$ and $v^- = (1,0)$. Hence the period lattice has the form $[2x, x+yi]$ with $2x = \Omega_0(f) = \Omega(f)$. Also $a_2 = 1$, so we have $L(f,1)/\Omega(f) = 1/2$. Hence we may compute the real period via $L(f,1)$ as before, and find $L(f,1) = 0.96666\ldots$, so that $\Omega(f) = 1.9333\ldots$. But the method we have used in the earlier examples to find the imaginary period will not work here, since for every prime $l \equiv 3 \pmod 4$, $l \neq 7$, we have $L(f\otimes l, 1) = 0$, since $\chi(-49) = \chi(-1) = -1$ where χ is the associated quadratic character modulo l.

Instead, we compute periods directly, as in Section 2.10. The cycle $(5) = \{0, 1/5\}$ is equal to $(11)+(2)$, from which it follows that $\langle (5), f\rangle = -x+yi$; the coefficients are the dot products of the vector $(1,1)$ with $v^\pm$. Now $\{0,1/5\} = \{0, M(0)\}$ with $M = \begin{pmatrix} 10 & 1 \\ 49 & 5 \end{pmatrix}$. Hence the simpler formula (2.10.5) gives

$$P_f(M) = \left\langle \left\{0, \frac{1}{5}\right\}, f \right\rangle = -x + yi = \sum_{n=1}^{\infty} \frac{a(n)}{n} e^{-2\pi n/49} \left(e^{2\pi i n x_2} - e^{2\pi i n x_1}\right)$$

where $x_1 = -5/49$ and $x_2 = 10/49$. Summing the first 100 terms as before, we find the values

$$x = 0.96666\ldots \qquad \text{and} \qquad y = 2.557536\ldots.$$

Of course, the value of x merely confirms the value we had previously obtained a different way. These values give, in turn,

$$c_4 = 104.99992\ldots \qquad \text{and} \qquad c_6 = 1322.9994\ldots,$$

which round to the exact invariants $c_4 = 105$ and $c_6 = 1323$ of the curve 49A1, which has equation $y^2 + xy = x^3 - x^2 - 2x - 1$.

Using the improved formula (2.10.8) with better convergence, gives (also using 100 terms)

$$x = 0.96665585\ldots \qquad \text{and} \qquad y = 2.55753099\ldots,$$

which lead to the better values

$$c_4 = 104.9999992\ldots \qquad \text{and} \qquad c_6 = 1322.99998\ldots.$$

In this computation, we have not exploited the presence of complex multiplication. Notice that, in fact, $y/x = \sqrt{7}$. Obviously if we had known this it would have given us an easier way of computing y from x, and hence from $L(f, 1)$. However not all newforms at square levels have complex multiplication. Some are twists of forms at lower levels (for example, 100A is the 5-twist of 20A, and 144B is the -3-twist of 48A), which means that we could find the associated curves more easily by twisting the earlier curve. Others first appear at the square level in pairs which are twists of each other (for example, 121A and 121C are -11-twists of each other, and 196B is the -7-twist of 196A). One could probably find both periods of all such forms by looking at suitable twists to moduli not coprime to the level, but we have not done this systematically, as the more direct method was adequate in all the cases we came across in compiling the tables.

In practice we always computed the Hecke eigenvalues for $p < 1000$ at least, with a larger bound for higher levels. In some cases, particularly when the target values of c_4 or (more usually) c_6 were large, and especially when a large twisting prime was needed, we needed to sum the series to several thousand terms before obtaining the vales of c_4 and c_6 to sufficient accuracy.

These four examples exhibit essentially all the variations which can occur. The only problem with the larger levels is one of scale, as the number of symbols and the dimensions of the spaces grow. A large proportion of the computation time, in practice, is taken up with Gaussian elimination. This is why we have tried wherever possible to reduce the size of the matrices which occur: first by carefully using the 2-term symbol relations to identify symbols in pairs as early as possible, and secondly by working in $H^+(N)$ during the stage where we are searching for Hecke eigenvalues. The symbol relation matrices are very sparse (with at most three entries per row); sparse matrix techniques, which we use in our implementation, help greatly here. For finding eigenvectors of the Hecke algebra, however, we use a completely general purpose exact Gaussian elimination procedure.

The second time-consuming stage is when we are computing a large number of Hecke eigenvalues, where we call a very large number of times the procedures to convert rational numbers (cusps) to M-symbols and look these up in tables to find their coordinates with respect to the symbol basis. It is vital that these procedures are written efficiently; during the preparation of the tables, many great improvements in the efficiency of the program were achieved over a period of several months.