How small are σ porous sets

and why are we interested in it

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The Problem Behind this Talk

Problem
Is is true that any three real-valued Lipschitz functions on an (infinite dimensional separable) Hilbert space have a common point of Fréchet differentiability?

Conjecture
Any finite (or even countable) collection of real-valued Lipschitz functions on a Banach space with separable dual has a common point of Fréchet differentiability.

Basic notions

Throughout this talk, functions will be real-valued, otherwise they will be called maps. X will be a separable Banach space.

A function $f\colon X \to \mathbb{R}$ is Lipschitz if there is $C < \infty$ so that

$$|f(x) - f(y)| \leq C\|x - y\| \text{ all } x, y \in X$$

Definition
A function $f\colon X \to \mathbb{R}$ is said to be Fréchet differentiable at a point x_0 if there is $x^* \in X^*$ so that

$$f(x_0 + u) = f(x_0) + x^*(u) + o(\|u\|),\ u \to 0.$$

Definition
The directional derivative of f at x_0 in direction $u \in X$ is

$$f'(x_0; u) := \lim_{t\to 0} \frac{f(x_0 + tu) - f(x_0)}{t}$$

provided that the limit exists.

More on differentiability

Recall

$$f'(x_0; u) := \lim_{t\to 0} \frac{f(x_0 + tu) - f(x_0)}{t} \qquad (\star)$$

Observation
The function f is Fréchet differentiable at x_0 if and only if

(1) *its directional derivative $f'(x_0; u)$ exists for every u and*
(2) *forms a bounded linear operator as a function of u and*
(3) *the limit in $(\star)$ is uniform for $\|u\| \leq 1$.*

Definition
If (1) holds, we say that f is directionally differentiable at x_0, if (1) and (2) hold, we say that f is Gâteaux differentiable at x_0.

Note. If f is Lipschitz and $\dim(X) < \infty$, (1) $\iff$ (3).

Note. In our problems, condition (2) is rather easy to achieve.

Existence of Gâteaux derivative

By Lebesgue, any Lipschitz $f : \mathbb{R} \to \mathbb{R}$ is differentiable a.e.

So for any Lipschitz $f : X \to \mathbb{R}$ and any $u \in X$, the set of points where f is non-differentiable in direction u belongs to the family $\mathcal{A}(u)$ of Borel sets that are null on every line parallel to u.

It follows that every Lipschitz $f : X \to \mathbb{R}$ is directionally (and also Gâteaux) differentiable except points of a set from the σ-ideal $\mathcal{A}$ generated by sets from $\mathcal{A}(u)$, $u \in X$.

One can prove that $\mathcal{A}$ is non-trivial, ie, $X \notin \mathcal{A}$.

Strengthenings (or weakenings) of these observations led (in the beginning of 70's) a number of authors to the following result, with various notions of "almost everywhere."

Theorem (Mankiewicz; Christensen; Aronszajn; Phelps)
Every real-valued Lipschitz function on a separable Banach space is Gâteaux differentiable almost everywhere.

Current strongest results on Gâteaux differentiability

Definition
For $u \in X$ and $\eta > 0$ denote by $\widetilde{\mathcal{A}}(u, \eta)$ the class of Borel sets $E \subset X$ having the property that $|\gamma^{-1}(E)| = 0$ for every Lipschitz curve γ with $\|\gamma'(t) - u\| \le \eta$.

$\widetilde{\mathcal{A}}$ is the σ-ideal generated by sets from $\widetilde{\mathcal{A}}(u, \eta)$, $u \in X$, $\eta > 0$.

Theorem (Zajíček, Preiss 2001)
Every real-valued Lipschitz function on a separable Banach space is Gâteaux differentiable $\widetilde{\mathcal{A}}$-almost everywhere.

