Abstract

We develop a novel rigorous approach to analyse the validity of continuum approximations for deterministic interacting particle systems. Some of our ideas have been used earlier in the context of annihilating Brownian spheres ([13]). We study the Boltzmann-Grad limit of ballistic annihilation, a topic which has has received considerable attention in the physics literature. Due to the deterministic nature of the evolution it is possible that correlations build up and the mean field approximation by the Boltzmann equation breaks down. We find a sharp condition on the initial distribution which ensures the validity of the Boltzmann equation and demonstrate the failure of the mean-field theory if the condition is violated.

Validity and non-validity of propagation of chaos

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The derivation of the continuum models of mathematical physics from atomistic descriptions is a longstanding and fundamental problem. One of the most notorious challenges is the question whether the Hamiltonian nature of the fundamental laws of motion (quantum mechanics, Newtonian mechanics) is compatible with the fact that the second principle of thermodynamics postulates that macroscopic systems are irreversible. An illustration of this question is provided by deterministic hard ball dynamics with random initial states. For high particle densities and suitably scaled diameters it is expected that the time-evolution of the density is close to the solution of the Boltzmann equation

$$\partial_t f + v \cdot \partial_u f = \int_{\mathbb{R}^d \times S^{d-1}} (f(u,\tilde{v}) f(u,\tilde{v}') - f(u,v) f(u,v')) |(v-v')\cdot\nu| \, \mathrm{d}v' \, \mathrm{d}\nu, \quad (1)$$

where $\tilde{v}, \tilde{v}'$ are obtained from v, v' by exchanging the respective components of v and v' in direction ν, that is

$$\tilde{v} = v + (v'-v)\cdot\nu\,\nu, \quad \tilde{v}' = v' + (v-v')\cdot\nu\,\nu,$$

and $f_t(u,v)$ is the density of presence at time t of particles at locations u with velocity v, see [12].

An important concept which sheds some light on the connection between the Boltzmann equation and hard ball dynamics is the propagation of chaos. This means that the distribution $p_N(u_1, v_1 \ldots, u_N, v_N, t)$ of N particles will loose its product structure for nonzero time t. However, the marginal distribution of the the first k particles should be very close to a product measure when the total number of particles N is large. A classical method to establish propagation of chaos is to express the evolution of k-particle marginals in terms of the $k+1$-particle marginals. This strategy is implemented in the BBGKY hierarchy. The weakness of this approach consists in the fact that establishing convergence of the resulting series is hard in many cases. O. Lanford succeeded in proving that in the case of hard ball dynamics the series that corresponds to the BBGKY hierarchy converges for small times to a solution of the Boltzmann equation ([8]). Unfortunately it cannot be shown that the time interval where the series is known to converge is larger than a small fraction of the mean free time, regardless of the initial data. This problem was partially overcome by [6] who managed to obtain a global result if the positions are in $\mathbb{R}^d$ and the initial density is sufficiently small. However, currently there is no result which covers the case where both data and time are large. It is arguable that the justification of the

Boltzmann equation (1) as a scaling limit of deterministic evolution constitutes a part of Hilbert's sixth problem ([5]).

In [9] the same strategy is applied to the simpler problem of coagulation. Here the spheres move along Brownian paths and two intact spheres annihilate each other if the distance between the centres drops below a. Although the series generated by the BBGKY hierarchy does not converge globally in time, Lang and Nguyen were able to give a rigorous justification of the corresponding Boltzmann equation by restarting the procedure at small positive time.

In this paper we consider kinetic annihilation, another simplification of hard ball dynamics which keeps two central features of the original evolution: The initial state is random, the evolution is deterministic. We assume that the initial phase space positions in the phase space $\mathbb{T}^d \times \mathbb{R}^d$ ($\mathbb{T}^d$ is the unit torus) form a Poisson point process with some intensity $\mu \in M(\mathbb{T}^d \times \mathbb{R}^d)$. As long as they are intact the centres of the spheres move along straight lines with constant velocity. When the centres of two spheres, which are still intact, come within distance a, then both spheres are destroyed.

We will consider the asymptotic behaviour of the system in the limit where the diameter a of the particles tends to 0 and the total intensity $n = \mu(\mathbb{T}^d \times \mathbb{R}^d)$ is linked to a by the Boltzmann-Grad relation

$$na^{d-1} = 1. \tag{2}$$

The central question in this paper is whether for small values of a the many-body evolution can be described by the gainless Boltzmann equation

$$\partial_t f + v \cdot \partial_u f = Q_-[f, f] \tag{3}$$

where $Q_-[f, g](v) = -\kappa_d f(v) \int_{\mathbb{R}^d} \mathrm{d}g(v')\, |v - v'|$ is the loss term of the hard-sphere collision kernel of the Boltzmann equation (1) and κ_d is the volume of the $d-1$ dimensional unit-ball. For the sake of simplicity we will restrict ourselves to the case where the initial density f_0 does not depend on u, in this case the transport term $v \cdot \partial_u f$ in equation (3) vanishes and $f_t(u, v) = f_t(v)$.

Solutions f of the Boltzmann equation (3) have two different but closely related interpretations. The macroscopic interpretation involves the empirical quantity $b(U, t) = \#(\omega(t) \cap U)$ where $U \subset \mathbb{T}^d \times \mathbb{R}^d$ is arbitrary (measurable) and $\omega(t) \subset \mathbb{T}^d \times \mathbb{R}^d$ is the realisation of the phase space positions of the particles at time t. Equation (3) is valid in the macroscopic sense if $b(U, t)/N$ converges to $\int_U \mathrm{d}u\, \mathrm{d}f_t(v)$ in probability as a tends to 0.

A microscopic interpretation is based on the single-particle marginals of the N-particle distribution. The gainless homogeneous Boltzmann equation is valid in the microscopic sense if

$$\lim_{a \to 0} \mathrm{Prob}((u_1(t), v_1(t)) \in U \text{ and particle 1 is intact at time } t) = \frac{1}{f_0(\mathbb{R}^d)} \int_U \mathrm{d}u\, \mathrm{d}f_t(v).$$

Since the distribution of the N particles is invariant under permutation it is irrelevant which particle index we use to define the microscopic validity.

It is well known that microscopic validity and a simple bound on correlations implies macroscopic validity. Ballistic annihilation has been studied extensively in the physics literature, see [4, 11, 2, 3, 1]. Kinetic annihilation dynamics can be used to model growth and coarsening of surfaces, see [7].

This paper contains an outline of the essential steps of the first mathematical proof of the microscopic validity of the Boltzmann equation as a scaling limit of kinetic annihilation (Theorem 1.2). The main idea is to determine the probability distribution of objects that are adapted to the system under consideration. In this case we work with marked trees that record the history of potential collisions which a tagged particle experiences. The trees have the property that their expected size remains finite as a tends to 0. This idea has been used earlier in [13] in the context of Brownian spheres. Note however that our results differ significantly from those in [13].

First of all, we are working with a *deterministic* evolution. The Boltzmann equation emerges because of random initial conditions. We obtain a limiting measure in the *phase space* $\mathbb{T}^d \times \mathbb{R}^d$, not in *position* space $\mathbb{T}^d$. Since we consider initial distributions which are u-independent we end up with measures on the velocity space $\mathbb{R}^d$.

Secondly, for a large subset of trees we obtain a very simple, explicit representation formula for the distribution of the trees (38). Thanks to this formula we are able to establish explicit $o(1)$-bounds of the total variation difference between empirical distribution $\hat{P}$ and the limiting measure P as a tends to 0.

