On level sets in the Heisenberg group

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Warwick, July 11, 2017

*Joint work with V. Magnani (UNIPI) and E. Stepanov (S.Pb UNIV. & STEKLOV)
The implicit function theorem on Lie groups

Implicit function theorem (Dini)

Regular level sets of a $C^1$ map between Euclidean spaces have a local $C^1$ parametrization.

Problem

What happens if we replace Euclidean spaces with more general Lie groups?

We study the simplest non-trivial case: $F : H \approx \mathbb{R}^3 \to \mathbb{R}^k$. 
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What happens if we replace Euclidean spaces with more general Lie groups?

We study the simplest non-trivial case: $F : \mathbb{H} \cong \mathbb{R}^3 \to \mathbb{R}^k$. 
The group $\mathbb{H}$ is a non-commutative Lie group, with two generators.

- On $\mathbb{H} = \mathbb{R}^3$, $x = (x^1, x^2, x^3)$, consider the two (horizontal) vector fields
  \[
  X_1(x) := \partial_1 - x^2 \partial_3 \quad X_2(x) := \partial_2 + x^1 \partial_3
  \]
  \[
  [X_1, X_2] = [\partial_1 - x^2 \partial_3, \partial_2 + x^1 \partial_3] = 2\partial_3 \quad \text{(Hörmander condition)}.
  \]
- Dual description: contact 1-form
  \[
  \theta = dx^3 + x^2 dx^1 - x^1 dx^2 \quad \Rightarrow \quad d\theta = -2dx^1 \wedge dx^2
  \]
- Horizontal tangent at $x \in \mathbb{H}$ is span $\{X_1(x), X_2(x)\} = \text{Ker} \, \theta_x$. 
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Heisenberg group: differential structure
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A (smooth) curve $\eta : I \to \mathbb{H}$ is horizontal if, for $t \in I$,
\[ \theta_{\eta_t}(\dot{\eta}_t) = \dot{\eta}_t^3 + \eta_t^2 \dot{\eta}_t^1 - \eta_t^1 \dot{\eta}_t^2 = 0. \]

Imposing $X_1(x)$ $X_2(x)$ are orthonormal $\Rightarrow$ CC-distance
\[ d(x, y) := \inf \left\{ \int_0^1 |\dot{\eta}_t| : \eta \text{ horizontal, } \eta_0 = x, \eta_1 = y \right\}. \]

Equivalence
\[ d(x, y) \approx |y^1 - x^1| + |y^2 - x^2| + |\vartheta_{xy}|^{1/2}, \]
where a “discrete” contact form appears
\[ \vartheta_{xy} := (y^3 - x^3) + x^2(y^1 - x^1) - x^1(y^2 - x^2). \]
(Recall $\theta = dx^3 + x^2 \, dx^1 - x^1 \, dx^2$.)
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Heisenberg group: regular maps

- We “measure” regularity of $F : \mathbb{H} \to \mathbb{R}^k$ in terms of horizontal derivatives

  \[ \nabla_h F(x) := (X_1 F(x), X_2 F(x)) \, . \]

  $p \in \mathbb{H}$ is non degenerate for $F$ if $\nabla_h F(p)$ has maximum rank.

- For $\alpha \in (0, 1)$, $F \in C^{1,\alpha}_h$ if $x \mapsto \nabla_h F(x)$ is (well-defined and) $\alpha$-Hölder continuous, (w.r.t. $d$). ($F \in C^1_h$ if just continuous).

- Fact: There are $F \in C^{1,\alpha}_h$ nowhere (Euclidean) differentiable on a set of positive Lebesgue measure.

Problem

Locally parametrize $F^{-1}(F(p))$ for $F \in C^1_h$ for non degenerate $p$'s.
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Literature on level sets in Heisenberg group, $F : H \to \mathbb{R}^k$

$k = 1$: algebraic splitting phenomenon (Ambrosio-SerraCassano-Vittone, . . .) ⇒ “intrinsic graphs”, parametrized surfaces via group operation.
(Interesting connection with non-linear PDE’s, recall talk by Katrin Fässler).


$k = 2$: Kozhevnikov (2011) ⇒ $\beta$-Hölder continuous curves ($\beta < 1/2$) via a sub-Riemannian Reifenberg-type argument.