Formally stronger results hold when $\widetilde{\mathcal{A}}$ is defined by: for every $u_k \in X$ with dense linear span, $E = \bigcup_k E_k$ where $E_k \in \widetilde{\mathcal{A}}(u_k, \eta_k)$ where $\eta_k > 0$;

and/or when $\widetilde{\mathcal{A}}(u, \eta)$ is replaced by the class $\widetilde{\mathcal{A}}_r(u, \eta)$ of Borel sets $E \subset X$ having the property that for every $\varepsilon > 0$ there is an open set $G \supset E$ such that $|\gamma^{-1}(G)| < \varepsilon$ for every Lipschitz curve γ with $\|\gamma'(t) - u\| \le \eta$.

Stronger assumptions, weaker differentiability

Definition
The modulus of asymptotic uniform smoothness of a space X is defined by

$$\bar{\rho}_X(t) := \sup_{\|x\|=1} \inf_{\dim X/Y<\infty} \sup_{\substack{y\in Y\\ \|y\|\le t}} \big(\|x+y\| - 1\big).$$

Example. For $X = \ell_p$, $\bar{\rho}_X(t) \asymp t^p$ as $t \searrow 0$.

Definition
A function $f : X \to \mathbb{R}$ is called ε-Fréchet differentiable if for every $\varepsilon > 0$ there are $x \in X$, $x^* \in X^*$ and $\delta > 0$ so that

$$\|f(x+u) - f(x) - x^*(u)\| \le \varepsilon\|u\|$$

if $\|u\| \le \delta$.

Current status of the conjecture

Theorem (Preiss 1990)
Every Lipschitz function on a Banach space with separable dual is Fréchet differentiable at least at one point.

Theorem (Lindenstrauss, Preiss 1996; Johnson, Lindenstrauss, Schechtman, Preiss 2002)
Every Lipschitz function on a separable Banach space that admits a norm with modulus of (asymptotic) smoothness $o(t)$ is ε-Fréchet differentiable.

Theorem (Lindenstrauss, Preiss 2003)
On some separable Banach spaces (such as c_0) every Lipschitz function is differentiable Γ-almost everywhere.

Theorem (Lindenstrauss, Tišer, Preiss 2008)
Every pair of Lipschitz functions on a separable Banach space that admits a norm with modulus of asymptotic smoothness $o\big(t^2 \log(1/t)\big)$ has a common point of Fréchet differentiability.

Definition of Γ_n-null sets

Denote by $\Gamma_n(X)$ the space of continuously differentiable mappings from $[0,1]^n$ to X.

Definition

A Borel set $E \subset X$ is Γ_n-null if

$$\{\gamma \in \Gamma_n(X) : |\gamma^{-1}(E)| > 0\}$$

is a first category subset of $\Gamma_n(X)$.

Notice that the Baire category theorem shows that Γ_n-null sets form a nontrivial σ-ideal of Borel subsets of X.

The definition makes sense also for $n = \infty$, in which case we leave out the index ∞. In this case, $\Gamma(X)$ is not a Banach space but only a Fréchet space.

Non-differentiability in $\mathbb{R}$

Observation

For every Lebesgue null set $E \subset \mathbb{R}$ there is a Lipschitz $f : \mathbb{R} \to \mathbb{R}$ which is non-differentiable at any point of E.

Proof.

Recursively find open $\mathbb{R} = G_0 \supset G_1 \supset G_2 \supset \cdots \supset E$ so small that G_{k+1} is small in every component of G_k. (For example, $|G_{k+1} \cap C| < 2^{-k-1}|C|$ for any component C of G_k.)

Let $\psi(x) = (-1)^k$ where k is the least index such that $x \in G_k$. Define $f(x) = \int \psi(x)\,dx$.

If $x \in E$ and (a,b) is a component of G_k containing x, then

$$\left| \frac{f(b) - f(a)}{b - a} - (-1)^k \right| \le 2^{-k},$$

so f is not differentiable at x. □

Porosity and directional porosity

Definition

A set E in X is said to be c-porous at $x \in E$, $0 < c < 1$, if for every $\varepsilon > 0$ there is a $z \in X \setminus E$ such that $\|x - z\| < \varepsilon$ and $B(z, c\|x - z\|) \cap E = \emptyset$.