Thirdly, since the only source of stochasticity are the initial values, it is less obvious that the initial chaos is propagated to such a degree that the limiting evolution can be described by a simple mean field theory which leads to the Boltzmann equation. We obtain novel necessary conditions on the absence of certain concentrations in the initial density (7) which are sufficient for the validity of the Boltzmann equation. A counter-example (Theorem 3.1) demonstrates that the mean field theory that underlies the Boltzmann equation is not consistent with the many body evolution if the concentrations are present in the initial density. This shows on the one hand, that our condition is actually sharp and on the other hand that a previously published justification of the Boltzmann equation by [3] requires the additional assumption that the initial velocity density is absolutely continuous with respect to the Lebesgue measure.

Since the Boltzmann equation is insensitive to these concentrations it is impossible to derive condition (7) from the mean-field theory itself.

Failure of the Boltzmann approximation of high-dimensional deterministic many-body systems has been previously observed in the case of ballistic annihilation for for $d = 1$, see [4] and for discrete velocity models of collisional dynamics, see [14]. In both cases the failure of the mean-field theory can be traced back to the the finiteness of the set of possible directions. Our analysis shows that the buildup of correlations is actually caused by concentrations in the initial distributions, not by the specifics of the evolution.

1 Validity

We consider n particles with initial values $(u_0(i), v_0(i)) \in \mathbb{T}^d \times \mathbb{R}^d$, $i = 1 \dots n$, which evolve by Newtonian dynamics

$$\begin{aligned} &u(i, t=0) = u_0(i),\ v(i, t=0) = v_0(i), \\ &\dot{u}(i,t) = v(i,t),\ \dot{v}(i,t) = 0. \end{aligned} \tag{4}$$

For each $t \in [0, \infty)$, $i \in \{1 \dots n\}$ there exists a unique scattering state $\beta_i^{(a)}(t) \in \{0, 1\}$ which satisfies the implicit relation

$$\beta^{(a)}(i,t) = \begin{cases} 1 & \text{if } \operatorname{dist}(z_i, z_{i'}, s) \geq a\beta^{(a)}(i', s) \text{ for all } s \in [0,t),\ i' \neq i, \\ 0 & \text{else} \end{cases} \tag{5}$$

with a modified distance function to ignore initial intersections

$$\operatorname{dist}((u,v),(u',v'),s) = |u - u' + s(v - v')| + a\chi_{[0,a]}(|u-u'|), \tag{6}$$

where $|.|$ is the metric on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

Definition 1.1 (Tagged Poisson point processes) *Let Ω be a measure space. The random variable $z \in \cup_{n=0}^{\infty}\Omega^n$ is a realisation of a Poisson point process (PPP) with density $\mu \in M_+(\Omega)$ if*

$$\operatorname{Prob}(z \in \Omega^n) = e^{-\mu(\Omega)}\frac{\mu(\Omega)^n}{n!}, \quad \operatorname{law}(z_i) = \mu/\mu(\Omega),$$

and $z_1, \dots, z_n$ are independent. Realisations of the tagged Poisson point process (tPPP) are obtained by adding an independent random variable z_0 with law $\mu/\mu(\Omega)$, i.e. for symmetric $A \subset \cup_{n=1}^{\infty}\Omega^n$ one obtains that

$$\operatorname{Prob}_{\text{tPPP}}((z_0, z) \in A) = \frac{1}{\mu(\Omega)e^{\mu(\Omega)}}\sum_{n=0}^{\infty}\frac{1}{n!}\int_{A\cap\Omega^{n+1}} \mathrm{d}\mu(z_0)\dots\mathrm{d}\mu(z_n).$$

Note that the tagged PPP is a symmetric point process. The motivation for working with this process is that the realisations of the tagged PPP without the tagged particle form a PPP and we obtain a very simple explicit formula for the distribution of trees, see (38), hence the complexity of the proof can be reduced. On the other hand, it seems that the formulae for the joint distribution of two trees are much more complicated, therefore we will only make statements which concern the law of a single, tagged particle.

Theorem 1.2 *(Justification of the gainless homogeneous Boltzmann equation) Let $f_0 \in PM_+(\mathbb{R}^d), d \geq 2$ be a momentum density with finite second moment ($\int_{\mathbb{R}^d} \mathrm{d}f_0(v)(1+|v|^2) < \infty$) which does not concentrate mass on lines*

$$f_0(v + \mathbb{R}\,w) = 0 \text{ for all } v, w \in \mathbb{R}^d. \tag{7}$$

Assume that the intensity of the tagged PPP is $\mu = n(\mathbf{1}_{\mathbb{T}^d} \otimes f_0)$, with n given by (2), then

$$\lim_{a\to 0} \mathrm{Prob}_{\mathrm{tPPP}}\Big(z(0,t)) \in A \text{ and } \beta^{(a)}(0,t) = 1\Big) = \int_A \mathrm{d}u \, \mathrm{d}f_t(v), \tag{8}$$

where $z(0,t), \beta(0,t)$ are position and status of the tagged particle at time t and $f : [0,\infty) \to M_+(\mathbb{R}^d)$ solves the gainless homogeneous Boltzmann equation

$$\partial_t f = Q_-[f,f], \quad f(t=0) = f_0. \tag{9}$$

The assumption that $\int_{\mathbb{R}^d} \mathrm{d}f_0(v) = 1$ is not necessary. We make it because it simplifies the notation in the proof.

Assumption (7) does not exclude the possibility that f_0 is concentrated on lower dimensional subsets, for example the uniform distribution on the sphere S^{d-1} is admissible, i.e. f_0 satisfies

$$\int \varphi(v) \, \mathrm{d}f_0(v) := \frac{1}{\mathcal{H}^{d-1}(S^{d-1})} \int_{S^{d-1}} \varphi(v) \, \mathrm{d}\mathcal{H}^{d-1}(v), \tag{10}$$

for all test-functions $\varphi \in C_c(\mathbb{T}^d \times \mathbb{R}^d)$, where $\mathcal{H}^d$ is the d-dimensional Hausdorff-measure.

2 Effective descriptions

2.1 The hierarchy of evolutions

Instead of expanding f_t into a power-series in t and matching coefficients, in a first step, we replace the initial value problem (9) by an infinite system using general initial distribution without concentrations

$$\dot{f}_k = Q_-[f_{k-1}, f_k], \quad f_{t=0,k} = f_0. \tag{11}$$

Since Q_- is quadratic, for fixed k the integro-differential equation (11) is in fact linear and non-autonomous. We can therefore work with the mathematically much more convenient mild formulation. The differential equation completely decouples in v and the equation for each v is a scalar linear nonautonomous ODE, which can be directly integrated to

$$f_{t,k} = \exp(-\textstyle\int_0^t L[f_{s,k-1}] \, \mathrm{d}s) f_0, \tag{12}$$

where $L[f](v) = \kappa_d \int \mathrm{d}f(v') \, |v - v'|$. We observe that $\mathrm{d}f_{t,k}(v)$ is absolutely continuous with respect to $\mathrm{d}f_0(v)$ due to the decoupling in v.

Lemma 2.1 *Let $f_0 \in M_{(1+|v|)^2}$ then f_k converges in $C^0_\rho([0,\infty), M_{1+|v|})$ to f for some $\rho > 0$ and $f \in C^1([0,\infty), M_{1+|v|})$ is the unique solution of (9).*

By $M_{1+|v|}$ and $M_{(1+|v|)^2}$ we mean the set of Radon measures with first and second moment, C_ρ denotes the continuous functions which grow not faster than $e^{\rho t}$. The proof of Lemma 2.1 together with a precise definition of the function spaces is standard.