For $k = 2$, parametrizations are quite implicit: is a “good calculus” missing?

Main results (Magnani-Stepanov-T., 2016): $k = 2$.

- Explicit “Level Set Differential Equation” (LSDE).
- Prove existence, uniqueness, and stability w.r.t. approximations for $F \in C^{1,\alpha}_h$ ($\alpha > 0$) using tools from Young integration (Rough paths).
- Prove area formula and (re)-obtain a coarea formula for $F \in C^{1,\alpha}_h$. 
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The Euclidean ODE argument

Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be $C^1$.

- Write $x = (x^1, x^2, x^3) \in \mathbb{R}^3$, $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, 3$,
- $F = (F^1, F^2)$ and

$$\nabla F = \begin{pmatrix} \partial_1 F^1 & \partial_2 F^1 & \partial_3 F^1 \\ \partial_1 F^2 & \partial_2 F^2 & \partial_3 F^2 \end{pmatrix} = (\nabla_{12} F, \nabla_3 F) \quad \text{with } \nabla_{12} F \text{ invertible.}$$

Differentiating $F(\gamma_t) = c$,

$$\begin{pmatrix} \dot{\gamma}_t^1 \\ \dot{\gamma}_t^2 \\ \dot{\gamma}_t^3 \end{pmatrix} = - (\nabla_{12} F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \dot{\gamma}_t^3.$$  

Impose $\dot{\gamma}_t^3 = 1$ and solve (Peano) for $(\gamma_t^1, \gamma_t^2)$. (Uniqueness of solutions?)
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Impose $\dot{\gamma}^3_t = 1$ and solve (Peano) for $(\gamma^1_t, \gamma^2_t)$. (Uniqueness of solutions?)
In $\mathbb{H} \sim \mathbb{R}^3$, recast the ODE

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in terms of the horizontal derivatives $X_1 F, X_2 F$ (change of coordinates):

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In analogy with $\dot{\gamma}_t^3 = 1$, set $\theta_{\gamma_t}(\dot{\gamma}_t) = 1 \Rightarrow$ non-horizontal, (vertical), curve.

Two difficulties:

1. The “vertical derivative” $\nabla_3 F$ may not be defined, even if $F \in C^{1,\alpha}$ with $0 < \alpha < 1$.

2. The intrinsic distance is $1/2$-Hölder along “vertical” directions $\Rightarrow \gamma$ is truly Hölder $\Rightarrow \dot{\gamma}_t$ is not defined.
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Here’s a “rule of thumb” to define

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\end{pmatrix} = - \left( \nabla_h F(\gamma_t) \right)^{-1} \nabla_3 F(\gamma_t) \theta_\gamma(\dot{\gamma}_t), \quad \theta_\gamma(\dot{\gamma}_t) = 1.
\]

1. “Integrating” \( \theta_\gamma(\dot{\gamma}_t) = 1 \) gives \( \vartheta_\gamma s \gamma_t = t - s \),

\[d(\gamma_s, \gamma_t) \approx |t - s|^{1/2} \Rightarrow \gamma \text{ is (intrinsically) } 1/2\text{-Hölder.}
\]

2. Writing \( \partial_3 F = [X_1, X_2]F = X_1(X_2 F) - X_2(X_1 F) \) gives

\( \partial_3 F \) is \((\alpha - 1)\)-Hölder.

3. The composition (!) \( \partial_3 F(\gamma_t) \) is then \( \frac{1}{2} \cdot (\alpha - 1)\)-Hölder.

4. Integration w.r.t. \( t \) increases regularity of “one degree” \( \Rightarrow \)

\( (\gamma^1, \gamma^2) \) is \( \left[ \frac{1}{2} \cdot (\alpha - 1) + 1 \right] = \frac{1 + \alpha}{2} \)-Hölder.