Definition

A set E in X is said to be directionally c-porous at $x \in E$, $0 < c < 1$, if there is a line L passing through x such that for every $\varepsilon > 0$ there is a $z \in (X \setminus E) \cap L$ such that $\|x - z\| < \varepsilon$ and $B(z, c\|x - z\|) \cap E = \emptyset$.

Definition of σ-(directionally) porous sets

Definition

- A set which is (directionally) c-porous at x for some $0 < c < 1$ is called (directionally) porous at x.
- A set is (directionally) porous if it is (directionally) porous at each of its points.
- A countable union of (directionally) porous sets is called σ-(directionally) porous.

Observation

E is (directionally) porous if and only if the function $x \to \mathrm{dist}(x, E)$ is not Fréchet (directionally, Gâteaux) differentiable at any point of E.

While in finite dimensional spaces every σ-porous set is σ-directionally porous, this is false in all infinite dimensional spaces.

Big σ-porous sets in infinite dimensional spaces

Example (Tišer, Preiss 1995)

Every infinite dimensional separable Banach space is the union of a σ-porous set and of a set that is null on every line.

The set from this example cannot be σ-directionally porous:
If it were, it would belong to $\mathcal{A}$, since a directionally porous set is null on all lines in the porosity direction.

To construct the example, recursively find points $x_1, x_2, \dots \in X$ that are 500-dense in X and x_{k+1} has distance 100 from the linear span of $x_1, \dots, x_k$ and define E as the complement of

$$\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} B(r_j x_k, r_j),$$

where $r_j \searrow 0$ sufficiently fast.

In uniformly convex spaces, a considerable strengthening of this example is due to E. Kopecká.

Non-differentiability in infinite dimensions

At the present time, we know three differently behaved classes of sets in infinite dimensional spaces for which we can construct non-differentiable functions.

(a) Preimages of Lebesgue null sets from $\mathbb{R}$ under linear projections (or under non-linear projections satisfying rather obvious requirements).

(b) σ-directionally porous sets.

(c) σ-porous, but not σ-directionally porous sets.

Notice that (a), (b) lead to Gâteaux non-differentiability while (c) leads to Fréchet non-differentiability.
By recent results of Alberti, Csörnyei and Preiss, we can replace $\mathbb{R}$ by $\mathbb{R}^2$ in (a). Or we can use any $\mathbb{R}^n$ and replace Lebesgue null by their description of sets of non-differentiability of Lipschitz function on $\mathbb{R}^n$.

The mysterious role of porous sets

Theorem (Lindenstrauss, Preiss 2003)

Every real-valued Lipschitz function on a Banach space X with separable dual is Fréchet differentiable Γ-almost everywhere provided that every porous set in X is Γ-null.

Notice that already when $X = \mathbb{R}$ there are many more non-differentiability sets than σ-porous sets: any null set can be the former, but the latter are necessarily of the first category.

Notice also that there are infinite dimensional spaces satisfying the conditions of this Theorem. (For example c_0. But the Hilbert space does not.)

Remark

In fact, for every Lipschitz $f : X \to \mathbb{R}$, where X has separable dual, there is a σ-porous set of "irregular" points and, if this set is Γ-null then f is Fréchet differentiable Γ-almost everywhere.

The "mean value" estimates

When a Fréchet differentiability result holds for a Lipschitz function $f : X \to \mathbb{R}$, we may expect that the corresponding monotonicity (or mean value) estimate holds:

(♣) If $u \in X$ is such that $f'(x; u) \le 0$ for every x at which f is Fréchet differentiable, then f decreases in direction u.

