Assignments for Mathematical Optimal Control Theory (MA4H5)
Sheet 7

1. Apply the Kalman filter to the problem of estimating an unknown constant $y \in \mathbb{R}$, i.e. $n = 1$. The state equation and the observations z are given by

$$\begin{cases} \dot{x}(s) &= 0, \quad t \leq s \leq T, \\ x(t) &= y, \\ z(s) &= x(s) + \eta(s), \end{cases}$$

η the noise that pollutes the measurement. Here m is arbitrary, say $m = 1$. The cost is given by

$$\int_t^T \left(C\alpha^2 + B(z(s) - x(s))^2\right) \mathrm{d}s + Dy^2.$$

Choose $D = 1$ and leave $B = b$ as a parameter. Show that the choice of C is immaterial. Determine the differential equation that is satisfied by $K^{-1}(t)$ and the estimator $x(T)$. Find a way to choose b depending on the noise η.

2. Compute explicitly the Kalman-filter for the case of $n = m = 1$,

$$\begin{cases} \dot{x}(s) &= \alpha, \quad t \leq s \leq T, \\ z(s) &= x(s) + \eta(s), \end{cases}$$

and $C = D = 1$, $B = b > 0$. Analyze the behavior of the filter for q large or small.

3. Consider the problem of minimizing the cost

$$P_{y,t}[\alpha] = \int_t^T \left(\alpha(s)^T C\alpha(s) + (Qx(s) - r(s))^T B(Qx(x) - r(s))\right) \mathrm{d}s + y^T Dy,$$

over α and y where

$$\dot{x} = Mx + G\zeta + N\alpha,$$

and ζ is a known control. Show that the minimum is achieved through the solution of

$$\begin{cases} \dot{z} &= Mz + L(Qz - r) + G\zeta, \\ z(t) &= 0, \end{cases}$$

where $L = -K^{-1}Q^T C$.

Hint: Simply convert the problem into an estimation problem for $\xi = x - \gamma$ and γ, ξ satisfy $\dot{\gamma} = M\gamma + G\zeta$ and $\dot{\xi} = Mx + N\alpha$.