5. \( \frac{1 + \alpha}{2} \)-Hölder continuity is consistent with assumption 1, closing the circle.
Here’s a “rule of thumb” to define

\[
\left( \begin{array}{c}
\dot{\gamma}_t^1 \\
\dot{\gamma}_t^2 \\
\end{array} \right) = - (\nabla_h F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \theta_{\gamma_t}(\dot{\gamma}_t), \quad \theta_{\gamma_t}(\dot{\gamma}_t) = 1.
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1. “Integrating” \( \theta_{\gamma_t}(\dot{\gamma}_t) = 1 \) gives \( \vartheta_{\gamma_s \gamma_t} = t - s \),

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Heuristics

Here’s a “rule of thumb” to define

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The “vertical” equation

We adopt the point of view of “local descriptions” by finite increments (Gubinelli’s approach to Rough paths), and use “physicists notation”

\[ \delta \gamma^i_{st} = \gamma^i_t - \gamma^i_s, \quad \text{for } s, t \in I, i \in \{1, 2, 3\}. \]

The equation

\[ \theta_{\gamma_t}(\dot{\gamma}_t) = \dot{\gamma}_t^3 + \gamma_t^2 \dot{\gamma}_t^1 - \gamma_t^1 \dot{\gamma}_t^2 = 1 \]

becomes our vertical equation

\[ \theta_{\gamma_s \gamma_t} = \delta \gamma^3_{st} + \gamma_s^2 \delta \gamma^1_{st} - \gamma_s^2 \delta \gamma^2_{st} = t - s + o(t - s). \]
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Instead of “differentiating”, we use finite differences $\Rightarrow$ horizontal Taylor expansion:

$$F(y) - F(x) - \nabla_h F(x) \left( \begin{array}{c} y^1 - x^1 \\ y^2 - x^2 \end{array} \right) - \nabla_3 F(x) \theta_{xy} = R_{xy}. $$

Imposing $F(\gamma_s) = F(\gamma_t)$ gives

$$\left( \delta \gamma_{st}^1, \delta \gamma_{st}^2 \right) = - \left( \nabla_h F(\gamma_s) \right)^{-1} R_{\gamma_s \gamma_t} + o(t - s).$$

To avoid multiplication, a better formulation is

$$\left( \delta \gamma_{st}^1, \delta \gamma_{st}^2 \right) = - \left( \nabla_h F(p) \right)^{-1} (R_{p \gamma_t} - R_{p \gamma_s}).$$
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Definition (LSDE)
Let $p$ be non degenerate for $F : \mathbb{H} \to \mathbb{R}^2$, $F \in C^1_h$. We say $\gamma : I \to \mathbb{H}$ is a solution to the level set differential equation (LSDE) if it is continuous and

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for every $s, t \in I$.

- The “horizontal equation” yields that $t \mapsto F(\gamma_t)$ is constant.
- The “vertical equation” gives that

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Theorem (Existence)

Let $\alpha > 0$ and $p$ be non degenerate for $F : \mathbb{H} \to \mathbb{R}^2$, $F \in C_h^{1, \alpha}$. Then, there exists $\delta > 0$ and $\gamma : [-\delta, \delta] \to \mathbb{H}$ solving the LSDE with $\gamma_0 = p$.

- Proof via Leray-Schauder fixed point on a subset of $C^{\frac{1+\alpha}{2}}([-\delta, \delta]; \mathbb{R}^3)$.
- Need of $\alpha > 0$: use Young integral (Sewing lemma) to move from

$$\dot{\gamma}_{s:t} = t - s + o(t - s)$$

to

$$\delta \gamma_{s:t}^3 = -\int_s^t \gamma_r^2 \, d\gamma_r^1 + \int_s^t \gamma_r^1 \, d\gamma_r^2 + (t - s).$$
Existence of solutions

Theorem (Existence)

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\[ \partial_{t\gamma} = t - s + o(t - s) \]

\[ \delta \gamma^{3}_{st} = - \int_{s}^{t} \gamma^{2}_{r} \, d\gamma^{1}_{r} + \int_{s}^{t} \gamma^{1}_{r} \, d\gamma^{2}_{r} + (t - s). \]
“Horizontal equation” $\Rightarrow$ solutions to the LSDE satisfy $t \mapsto F(\gamma_t)$ constant.

