Behaviour of Lipschitz functions on negligible sets

G. Alberti[1] M. Csörnyei[2] D. Preiss[3]

[1]Università di Pisa
[2]University College London
[3]University of Warwick

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Motivation

Our goal is to understand phenomena that occur because of behaviour of Lipschitz functions around the points of null sets and to answer problems that these results left open.

- Rademacher's Theorem that every Lipschitz $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable a.e. has a strange refinement: For $n \geq 2$ there is a Lebesgue null $E \subset \mathbb{R}^n$ such that every Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point of E. (Preiss 1990)
- Non-existence of Lipschitz solutions of equations such as $\operatorname{div}(f) = \psi$, where ψ is bounded and measurable. (Burago–Kleiner, McMullen, Preiss, . . . 1997–)
- The singular part of the gradient of vector-valued BV maps on $\mathbb{R}^n$ has rank one. (Alberti 1993)
- Every planar set of positive measure can be mapped by a Lipschitz map onto a disk. (Two dimensional case of Laczkovich's problem. As pointed out by Peter Jones, this is shown by the map of Uy, 1979. Proofs relevant here: Preiss 1992 (unpublished), Matoušek 1997)

Outline

I will concentrate on the description of non-differentiability sets of Lipschitz maps on $\mathbb{R}^n$ and explain the development of the ideas, current results, and the main remaining open problem.

Reformulating some observations on Lipschitz maps, I arrive to the notion of rather mysterious tangent fields to null sets at the beginning and get a full description of non-differentiability sets of Lipschitz maps on $\mathbb{R}^n$ at the end.

The relation to existence of solutions of $\operatorname{div}(f) = \psi$ will be touched only briefly. If time permits, I will point out how the tangent fields provide a new understanding of Alberti's theorem, answer Mokobodzki's problem on existence of weak derivatives and/or discuss very interesting combinatorial connections.

The planar case was covered in Alberti's talk at the Fourth European Congress of Mathematics. It will be treated here only when necessary or when it simplifies the exposition.

Non-differentiability in $\mathbb{R}$

Observation

For every Lebesgue null set $E \subset \mathbb{R}$ there is a Lipschitz $f : \mathbb{R} \to \mathbb{R}$ which is non-differentiable at any point of E.

Proof.

Recursively find open $\mathbb{R} = G_0 \supset G_1 \supset G_2 \supset \cdots \supset E$ so small that G_{k+1} is small in every component of G_k. (For example, $|G_{k+1} \cap C| < 2^{-k-1}|C|$ for any component C of G_k.)

Let $\psi(x) = (-1)^k$ where k is the least index such that $x \in G_k$. Define $f(x) = \int \psi(x)\, dx$.

If $x \in E$ and (a, b) is a component of G_k containing x, then

$$\left| \frac{f(b) - f(a)}{b - a} - (-1)^k \right| \leq 2^{-k},$$

so f is not differentiable at x. □

The same example in $\mathbb{R}^2$ and the equation $\operatorname{div}(f) = \psi$

Using essentially the same approach, we can find, given any set $E \subset \mathbb{R}^2$ of measure zero, a bounded measurable $\psi : \mathbb{R}^2 \to \mathbb{R}$ so that for every $x \in E$ there are arbitrarily small radii r, s for which

$$\int_{\|y-x\|<r} \psi(y)\, dy > r^2 \quad \text{and} \quad \int_{\|y-x\|<s} \psi(y)\, dy < -s^2.$$

Of course, we cannot dream of a function $f : \mathbb{R}^2 \to \mathbb{R}$ whose derivative is ψ, but we can, for example, try to consider a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\operatorname{div}(f) = \psi$.

The divergence theorem would imply that f is non-differentiable at any point of E. (This remark is due to V. Šverák.)

This program doesn't work. The reason is that, for large enough null set E, the function ψ provides an example of a bounded measurable function for which the equation $\operatorname{div}(f) = \psi$ has no Lipschitz solution.