Now we have to translate this idea into the context of deterministic many-body dynamics. To limit the complexity of the notation we will from now on assume that everything except the constants depends on a without displaying the dependency. For every realisation of the n-body evolution the random variable $\beta(i,t) \in \{0,1\}$, which encodes the scattering state of particle $i \in \{1 \dots n\}$ at time $t \in [0,\infty)$ satisfies the implicit relation (5). The computation of β can be simplified by introducing a hierarchy of artificial evolutions indexed by $k \in \mathbb{N}$. We assume that the initial values of the particles at all levels are identical. The particles at level $k = 1$ are simply transported and do not interact with anything. The particles at level $k > 1$ interact only with the particles at level $k-1$, but not with each other. For each $k \in \mathbb{N}$ and $i \in \{1 \dots n\}$ the scattering state $\beta_k(i,t) \in \{0,1\}$ is defined in the following way

$$\beta_k(i,t) = \begin{cases} 1 & \text{if } \operatorname{dist}(z_i, z_{i'}, s) \geq a\beta_{k-1}(i', s) \text{ for all } s \in [0,t), i' \neq i, \\ 0 & \text{else,} \end{cases} \tag{13}$$

$$\beta_1(i) \equiv 1, \tag{14}$$

with dist as in (6).

Remark 2.2 *While the determination of the collision-state $\beta(i,t)$ is a complicated problem, the state $\beta_k(i,t)$ emerges via a very simple calculation from $\beta_{k-1}(\cdot,t)$.*

Lemma 2.3 *For all realisations of the initial conditions $\omega \in \cup_{n=1}^{\infty} (\mathbb{T}^d \times \mathbb{R}^d)^n$ both $\beta_k(i,t)$ and $\beta(i,t)$ are well defined and*

$$\lim_{k\to\infty} \beta_k(i,t) = \beta(i,t) \tag{15}$$

point-wise in i and uniformly in t.

2.2 The concept of marked trees

The translation of the n-body evolution into scattering states β is greatly facilitated by the concept of trees. In the collision tree with root (u,v) we will collect information of collisions and potential collisions up to time t for a particle with initial data u, v.

As an example assume that $n = 4$ and consider the scenario in fig. 1 where the letters A, B, C, D are the labels of the four particles, the empty circles are the initial positions and the arrows are the initial velocities. Consequentially the arrow-tips indicate the positions of the particles at time $t = 1$. To determine whether a certain particle has been scattered before time $t = 1$ it suffices to analyse the associated collision tree which is constructed as follows: The particle

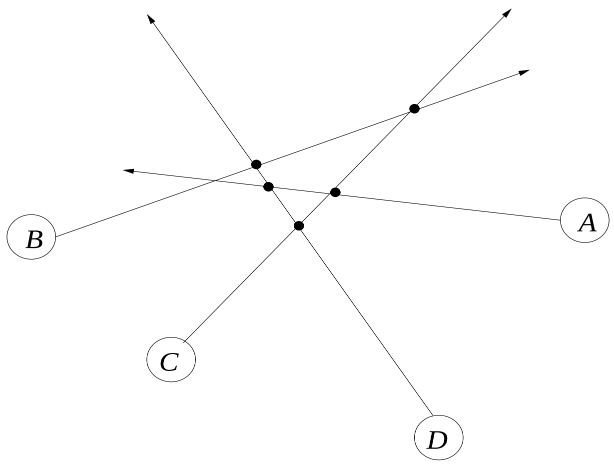

Figure 1: Initial positions and velocities of four particles. The bullets indicate the positions where the particles are potentially scattered. The shown configuration is not very likely and consequentially the collision trees are quite complex.

of interest is the root with initial data (u, v). The particles which are potentially scattered by the root are added as leaves, i.e. a particle with initial data (u', v') is added, if $|u + sv - (u' + sv')| \leq a$ for some $s \in [0, t]$. This procedure is recursively applied to every leaf but we consider only potential scattering events which are upstream, i.e. before the event which is responsible for adding the leaf. The four collision trees associated to the scenario in fig. 1 are shown in fig. 2. The extraction of the collision trees amounts to a significant reduction of the complexity of the problem. In general, the number of potential scattering events (bullets) is proportional to N but thanks to the Boltzmann-Grad-scaling (2) the number of nodes in the individual trees is a Poissonian random number with an intensity which is asymptotically independent of N and grows exponentially with t, see Proposition 2.7.

We convert now the example into a general concept.

Definition 2.4 *Let $\mathbb{N} = \{1, 2, \ldots\}$. The height of a node (or multi-index) $l \in \mathbb{N}^i$ is defined by $|l| := i$, the parent node of $l \in \mathbb{N}^i$ is $\bar{l} = (l_1, \ldots, l_{i-1})$. Let $\mathcal{F} = \cup_{i=1}^{\infty} \mathbb{N}^i$ be the set of multi-indices. We say that $m \subset \mathcal{F}$ is a tree with root ($m \in \mathcal{T}$), if*

1. *$\#m < \infty$,*
2. *$m \cap \mathbb{N} = \{1\}$,*
3. *$\bar{l} \in m$ for all $l \in m \setminus \mathbb{N}$,*
4. *$l - 1 \in m$ for all $l \in m$ such that $l \neq (*, \ldots, *, 1)$,*

where $l - 1 = l - (0, \ldots, 0, 1)$. We say that a tree m has at most height k ($m \in \mathcal{T}_k$) if $m \cap \mathbb{N}^{k+1} = \emptyset$.

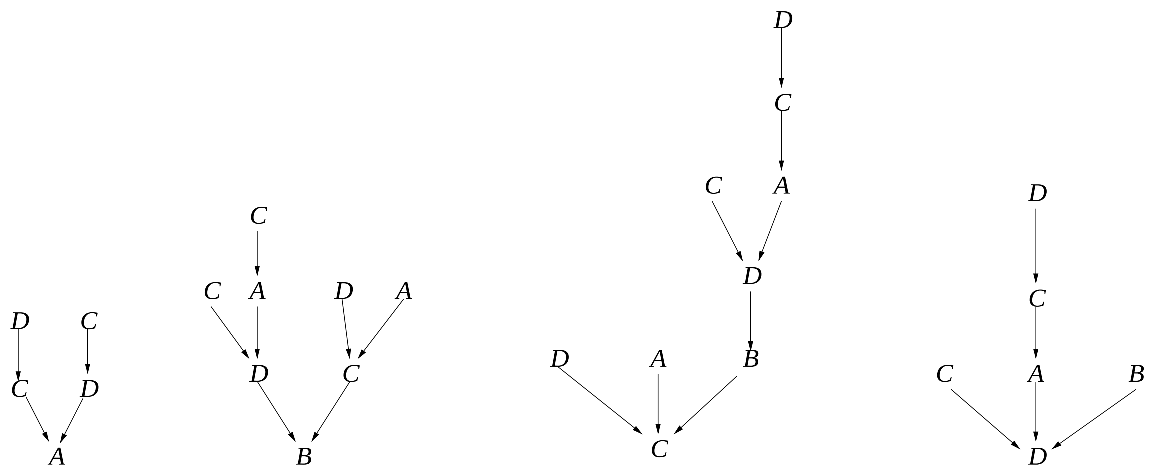

Figure 2: Collision trees of the four particles with initial positions and collision structure given in fig. 1. At time $t = 1$ particles C and D have been scattered, particles A and B have not. Note that the labels of the particles which generate the potential scattering events are only included in the picture in order to illustrate the translation of fig. 1 into collision trees. The scattering state of the particle at the root is completely determined by the tree structure, the labels of the tree nodes are irrelevant. For example, the tree of particle B does not contain enough information to decide whether particle A is scattered.