**Theorem (surjectivity)**

Let $p$ be non degenerate for $F : \mathbb{H} \to \mathbb{R}^2$, $F \in C^1_h$. Let $\gamma : I \to \mathbb{H}$ solve the LSDE with $\gamma_0 = p$. Then, there exists $\varepsilon > 0$ such that

$$F^{-1}(F(p)) \cap B_\varepsilon(p) = \gamma(I) \cap B_\varepsilon(p).$$

No need of $C^{1,\alpha}_h$ (but we do not know how to get existence . . .)

**Proof** is a combination of two lemmas:

- “Horizontal injectivity” (due to non degeneracy of $p$) $\Rightarrow$ we attach a region of injectivity (for the level set) at every $\gamma_t$;
- As $t$ varies, such regions at $\gamma_t$ cover a neighbourhood of $p$. 

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Surjectivity of solutions (on the level set)

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Lemma (Local uniqueness)

Any two solutions $\gamma$, $\tilde{\gamma}$ to the LSDE with $\gamma_0 = \tilde{\gamma}_0 = p$ coincide on a neighbourhood of $t = 0$.

Proof: Since both $\gamma$, $\tilde{\gamma}$, parametrize $F^{-1}(F(p))$, one has

$$\gamma_t = \tilde{\gamma}_\varphi(t).$$

The “vertical equation” gives

$$t - s + o(t - s) = \vartheta_{\gamma_s \gamma_t} = \vartheta_{\tilde{\gamma}_\varphi(s) \tilde{\gamma}_\varphi(t)} = \varphi(t) + \varphi(s) + o(\varphi(t) - \varphi(s)).$$

Divide by $t - s$ and let $s \to t \Rightarrow$

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$$\gamma t = \bar{\gamma} \varphi(t).$$

The “vertical equation” gives

$$t - s + o(t - s) = \vartheta_{\gamma s \gamma t} = \vartheta_{\bar{\gamma} \varphi(s) \bar{\gamma} \varphi(t)} = \varphi(t) + \varphi(s) + o(\varphi(t) - \varphi(s)).$$

Divide by $t - s$ and let $s \to t \Rightarrow$

$$\frac{d\varphi}{dt} = 1 \quad \Rightarrow \quad \varphi(t) = t.$$
Lemma (Local uniqueness)

Any two solutions $\gamma, \bar{\gamma}$ to the LSDE with $\gamma_0 = \bar{\gamma}_0 = p$ coincide on a neighbourhood of $t = 0$.

Proof: Since both $\gamma, \bar{\gamma}$, parametrize $F^{-1}(F(p))$, one has

$$\gamma(t) = \bar{\gamma}_\varphi(t).$$

The “vertical equation” gives

$$t - s + o(t - s) = \vartheta_{\gamma_s \gamma t} = \vartheta_{\bar{\gamma}_\varphi(s) \bar{\gamma}_\varphi(t)} = \varphi(t) + \varphi(s) + o(\varphi(t) - \varphi(s)).$$

Divide by $t - s$ and let $s \to t \Rightarrow$

$$\frac{d\varphi}{dt} = 1 \Rightarrow \varphi(t) = t.$$
Further results and open problems

**Theorem (Area formula)**

Let $\gamma : I \rightarrow \mathbb{H}$ solve the LSDE. Then, for every interval $[a, b] \subseteq I$,

$$S^2(\gamma([a, b])) = \mathcal{L}^1([a, b]).$$

Actually we prove a more general Area formula for nice “vertical curves”.

**Theorem (Coarea formula)**

Let $F : \mathbb{H} \rightarrow \mathbb{R}^2$, $F \in C^{1,\alpha}_h$. Then for $A \subseteq \mathbb{H}$,

$$\int_A J_h F \, d\mathcal{L}^3 = \int_{\mathbb{R}^2} S^2 \left( A \cap F^{-1}(z) \right) d\mathcal{L}^2(z).$$

Proof uses area formula and blow-up argument. (Case $\alpha = 0$ is open).
### Open problems

- Relax $\alpha > 0$ condition:
  - compactness as $\alpha \to 0$,
  - generic level set,
  - other notions of integrals?

- Use parametrized “vertical” curves to build (counter-)examples.

- Study surfaces:
  - on splitting case – no need of Young integrals
  - Frobenius theorems in Hölder regularity (Euclidean setting, Banach fixed point)
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