Non-differentiability and directional non-differentiability

Recall that the directional (or partial) derivative of $f : \mathbb{R}^n \to \mathbb{R}^m$ at $x \in \mathbb{R}^n$ in direction $u \in \mathbb{R}^n$ is

$$f'(x; u) := \lim_{t \to 0} \frac{f(x + tu) - f(x)}{t}$$

provided that the limit exists.

Although differentiability is not the same as existence of enough partial derivatives, the set of points at which these two notions differ is relatively easy to control. An example showing how this is achieved is given by the following statement which is valid also in infinite dimensional separable Banach spaces.

Theorem (Zajíček, Preiss 2001)

The set of points at which a Lipschitz function is differentiable at a spanning set of directions but is not differentiable is σ-porous.

Definition of σ-porous sets

- A set E in X is said to be c-porous at $x \in E$, $0 < c < 1$, if for every $\varepsilon > 0$ there is a $z \in X \setminus E$ such that $\|x - z\| < \varepsilon$ and $B(z, c\|x - z\|) \cap E = \emptyset$.
- A set is porous if it is porous at each of its points.
- A countable union of porous sets is called σ-porous.

A remark particularly relevant to the content of this talk is that E is porous at $x \in E$ if and only if the function $x \to \operatorname{dist}(x, E)$ is non-differentiable at x.

Notice however that σ-porous sets are of the first category, and so they do not fully describe non-differentiability sets of Lipschitz functions (already on $\mathbb{R}$).

There are even compact subsets of $\mathbb{R}$ of measure zero (and so also of the first category) that are not σ-porous. (Zajíček 1976)

Differentiability in no direction

In what follows, I will pretend that differentiability and differentiability in a spanning set of directions are equivalent.

Although the treatment of the exceptional σ-porous set is not always trivial, it is a minor problem compared to other difficulties one encounters in this area.

There is another type of sets that look very negligible, sets of points at which a Lipschitz function may be differentiable in no direction. These sets form a σ-ideal of "super-negligible" sets for differentiability problems. We will call them uniformly purely unrectifiable.

Notice that uniformly purely unrectifiable sets are purely unrectifiable (or better, purely 1-unrectifiable), i.e., null on every rectifiable curve.

A mysterious vectorfield

Let us see where the directional derivative arguments lead to. For simplicity, consider just the planar case.
Let $E \subset \mathbb{R}^2$ be a Lebesgue null set contained in the set of points of non-differentiability of a Lipschitz $f : \mathbb{R}^2 \to \mathbb{R}^m$.
The key observation is:

Fact. For all $x \in E$ except for those belonging to a uniformly purely unrectifiable set $N \subset E$, there is a unique differentiability direction $\tau(x)$ of f.

Fact. The vector field τ is Borel measurable and, up to a uniformly purely unrectifiable set, is uniquely determined by E. (In other words, it is independent of f!)

Reason. If E is contained in the non-differentiability set of both $f : \mathbb{R}^2 \to \mathbb{R}^m$ and $g : \mathbb{R}^2 \to \mathbb{R}^k$, the τ defined by (f, g) must coincide with the ones defined by f or g whenever f, g and (f, g) have a unique direction of differentiability.

A very wrong argument

We have seen that if every Lebesgue null set $E \subset \mathbb{R}^2$ is a set of non-differentiability, then every such set carries a uniquely (modulo purely unrectifiable sets) determined tangent direction. This seems to be in immediate contradiction with several arguments:

- How can it be unique? We can rotate the set!
- By Davies's theorem, every null set E is contained in another null set F such that through every point of E there pass lines lying in F whose directions are dense. How can the original set E remember its unique directions?

Perhaps such tangent field doesn't exist and this is the reason why a particular null set in $\mathbb{R}^2$ contains a point of differentiability of every Lipschitz $f : \mathbb{R}^2 \to \mathbb{R}$? **NO!**

Theorem. *For every Lebesgue null set $E \subset \mathbb{R}^2$ there is a Lipschitz $f : \mathbb{R}^2 \to \mathbb{R}^2$ non-differentiable at any point of E.*

The tangent field argument in its full beauty

Definition. Let $\mathcal{N}_{n,k}$ be the σ-ideal of subsets of $\mathbb{R}^n$ generated by sets for which there is a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ differentiable in at most k linearly independent directions.