Let $Y = \{(u, v, s, \nu) \in \mathbb{T}^d \times \mathbb{R}^d \times [0, \infty) \times S^{d-1}\}$ be the space of initial values and collision parameters. The set of marked trees is given by

$$\mathcal{MT} = \Bigg\{ (m, \phi) \;\Bigg|\; m \in \mathcal{T}, \phi : m \to Y \text{ with the property } s_l \in [s_{l-1}, s_{\bar{l}}]$$
$$\text{and } \nu_l = \tfrac{1}{a}(u_{\bar{l}} - u_l + s_l(v_{\bar{l}} - v_l)) \text{ for all } l \in m \setminus \mathbb{N} \Bigg\},$$

where $s_{(*,\ldots *,0)} = 0$. For each skeleton $m \in \mathcal{T}$ we define the set

$$\mathcal{E}(m) = \{(\tilde{m}, \phi) \in \mathcal{MT} \mid \tilde{m} = m\}, \tag{16}$$

which contains all trees with skeleton m. We stipulate a strict order of the set of nodes m:

$$l < l' \text{ if either } |l| < |l'| \text{ or } (|l| = |l'| \text{ and } \bar{l} < \bar{l}') \text{ or } (\bar{l} = \bar{l}' \text{ and } l_{|l|} < l'_{|l|}) \tag{17}$$

This order is induced by the link between the collision time and the indices $l \in m$.

For example, $\{(1), (1, 1), (1, 2), (1, 3), (1, 1, 1), (1, 1, 2)\} \in \mathcal{T}_3$, but $\{(1), (2, 1)\}$ is not a tree skeleton. The assumption $s_l \in [s_{l-1}, s_{\bar{l}}]$ implies that for all nontrivial permutations $\pi \in S_{\#m} \setminus \mathrm{Id}$ (S_n is the set of permutations of n symbols) and all trees $\Phi = (m, \phi) \in \mathcal{MT}$ the permuted tree $\Phi^\pi = (m, \phi^\pi)$ with $\phi^\pi_l = \phi_{\pi(l)}$ is not a tree in the sense of Definition 2.4.

The value ν_1 has no relevance. To circumvent this problem we fix a point $\nu^* \in (S^{d-1})$, define

$$\mathcal{MT}^* = \{\Phi \in \mathcal{MT} \mid \nu_1 = \nu_1^* \}.$$

and will in future denote $\mathcal{MT}^*$ by $\mathcal{MT}$.

It is clear from the definition that for each tree $m \in \mathcal{T}$ there exists a function $r : m \to \mathbb{N} \cup \{0\}$ which counts the number of direct successors, i.e.

$$r_l = \#\{l' \in m \mid \bar{l}' = l\}.$$

Remark 2.5 *Graph theoretical description of collisions in a hard-sphere gas can lead to many different graphs, which are not necessarily trees. The advantage of our definition is that this graph will always be a tree. Particles might appear several times in a tree, as in fig. 2. This will not destroy the tree structure, as these are due to different collision events. Multiple collisions, which are well-defined in our setting, can lead to identical branches within the tree, but the definition $\mathcal{T}$ will discriminate between these and the graph of collisions is still a tree.*

The scattering state $\beta : m \to \{0, 1\}$ is determined uniquely by the skeleton, i.e. the labels of the particles are immaterial, but the actual computation is not completely trivial. The most important aspect of the computation of β is that the scattering information flows from the leaves to the root, i.e. the scattering state of a node is completely determined by the state of the nodes above, the nodes below are irrelevant.

We will construct now two families of probability measures $P_{t,k}, \hat{P}_{t,k} \in PM(\mathcal{MT})$. The empirical distribution $\hat{P}_{t,k}$ is induced by the many-body dynamics and will be constructed recursively in Section 2.4. The mean-field distribution $P_{t,k}$ is given by an explicit formula (18). The link between $P_{t,k}$ and $\hat{P}_{t,k}$ is provided by the set of good trees $\mathcal{G}(a) \subset \mathcal{MT}$ (Definition 2.9) which has the properties that restriction of $\hat{P}_{t,k}$ on $\mathcal{G}(a) \cap \mathcal{MT}$ converges to $P_{t,k}$ and $P_{t,k}(\mathcal{G}(a))$ goes to 1 as a tends to 0 (Proposition 2.11).

This is the crucial step which eventually yields the justification of the mean-field theory. In other words, the main task consists in analysing the mean-field measure $P_{t,k}$, the empirical distribution $\hat{P}_{t,k}$ enters only when we prove that $P_{t,k}$ is consistent with $\hat{P}_{t,k}$.

2.3 The mean-field distribution $P_{t,k}$

We construct now the mean-field distribution of trees $P_{t,k} \in PM(\mathcal{MT})$. Let $\Omega \subset \mathcal{MT}$ and $t \in [0, \infty)$. The mean field probability that the observed tree is in Ω is given by

$$P_{t,k}(\Omega) = \sum_{m \in \mathcal{T}_k} \int_{\Omega \cap \mathcal{E}(m)} e^{-\sum_{j<k} \Gamma_j(\Phi)} \, \mathrm{d}\lambda^m(\phi) \tag{18}$$

where

$$\begin{aligned}
\Gamma_j(\Phi) &= \sum_{l \in m, |l|=j} \gamma_l(\Phi), \\
\gamma_l(\Phi) &= \int_0^{s_l} L[f_0](v_l) \, \mathrm{d}s' = s_l \, L[f_0](v_l) \geq 0 \text{ is the collision rate of particle } l, \\
\lambda^m(\phi) &= \mu(z_1) \otimes \delta(s_1 - t)] \otimes \prod_{l \in m \setminus \mathbb{N}} \left[((v_l - v_{\bar{l}}) \cdot \nu_l)_+ \, \chi_{[s_{l-1}, s_{\bar{l}}]}(s_l) \, \mathrm{d}f_0(v_l) \, \mathrm{d}\nu_l \, \mathrm{d}s_l \right], \qquad (19) \\
\mu(u, v) &= \mathbf{1}_{\mathbb{T}^d}(u) \otimes f_0(v).
\end{aligned}$$

Remark 2.6 *1. Note that the positions u_l are completely determined by (u_1, v_1) and $(v_l, s_l, \nu_l)_{l \in m \setminus \{1\}}$. Since we have assumed that ν_1 is fixed, the value of $P_{t,k}(\Omega)$ is well-defined.*

2. It is noteworthy that the measures $P_{t,k}$ depend on time only via the parameter t. In other words, time plays the role of a parameter which propagates through the tree and qualifies the local branching structure.

3. For some event $\Omega \subset \mathcal{MT}_k$ the probability $P_{t,k'}(\Omega)$ is independent of k' if $k' > k$. Equivalently, $P_{t,k_1}(\Omega \cap \mathcal{E}(m)) = P_{t,k_2}(\Omega \cap \mathcal{E}(m))$, if the height of m is strictly smaller than $\min\{k_1, k_2\}$.