For functions on spaces with separable dual this is actually true.
Similarly, assuming the Conjecture is true, we may expect that the mean value estimate (which is now a corollary of the divergence theorem) holds also for several functions. This is somewhat technical to state, but we may simplify our life by observing that (♣) holds for Gâteaux differentiability and so state it as:

(♠) For every $y \in X$ at which f is Gâteaux differentiable, $u \in X$ and $\varepsilon > 0$ there is $x \in X$ at which f is Fréchet differentiable and $f'(x; u) > f'(y; u) - \varepsilon$.

We will employ a similar simplification for several functions.

The “mean value” variant of the problem

Problem

Given a Banach space with separable dual and an integer n, when is it true that for any $f_1, \dots, f_n : X \to \mathbb{R}$, $u_1, \dots, u_n \in X$ and $\varepsilon > 0$ and for every $y \in X$ at which all f_i are Gâteaux differentiable there is $x \in X$ at which all f_i are Fréchet differentiable and

$$\sum_{i=1}^n f_i'(x; u_i) > \sum_{i=1}^n f_i'(y; u_i) - \varepsilon?$$

First answer (Tišer, Preiss 1995)

If $X = \ell_p$ and $n > p$, the above fails even with ε-Fréchet derivative.
In particular, it fails when X is an infinitely dimensional Hilbert space and $n \geq 3$.

“Mean value” problem and porous sets

Recent work of Lindenstrauss, Preiss and Tišer gives a reasonably satisfactory answer to the mean value version of the problem for ε-Fréchet derivatives. One direction is a considerably refined version of the example mentioned on the previous slide. The key to the other direction is

Theorem

Suppose that the Banach space X with separable dual has the property that every c-porous set in X can be covered by a union of a σ-directionally porous set and a Γ_n-null G_δ set.
Then for any $f_1, \dots, f_n : X \to \mathbb{R}$, $u_1, \dots, u_n \in X$ and $\varepsilon > 0$ and for any $y \in X$ at which all f_i are Gâteaux differentiable there is $x \in X$ at which all f_i are Gâteaux differentiable, ε-Fréchet differentiable and

$$\sum_{i=1}^n f_i'(x; u_i) > \sum_{i=1}^n f_i'(y; u_i) - \varepsilon$$

Porous sets and Γ_n-null sets

Theorem

Let X be a separable Banach space with

$$\bar{\rho}_X(t) = o(t^n \log^{n-1}(1/t)) \text{ as } t \to 0.$$

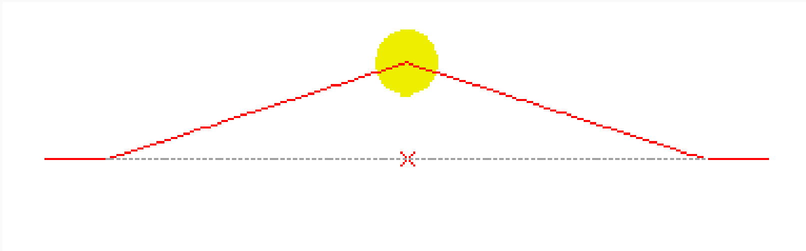
Then every porous set in X is contained in a union of a σ-directionally porous set and a Γ_n-null G_δ set.

Combined with the previous theorem, we get an answer to the “mean value” version of the problem for ε-Fréchet derivatives.
Funnily enough, directionally porous sets cause us more serious troubles than the porous sets, except when $n \leq 2$.

Theorem

Every σ-directionally porous subset of any Banach space is Γ_1-null as well as Γ_2-null.

The main argument



Run a curve through the hole and notice that, at least in the Hilbert space case, the preimage of the hole has much bigger measure than the prolongation of the curve.
Repeating the process should lead to a curve with length almost equal to the length of the starting curve but almost never passing through the porous set.
In reality, we try to minimise a suitable perturbation of $\gamma \to |\gamma^{-1}(\overline{E})|$; the perturbation being related to length.