So $\mathcal{N}_{n,0}$ are exactly the uniformly purely unrectifiable sets while $\mathcal{N}_{n,n-1}$ are the non-differentiability sets we are mainly interested in.

Theorem. *Whenever $E \in \mathcal{N}_{n,k}$, there is $\tau : E \to G(n, k)$ such that for all $x \in E$ except those belonging to an $\mathcal{N}_{n,k-1}$ set, every Lipschitz $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable in the direction $\tau(x)$. Moreover, this property determines τ uniquely up to a set from $\mathcal{N}_{n,k-1}$.*

We shall say that τ is the k-dimensional tangent field of E.

Smallness of sets of directional non-differentiability

The set of (directional) nondifferentiability of $f : \mathbb{R}^n \to \mathbb{R}$ can be written as a countable union of sets E, for each of which we may find direction u and numbers $a < b$ such that

$$\liminf_{t \to 0} \frac{f(x + tu) - f(x)}{t} < a < b < \limsup_{t \to 0} \frac{f(x + tu) - f(x)}{t}.$$

Since our f is Lipschitz, such set E is null not only on every line in direction u, but also on every curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ provided that $\|\gamma' - u\|$ is small enough.

We can do slightly better: If $\delta > 0$ is small enough, for every $\varepsilon > 0$ there is an open set $G \supset E$ such that the length of $G \cap \gamma$ is $< \varepsilon$ for every curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ with $\|\gamma' - u\| < \delta$.

We will use this observation to introduce a new way of measuring size of sets in $\mathbb{R}^n$.

Width of sets in $\mathbb{R}^n$

Definition. Let C be a proper closed convex cone with nonempty interior in $\mathbb{R}^n$.

- The C-width of an open set $G \subset \mathbb{R}^n$ is the supremum of the lengths of $G \cap \gamma$ over all curves with $\gamma'(t) \in C$.
- The C-width of an arbitrary set E is the infimum of the C-widths of all open sets containing E.

Remark. For technical as well as aesthetic reasons, our definition of width is more complicated. In particular, it is normalized so that, if $C = \{(x, y) \in \mathbb{R}^2 : y \geq |x|\}$, the set

$$G = \{(x, y) \in \mathbb{R}^2 : \varphi(x) < y < \varphi(x) + w\}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$, $\mathrm{Lip}(\varphi) \leq 1$ and $w > 0$, has width w.

We have argued that a non-differentiability set of a Lipschitz function can be covered by countably many sets, each of which has width zero with respect to some cone.

First attempt at understanding non-differentiability sets

We do not know if the non-differentiability sets of Lipschitz functions are exactly described by the property that they can be covered by countably many sets, each of which has width zero with respect to some cone.

We can, however, show that the non-differentiability sets have a (formally) stronger property that for every $\tau > 0$ they can be covered by (a finite number of) sets each of which has width zero with respect to some cone that is only τ-far from a halfspace.

These ideas also lead to a full description of sets from $\mathcal{N}_{n,k}$ (the σ-ideal generated by sets at which f can be differentiable in at most k linearly independent directions). We will state these results in a slightly different language.

Back to tangent fields

Definition. If $E \subset \mathbb{R}^n$, the mapping $\tau : E \to G(n, k)$ is called a k-dimensional tangent field of E if for every proper closed convex cone C the set $\{x \in E : \tau(x) \cap C = \{0\}\}$ has C-width zero.