We can simplify the measure $P_{t,k}$ by integrating over the collision parameters $\nu_l \in S^{d-1}$, $l \in m$. Let $\hat{Y} = \mathbb{R}^d \times [0, \infty)$ be the reduced set of collision data. For every $\Omega \subset \mathcal{T}(\hat{Y})$ we find that when still denoting the collision data as ϕ

$$\bar{P}_{t,k}(\Omega) = \sum_{m \in \mathcal{T}_k} \int_{\Omega \cap \mathcal{E}(m)} \mathrm{d}\bar{\lambda}^m(\phi)\, e^{-\sum_{j<k} \Gamma_j(\Phi)} \tag{20}$$

with

$$\bar{\lambda}^m(\phi) = f_0(v_1) \otimes \delta(s_1 - t) \otimes \prod_{l \in m \setminus \mathbb{N}} \left[\kappa_d\, |v_l - v_{\bar{l}}|\, \chi_{[s_{l-1}, s_{\bar{l}}]}(s_l)\, \mathrm{d}f_0(v_l)\ \mathrm{d}s_l\,\right].$$

The measures $P_{t,k}$ have the remarkable property that the expectation of certain random variables can be computed exactly or estimated accurately.

Proposition 2.7 *For a tree $m \in \mathcal{T}$ the number of non-root nodes is given by $R(m) = \sum_{r \in m} r_l = \#m - 1$. The expected value of R satisfies the estimate uniformly in k*

$$\mathbb{E}(R) \leq K_{\mathrm{ini}} \exp(\kappa_d K_{\mathrm{ini}} t), \tag{21}$$

with $K_{\mathrm{ini}} = \int_{\mathbb{R}^d} \mathrm{d}f_0(v)\,(1 + |v|)^2$ and

$$P_{t,k+1}\,(v_1 \in \Omega \text{ and } \beta_1(t) = 1) = \mathrm{d}f_{t,k}(v) \tag{22}$$

holds, where $f_{t,k}$ is the solution of system (12).

The proof relies on the recursive structure of the definition of the measure $P_{t,k}$ and can be found in [10].

2.4 The empirical distribution $\hat{P}_{t,k}$

We return now to the hierarchy of many body evolutions described in Section 2.1. The initial values of the particles form a random set $\omega \subset \mathbb{T}^d \times \mathbb{R}^d$ and it is assumed that the law of ω is the Poisson point process with density $N\mu$, where $\mu = \mathbf{1}_{\mathbb{T}^d} \otimes f_0 \in PM(\mathbb{T}^d \times \mathbb{R}^d)$. Hence, the size $n = \#\omega$ is Poissonian random variable with intensity N. As explained in Section 2.2, the family of probability measures $\hat{P}_{t,k} \in PM(\mathcal{MT})$ is the empirical distribution of the tree Φ which is generated by the many-body evolution and has a randomly chosen (tagged) particle as its root. This tree is only well defined if $n > 0$, i.e. ω is non-empty. For this reason we define $\hat{P}_{t,k}(\Omega)$ as the conditional probability that the tree is contained in the set Ω, given that $n = \#\omega > 0$.

A particularly simple method of sampling from this conditional distribution consists in drawing a realisation of ω according to the unconditioned Poisson point process, and an independent random variable $z \in \mathbb{T}^d \times \mathbb{R}^d$ with law $\mu(z) = \mathbf{1}_{\mathbb{T}^d}(u) \otimes f_0(v)$ which is the initial value of the tagged particle. It can be checked without difficulty that the joint distribution of ω and z is the previously defined conditional distribution.

The trees generated by this procedure are denoted by $\Phi(t,k) = (m(t,k), \phi) \in \mathcal{MT}_k$, where $m(t,k) \in \mathcal{T}_k$ is the skeleton and $\phi : m(t,k) \to Y$ specifies the initial values, the collision times and the impact parameters. The measures $\hat{P}_{t,k}$ are the image measure of $\mathrm{Prob}_{\mathrm{tPPP}}$ induced by the many-particle flows so that for each $\Omega \subset \mathcal{MT}$ we obtain

$$\hat{P}_{t,k}(\Omega) := \mathrm{Prob}_{\mathrm{tPPP}}((m(t,k),\phi) \in \Omega). \tag{23}$$

The tree measures $\hat{P}_{t,k}$ are derived from $\mathrm{Prob}_{\mathrm{tPPP}}$, but $\mathrm{Prob}_{\mathrm{tPPP}}$ cannot be derived from $\hat{P}_{t,k}$.

By construction, for fixed ω the skeleton m is monotonously increasing in t and k, and for fixed $l \in m$ the data ϕ_l does not depend on t or k. This is equivalent to saying that the j-marginal of $\hat{P}_{t,k}$ (trees of hight $j \leq k$) is given by $\hat{P}_{t,j}$, i.e.

$$\hat{P}_{t,k}\left(\left(m(t,k) \cap (\cup_{i=1}^{j}\mathbb{N}^i), (\phi_l)_{|l|\leq j}\right) \in \Omega\right) = \hat{P}_{t,j}((m(t,j),(\phi_l)_{|l|\leq j}) \in \Omega) \tag{24}$$

for all $\Omega \subset \mathcal{MT}_j$, $k \geq j$.

We will use formula (24) to construct an alternative characterisation of $\hat{P}_{t,k}$ which reflects the iterative process that underlies the definition of $m(t,k)$. Using this alternative characterisation one can easily establish total-variation bounds for $P_{t,k} - \hat{P}_{t,k}$. Since the time t is arbitrary but fixed we will often write $\hat{P}_k$ instead of $\hat{P}_{t,k}$.

Let $(m',\phi') \in \mathcal{MT}_{k-1}$ and let $\hat{P}_k(\,\cdot\,|\,(m',\phi')) \in PM(\mathcal{MT}_k)$ be the conditional distribution of $\hat{P}_k$ in the sense that

$$\begin{aligned}\hat{P}_k(\Omega\,|\,(m',\phi')) := \hat{P}_k\Big((m(k),\phi) \in \Omega \mid m \cap \mathbb{N}^j = m' \cap \mathbb{N}^j \text{ for all } j \in \{1\dots k-1\}\\ \text{and } \phi_l = \phi'_l \text{ for all } l \in m \text{ such that } |l| < k\Big).\end{aligned}$$

Formula (24), which characterises the j-marginals of $\hat{P}_{t,k}$, yields the following recurrence relation for $\hat{P}_k$:

$$\hat{P}_k(\Omega) = \int_{\mathcal{MT}_{k-1}} \mathrm{d}\hat{P}_{k-1}(\Phi')\,\hat{P}_k(\Omega\,|\,\Phi'). \tag{25}$$

Repeating this step $k-1$ times we obtain the following iterative representation of $\hat{P}_k$:

$$\begin{aligned}\hat{P}_k(\Omega) = \int_{\mathcal{MT}_1} \mathrm{d}P_1(\Phi_1) \int_{\mathcal{MT}_2} \mathrm{d}\hat{P}_2(\Phi_2\,|\,\Phi_1)\\ \dots \int_{\mathcal{MT}_{k-1}} \mathrm{d}\hat{P}_{k-1}(\Phi_{k-1}\,|\,\Phi_{k-2})\,\hat{P}_k(\Omega\,|\,\Phi_{k-1}),\end{aligned} \tag{26}$$

where

$$P_1(z_1) = \mu(z_1) \in PM\left(\mathbb{T}^d \times \mathbb{R}^d\right) \tag{27}$$

is the distribution of initial value for the root particle.