Two norms on $\Gamma_n(X)$

We consider $\Gamma_n(X)$ with two metrics: d induced by the C^1-norm and d_0 induced by the maximum norm.
First important point is that the function $f(\gamma) := |\gamma^{-1}(\overline{E})|$ that we wish to "minimise" is Baire-1 in the metric d_0.
Second important point is that, arguing by contradiction, we manage to find somewhere d-dense, d_0-G_δ subset M of $\Gamma_n(X)$ on which $|\gamma^{-1}(E)| > c > 0$.
We wish to find a minimum of perturbed f on M. Since d_0 continuous perturbations are OK, this would be trivial were M d_0 complete. It is not, but the following weaker notion suffices.

Definition. A space (M, d, d_0), where d_0 is a pseudometric continuous with respect to the metric d, is (d, d_0)-complete if there are functions $\delta_j(x_0, \ldots, x_j) : M^{j+1} \to (0, \infty)$ such that every d-Cauchy sequence $(x_j)_{j=0}^\infty$ converges provided that

$$d_0(x_j, x_{j+1}) \le \delta_j(x_0, \ldots, x_j) \text{ for each } j = 0, 1, \ldots.$$

Sufficient lower semicontinuity of the measure function

The function f to be "minimised", being Baire-1 in the metric d_0 (in fact it is upper semicontinuous), has the following property.

Definition

Suppose that (M, d) is a metric space and d_0 a continuous pseudometric on M. We say that a function $f : M \to \mathbb{R}$ is (d, d_0)-lower semi-continuous if there are functions $\delta_j(x_0, \ldots, x_j) : M^{j+1} \to (0, \infty)$ such that

$$f(x) \le \liminf_{j\to\infty} f(x_j)$$

whenever $x_j \in M$ d-converge to x and

$$d_0(x_j, x_{j+1}) \le \delta_j(x_0, \ldots, x_j) \text{ for each } j = 0, 1, \ldots.$$

We use the smoothness assumption on X to define suitable perturbation functions (similar to length) and are ready to deduce the statement from a variational principle.

A variational principle

Let (M, d) be a metric space and $F_j : M \times M \to [0, \infty]$ be d-lower semi-continuous in the second variable with $F_j(x, x) = 0$ for all $x \in M$ and, for some $r_j \searrow 0$,

$$\inf_{d(x,y)>r_j} F_j(x, y) > 0$$

Suppose further that d_0 is a continuous pseudometric on M, M is (d, d_0)-complete and $f : M \to \mathbb{R}$ is (d, d_0)-lower semi-continuous function and bounded from below.
Then, given any $x_0 \in M$ and any sequence of positive numbers $(\varepsilon_j)_{j=0}^\infty$ such that $f(x_0) \le \varepsilon_0 + \inf_{x\in M} f(x)$, one may find a sequence $(x_j)_{j=1}^\infty$ in M converging in the metric d to some $x_\infty \in M$ and a d_0 continuous function $\phi : M \to \mathbb{R}$ such that the function

$$h(x) := f(x) + \phi(x) + \sum_{j=0}^\infty F_j(x_j, x)$$

attains its minimum on M at $x = x_\infty$.

Selected open problems

Problem

Is there a real-valued Lipschitz function on a separable Hilbert space (or at least on some Banach space with separable dual) whose set of points of Gâteaux differentiability is σ-porous?

Problem

Can an infinite dimensional separable Hilbert space (or just some Banach space with separable dual) be decomposed as a union of a σ-porous set and an $\widetilde{\mathcal{A}}$ or $\widetilde{\mathcal{A}}_r$ set?

Problem

Are porous subsets of $\mathbb{R}^{n+1}$ Γ_n-null?

Problems

Little is known about relation between various definitions of $\widetilde{\mathcal{A}}$. Are the weaker and stronger definitions equivalent? Is $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_r$? In $\mathbb{R}^n$, is $\widetilde{\mathcal{A}}$ =Lebesgue null?