2.5 Convergence of $\hat{P}_k$ to P_k

Having constructed an iterative characterisation of $\hat{P}_k$ we will now show that it is very similar to the mean field measure P_k in a precise way. The key is to identify the mechanisms by which the two probability distributions fail to be equal. In this part of the paper we will work with the phase-space representation of the trees: $z_l = (u_l, v_l) \in \mathbb{T}^d \times \mathbb{R}^d$.

Remark 2.8 *There are only two reasons why $\hat{P}_k$ fails to coincide with P_k in the limit $a \to 0$:*

1. *The cylinders which are covered by the paths of the particles might contain self-intersections due to the periodic boundary conditions: $v - v' \in R(t, a)$ with*

$$R(t, a) = \left\{ v \in \mathbb{R}^d \mid \min\{|s\, v - \xi| \mid s \in [0, t], \xi \in \mathbb{Z}^d \setminus \{0\}\} \leq a \right\}. \quad (28)$$

2. *Particles might appear at different positions within a tree, i.e. the map $z : m \to \mathbb{T}^d \times \mathbb{R}^d$ might be not injective.*

The set $R(t, a)$, which can easily seen to be nonempty, is relevant due to periodic boundary conditions, which will lead to self-intersections of the cylinders. This happens, if $v - v_j$ is sufficiently close to a velocity v^*, where the components of $v_1^*, \ldots, v_d^*$ are rationally dependent, i.e. $\eta \cdot v^* \in \mathbb{Z}$ with $\eta \in \mathbb{Z}^d$, but only if $|\eta| \leq t$. The effect is not present in a setting where $(u, v) \in \mathbb{R}^d \times \mathbb{R}^d$.

The second effect is caused by the notorious recollisions. These dependencies disappear as the diameter a tends to zero.

Motivated by Remark 2.8 we define the set of "good" trees.

Definition 2.9 *For each $a_0 > 0$ the set of "good" trees $\mathcal{G}(a_0) \subset \mathcal{MT}$ consists of those trees $(m, \phi) \in \mathcal{MT}$ with the property that for all $0 < a \leq a_0$ and all $l \in m$*

$$v_{\bar{l}} - v_l \in \mathbb{R}^d \setminus R(t, a) \qquad \text{(all parent-child-pairs are non-resonant)}, \quad (29)$$

$$z_l \notin \cup_{\substack{l' < l \\ l' \neq \bar{l}}} C_{l'} \qquad \text{(no particle has more than one parent)}, \quad (30)$$

where we associate to each node $l \in m$ the set of colliding initial values

$$C_l = \left\{ z' \in \mathbb{T}^d \times \mathbb{R}^d \,\middle|\, \min_{s' \in [0, s_l]} |\mathrm{dist}(z_l, z', s')| \leq a \right\},$$

and dist *as in (6) ignores overlap in the initial data.*

Note that $\mathcal{G}(a_0) \subset \mathcal{MT}$ is a family of sets which increases monotonely with decreasing a_0. An elementary calculation yields that for all $v' \in \mathbb{R}^d \setminus (v_l + R(t, a))$

$$N\, \mathcal{H}^d \left(C_l \cap (\mathbb{T}^d \times \{v'\}) \right) = \kappa_d |v_l - v'| s_l. \quad (31)$$

The significance of $\mathcal{G}(a_0)$ is given by the following results

$$\lim_{a_0\to 0}\inf_k P_k(\mathcal{G}(a_0)) = 1, \tag{32}$$

$$\lim_{a\to 0}\sup\left\{\left|\hat{P}_k(\Omega) - P_k(\Omega)\right| \,\middle|\, k\in\mathbb{N},\ \Omega\subset\mathcal{G}(a_0)\right\} = 0 \text{ for fixed } a_0, \tag{33}$$

which are given in Proposition 2.11. For this we need a more explicit characterisation of the distributions $\hat{P}_k(\cdot\,|\,\Phi_{k-1})$ and $\hat{P}_k(\cdot)$.

We recall the following fundamental independence-principle of Poisson-point processes which allows us to compute certain conditional probabilities explicitly.

Lemma 2.10 *Let the random set $\omega\subset\mathbb{T}^d\times\mathbb{R}^d$ be distributed according to a Poisson point-process with density μ, $\bar{\mathcal{C}},\mathcal{C}\subset\mathbb{T}^d\times\mathbb{R}^d$ and $A\subset\cup_{r=0}^{\infty}(\mathcal{C}\setminus\bar{\mathcal{C}})^r$ be symmetric. Then we obtain the following formula for the conditional probability of the event A:*

$$\mathrm{Prob}_{\mathrm{tPPP}}\left(\omega\cap\mathcal{C}\in A\,\middle|\,\omega\cap\bar{\mathcal{C}}=\emptyset\right) = \exp\left(-\mu(\mathcal{C}\setminus\bar{\mathcal{C}})\right)\sum_{r=0}^{\infty}\frac{1}{r!}\int_{A\cap\mathcal{C}^r}\mathrm{d}\mu^r(z), \tag{34}$$

where $\mu^r = \underbrace{\mu\otimes\ldots\otimes\mu}_{r\ \textit{terms}}$

To apply Lemma 2.10 we have to work with the phase space representation of trees. Owing to the decomposition $\Omega = \dot{\cup}_{m\in\mathcal{T}}\mathcal{E}(m)\cap\Omega$ we can assume that $\Omega\subset\mathcal{E}(m)$ for some $m\in\mathcal{T}$.

Note that for a general tree $\Phi=(m,\phi)\in\mathcal{MT}$ the number of nodes $\#m$ can be bigger than the number of particles involved in the collisions, i.e. it is possible that the map $z: m\to\mathbb{T}^d\times\mathbb{R}^d$ is not injective and $z_l = z_{l'}$ for some pair $l,l'\in m$, $l\neq l'$. This scenario corresponds to a bad tree where the same particle appears twice in the tree or one node has two parent nodes. For this reason we restrict our attention to sets Ω which are subsets of $\mathcal{G}(a)$. The excluded set has nonzero probability, however we will show that the probability of $\mathcal{MT}\setminus\mathcal{G}(a)$ tends with a to 0. By construction for all trees in Ω the map $l\mapsto z_l$ is injective.

The order defined by (17) induces a representation of the events $\Omega\subset\mathcal{MT}$ in phase-space coordinates:

$$A(\Omega)\subset(\mathbb{T}^d\times\mathbb{R}^d)^{\#m}.$$

In the same spirit one obtains a one-to-one correspondence between the the initial values of particles associated with the tree-nodes at height k and subsets of $(\mathbb{T}^d\times\mathbb{R}^d)^{\#m\cap\mathbb{N}^k}$:

$$Z_k = ((z_l)_{|l|=k}\in(\mathbb{T}^d\times\mathbb{R}^d)^{\#(m\cap\mathbb{N}^k)}.$$

We will also need the conditional events

$$A_k(\Omega,\Phi) = \left\{Z_k\in(\mathbb{T}^d\times\mathbb{R}^d)^{\#(m\cap\mathbb{N}^k)}\,|\,(Z_k,\Phi)\in\Omega\right\},$$

where $\Phi \in \mathcal{MT}_{k-1}$ and $(Z_k, \Phi) \in \mathcal{MT}_k$ is the tree obtained by attaching the leaves Z_k to the topmost nodes of Φ.

Recall that the density of the Poisson-point process which generates the initial positions of the particles is given by $N\mu$ where

$$\int \mathrm{d}\mu(z)\,\varphi(z) = \int_{\mathbb{R}^d} \mathrm{d}f_0(v) \int_{\mathbb{T}^d} \mathrm{d}u\,\varphi(u,v)$$

for every test-function $\varphi \in C_c(\mathbb{T}^d \times \mathbb{R}^d)$.

Before applying Lemma 2.10 we have to specify the sets $\mathcal{C}$ and $\bar{\mathcal{C}}$. Fix $a_0 > 0$ and let $\Phi \in \mathcal{MT} \cap \mathcal{G}(a_0)$. We are interested in the distribution of those trees which coincide with Φ up to level k. Clearly, the initial positions of the particles at height $k+1$ are contained in the set

$$\mathcal{C}_k(\Phi) := \bigcup_{l \in m_k \cap \mathbb{N}^k} C_l(\phi) \subset \mathbb{T}^d \times \mathbb{R}^d,$$

with $\Phi = (m, \phi)$. In order to apply formula (34) we have to identify the conditioning of the distribution $\omega \cap \mathcal{C}_k(\Phi)$. Define the collection of cylinders

$$\bar{\mathcal{C}}_k(\Phi) := \bigcup_{|l|<k} C_l(\phi) \subset \mathbb{T}^d \times \mathbb{R}^d$$

which contains those initial values that would affect the lower nodes. By construction the information on the point process ω that we have accumulated so far is given by $\omega \cap \bar{\mathcal{C}}_k(\Phi) = \{z_l \mid |l| \le k\}$. Furthermore, since $\Phi \in \mathcal{G}(a_0)$ we have that $\omega \cap \mathcal{C}_k(\Phi) \cap \bar{\mathcal{C}}_k(\Phi) = \emptyset$. This implies that for each $\Omega \subset \mathcal{MT} \cap \mathcal{G}(a_0)$ and $\Phi \in \mathcal{MT}_k \cap \mathcal{G}(a_0)$ that

$$\hat{P}_{k+1}(\Omega \,|\, \Phi) = \mathrm{Prob}_{\mathrm{tPPP}}(\mathcal{C}_k(\Phi) \cap \omega \in \mathrm{sym}(A_k(\Omega, \Phi)) \mid \mathcal{C}_k(\Phi) \cap \bar{\mathcal{C}}_k(\Phi) \cap \omega = \emptyset).$$

where $\mathrm{sym}(A)$ is the symmetrisation of the set A, i.e. $(z_1, \ldots, z_n) \in \mathrm{sym}(A)$ if there exists a permutation $\pi \in S_n$ such that $(z_{\pi(1)}, \ldots, z_{\pi(n)}) \in A$; in particular $A \subset \mathrm{sym}(A)$. This is the crucial step where the complicated dependency on the past of the many-body evolution is reduced to a simple conditional expectation of the Poisson point process. Since $A(\Omega, \Phi) \cap \underbrace{\bar{\mathcal{C}}_k(\Phi) \times \ldots \times \bar{\mathcal{C}}_k(\Phi)}_{r \text{ terms}} = \emptyset$ for each r we can use formula (34) and deduce that

$$\hat{P}_{k+1}(\Omega \,|\, \Phi) = e^{-\hat{\Gamma}_k(\Phi)} \frac{1}{r!} \int_{\mathrm{sym}(A_{k+1}(\Omega,\Phi))} \mathrm{d}\mu^r(Z_{k+1})$$

where

$$\hat{\Gamma}_k(\Phi) = \mu(\hat{\mathcal{C}}_k(\Phi)) \tag{35}$$

and $\hat{\mathcal{C}}(k) = \mathcal{C}_k(\Phi) \setminus \bar{\mathcal{C}}_k(\Phi)$. Recall the convention that the value of the integral over $(\mathbb{T}^d \times \mathbb{R}^d)^0$ is 1.

Since each permutation of the labels $l \in m$ destroys the tree structure we obtain that if $z_\pi \in A$ and $z \in A$, then necessarily π is the identity transformation, i.e. $z_\pi = z$. This implies that if we replace in the above formula $\mathrm{sym}(A)$ by the non-symmetric set A we have to drop the term $\frac{1}{r!}$.

$$\hat{P}_{k+1}(\Omega \mid \Phi) = e^{-\hat{\Gamma}_k(\Phi)} \int_{A_{k+1}(\Omega,\Phi)} \mathrm{d}\mu^r(Z_{k+1}). \tag{36}$$

Plugging the expression (36) for the conditional expectation $\hat{P}_{k+1}(\cdot \mid \Phi)$ into equation (26) yields that

$$\begin{aligned}\hat{P}_k(\Omega) = &\int_{\mathbb{T}^d\times\mathbb{R}^d} \mathrm{d}P_1(\phi_1(Z_1))\, e^{-\hat{\Gamma}_1(\Phi_1(Z_1))} \int_{(\mathbb{T}^d\times\mathbb{R}^d)^{r_2}} \mu^{r_2}(\Phi_2(Z_2))\\
&\qquad \dots e^{-\hat{\Gamma}_{k-1}(\Phi_{k-1}(Z_1\dots Z_{k-1}))} \int_{A_k(\Omega,\Phi_{k-1}(Z_1\dots Z_{k-1}))} \mathrm{d}\mu^{r_k}(Z_k)\\
= &\sum_{m\in\mathcal{T}_k} \int_{A(\Omega)} \mathrm{d}\mu^{\#m}(z)\, e^{-\sum_{j<k}\hat{\Gamma}_j(\Phi(z))}. \end{aligned}\tag{37}$$

The intermediate step in the computation above relies on the additional assumption that $m \in \mathcal{T}_k \setminus \mathcal{T}_{k-1}$. In general we have to be more careful concerning the domains of integration, but the the final formula is unaffected.

We return now to the collision representation of the trees. This means that the variables $(z_l)_{l\in m}$ are replaced by $(u_1, v_1)\times(s_l, \nu_l, v_l)_{l\in m\setminus\{1\}}$.The determinant of the derivative of this transformation is given by

$$\det D_\Phi z(\Phi) = \prod_{l\in m\setminus\{1\}} \left(a^{d-1}[\nu_l\cdot(v_l - v_{\bar{l}})]_+.\right)$$

Thus changing coordinates in the integrals we obtain that for each $m \in \mathcal{T}$

$$\begin{aligned}&\int_{A(\Omega)} e^{-\sum_{j<k}\hat{\Gamma}_j(\Phi(z))}\,\mathrm{d}\mu^{\#m}(z)\\
=&\int_\Omega \mathrm{d}P_1(z_1)\, e^{-\sum_{j<k}\hat{\Gamma}_j(\Phi)}\\
&\qquad\prod_{l\in m\setminus\{1\}} \left(N\,\mathrm{d}f_0(v_l)\,\mathrm{d}\nu_l\,\mathrm{d}s_l\,\chi_{[s_{l-1},s_{\bar{l}}]}(s_l)\,a^{d-1}\,[(v_l - v_{\bar{l}})\cdot\nu_l]_+\right)\\
\overset{(2)}{=}&\int_\Omega \mathrm{d}P_1(z_1)\, e^{-\sum_{j<k}\hat{\Gamma}_j(\Phi)} \prod_{l\in m\setminus\{1\}} \left(\mathrm{d}f_0(v_l)\,\mathrm{d}\nu_l\,\mathrm{d}s_l\,\chi_{[s_{l-1},s_{\bar{l}}]}(s_l)\,[(v_l - v_{\bar{l}})\cdot\nu_l)]_+\right)\\
=&\int_\Omega \mathrm{d}\lambda^m(\phi)\, e^{-\sum_{j<k}\hat{\Gamma}_j(\Phi)},\end{aligned}$$

Thus we have shown that for all $\Omega \subset \mathcal{G}(a)$

$$\hat{P}_k(\Omega) = \sum_{m\in\mathcal{T}_k} \int_{\Omega\cap\mathcal{E}(m)} e^{-\sum_{j<k}\hat{\Gamma}_j(\Phi)}\,\mathrm{d}\lambda^m(\phi). \tag{38}$$

and

$$P_k(\Omega) = \hat{P}_k(\Omega) + e_k(\Omega), \tag{39}$$

where the error has the form

$$e_k(\Omega) = \sum_{m \in \mathcal{T}_k} \int_{\Omega \cap \mathcal{E}(m)} \mathrm{d}\lambda^m(\phi) \left(e^{-\sum_{j<k} \Gamma_j(\Phi)} - e^{-\sum_{j<k} \hat{\Gamma}_j(\Phi)} \right). \tag{40}$$

Since $\hat{\Gamma}_j(\Phi) \leq \Gamma_j(\Phi)$ the difference $e_k(\cdot)$ is a non-negative measure.

Now we are in a good position to prove that equations (32) and (33) hold.

Proposition 2.11 (Similarity of $\hat{P}_k$ and P_k) *Let $\mathcal{G}(a)$ the set of good trees from Definition 2.9, and $\Omega \subset \mathcal{G}(a_0)$. Then equations (32) and (33) hold.*

The proof requires elementary but tedious estimates of sets $C_l \cap C_{l'}$ and can be found in [10].

Proof of Theorem 1.2

We first demonstrate that the distribution of a single tagged particle satisfies the Boltzmann equation. Let $A \subset \mathbb{T}^d \times \mathbb{R}^d$ and define $\Omega(A) \subset \mathcal{MT}$ by

$$\Omega(A) = \{\Phi \in \mathcal{MT} \mid \beta_1(m) = 1 \text{ and } z_1 \in A\}.$$

With this notation we obtain that for every $a_0 > 0$

$$\begin{aligned}
&\left| \lim_{a\to 0} \lim_{k\to\infty} \hat{P}_{t,k}(\Omega) - \int_A \mathrm{d}u \, \mathrm{d}f_t(v) \right| \\
&\overset{\text{Lem. 2.1}}{=} \lim_{a\to 0} \lim_{k\to\infty} \left| \hat{P}_{t,k}(\Omega) - \int_A \mathrm{d}u \, \mathrm{d}f_{t,k-1}(v) \right| \\
&\overset{\text{Prop. 2.7}}{=} \lim_{a\to 0} \lim_{k\to\infty} \left| \hat{P}_{t,k}(\Omega) - P_{t,k}(\Omega) \right| \\
&= \lim_{a\to 0} \lim_{k\to\infty} \Big| \hat{P}_{t,k}(\Omega \cap \mathcal{G}(a_0)) - P_{t,k}(\Omega \cap \mathcal{G}(a_0)) \\
&\qquad - P_{t,k}(\Omega \setminus \mathcal{G}(a_0)) + \hat{P}_{t,k}(\Omega \setminus \mathcal{G}(a_0)) \Big| \\
&\overset{(33)}{\leq} \lim_{a\to 0} \lim_{k\to\infty} P_{t,k}(\mathcal{MT} \setminus \mathcal{G}(a_0)) + \lim_{a\to 0} \lim_{k\to\infty} \hat{P}_{t,k}(\mathcal{MT} \setminus \mathcal{G}(a_0))
\end{aligned}$$

Now using equation (33) again for $\tilde{\Omega} := \mathcal{MT} \cap \mathcal{G}(a_0)$ and that $\hat{P}_{t,k}$ and $P_{t,k}$ are probability measures, we also obtain that $\lim_{a\to 0} \hat{P}_{t,k}(\mathcal{MT} \setminus \mathcal{G}(a_0)) = P_{t,k}(\mathcal{MT} \setminus \mathcal{G}(a_0))$. Now proceeding

$$\leq 2 \lim_{k\to\infty} P_{t,k}(\mathcal{MT} \setminus \mathcal{G}(a_0)),$$

we send now a_0 to 0, apply (32) and obtain that $\lim_{a_0\to 0} \lim_{k\to\infty} P_{t,k}(\mathcal{MT} \setminus \mathcal{G}(a_0)) = 0$, hence $\lim_{a\to 0} \lim_{k\to\infty} \hat{P}_{t,k}(\Omega) = \int_A \mathrm{d}u \, \mathrm{d}f_t(v)$.

The proof of Theorem 1.2 is complete. □

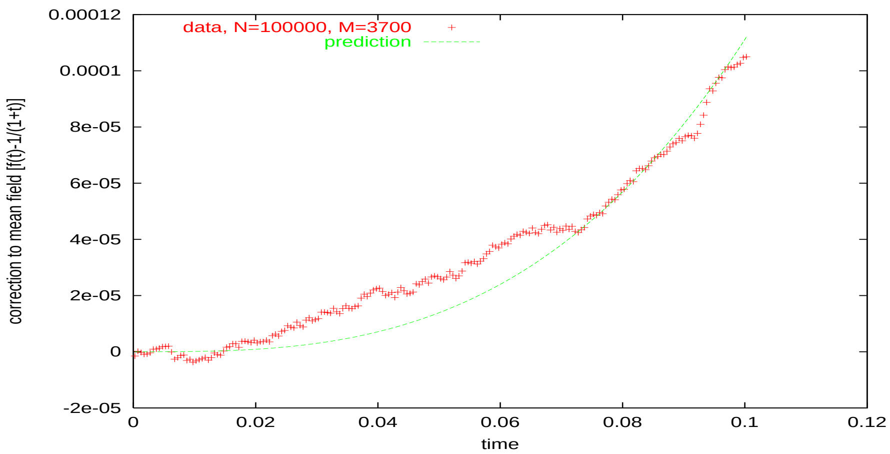

Figure 3: Comparison between the empirical probability of colliding and the mean-field prediction. The dashed line is the cubic parabola $t \mapsto \frac{1}{9}t^3$, the signs '+' mark the difference between the number of non-collided particles at time t divided by N and the mean-field prediction $\frac{1}{1+t}$.

3 Concentrations and non-validity

We illustrate now that the mean field theory does not capture the many-particle dynamics if the initial distribution f_0 exhibits strong concentrations. To simplify the long calculations in the proof we assume that $d = 2$, but similar results are expected to hold in the case $d = 3$.

Theorem 3.1 *Let $v \in \mathbb{R}^2$ be nonresonant ($\alpha \cdot v \notin \mathbb{Z}$ for all $\alpha \in \mathbb{Z}^d$) such that $|v| = 1$ and set $f_0 = \frac{1}{2}(\delta(\cdot - v) + \delta(\cdot + v))$. If $\hat{Q}(t) = \lim_{a \to 0} \lim_{k \to \infty} \hat{P}_{t,k}(\beta_1 = 1)$ denotes the empirical probability that a tagged particle does not collide, then*

$$\lim_{t \to 0} \frac{1}{t^3} \left(\hat{Q}(t) - \int_{\mathbb{R}^2} \mathrm{d}f_t(v) \right) = \frac{1}{9}, \tag{41}$$

where $f_t = \frac{1}{1+t} f_0$ is the unique solution of the Boltzmann equation (9) which satisfies the initial condition $f_{t=0} = f_0$.

A numerical simulation (fig. 3) confirms the prediction (41). The proof is based on a simple but lengthy calculation and can be found in [10].

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