Theorem. *The σ-ideal $\mathcal{N}_{n,k}$ of subsets of $\mathbb{R}^n$ generated by sets for which there is a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ differentiable in at most k linearly independent directions coincides with the family of those subsets of $\mathbb{R}^n$ that admit a k-dimensional tangent field.*

Theorem. *The k-dimensional tangent field of E (as defined above) is determined uniquely up to an $\mathcal{N}_{n,k-1}$-set and coincides with the tangent field defined via Lipschitz functions. In particular, every Lipschitz $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable in the direction $\tau(x)$, for $\mathcal{N}_{n,k-1}$ almost every $x \in E$.*

Description of sets of non-differentiability

Theorem. *For every set $E \subset \mathbb{R}^n$, the following are equivalent.*

- *There is a Lipschitz $f : \mathbb{R}^n \to \mathbb{R}^n$ that is non-differentiable at any point of E.*
- *There is a sequence (possibly infinite) of Lipschitz functions $f_j : \mathbb{R}^n \to \mathbb{R}$ such that at every point of E at least one of the f_j is non-differentiable.*
- *The set E admits an $n-1$-dimensional tangent field.*
- *If $n \leq 2$: E has Lebesgue measure zero.*

Corollary. *The following statements about a (σ-finite Borel) measure in $\mathbb{R}^n$ are equivalent.*

- Every Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable μ almost everywhere.
- Every set admitting an $n-1$-dimensional tangent field is μ null.

The key open problem

The key open problem. *Is the restriction to $n \le 2$ necessary?* In other words, is it true that for all n that

- Lebesgue null sets coincide with $\mathcal{N}_{n,n-1}$;
- every Lebesgue null set in $\mathbb{R}^n$ admits an $n-1$ dimensional tangent field?

Problem. Is it true that for every $m < n$ there is a set $E \in \mathcal{N}_{n,m}$ such that every Lipschitz $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at some point of E?

The answer is "yes" for $m = 1$ and any n; this is (essentially) the result mentioned in the beginning.

It seems that the methods developed by Lindenstrauss, Tišer and Preiss in their study of differentiability problems in infinite dimensional spaces can show "yes" also for $m = 2$. But these methods cannot work for $m \ge 3$.

Measures with unique $n-1$-dimensional tangent field

For what measures is the $n-1$-dimensional tangent field uniquely defined a.e?

(a) The measure has to be concentrated on $\mathcal{N}_{n,n-1}$ and
(b) absolutely continuous with respect to $\mathcal{N}_{n,n-2}$.

Observe that sets from $\mathcal{N}_{n,n-2}$ are purely $n-1$-unrectifiable. So for (b) it suffices that the measure be absolutely continuous with respect to purely $n-1$-unrectifiable sets.

The requirement (a) would be equivalent to singularity if $\mathcal{N}_{n,n-1}$ coincided with Lebesgue null sets, which we do not know. But the methods used to prove it when $n = 2$ are powerful enough to show that singularity plus absolute continuity with respect to purely $n-1$-unrectifiable sets suffices:

Theorem. *Every Lebesgue null set in $\mathbb{R}^n$ is a union of a set from $\mathcal{N}_{n,n-1}$ and a purely $n-1$-unrectifiable set.*

Understanding Alberti's theorem

Alberti's theorem can be understood as saying that certain class of (positive) measures in $\mathbb{R}^n$, namely those that arise as singular parts of derivatives of BV functions, have a.e. uniquely defined normal directions.

Since it is known that they are absolutely continuous with respect to the purely $n-1$-unrectifiable sets, and since they are singular by definition, they have uniquely defined $n-1$-dimensional tangent fields in our sense.

One can now adapt Alberti's argument of "rectifiably representable measures" to show that the gradient directions of the derivative of a BV function are, almost everywhere with respect to its singular part, orthogonal to our tangent field.

Width and construction of Lipschitz functions

For an open set G of (small) C-width w (in a suitably normalized definition) and a unit vector e from the interior of C, we can construct a function ω such that

(a) $\mathrm{Lip}(\omega)$ is bounded by a constant depending on C and e;
(b) $\omega(y) \ge \omega(x)$ if $y - x \in C$;
(c) $\omega(x + te) = \omega(x) + t$ if the segment $[x, x + te]$ lies in G;
(d) $0 \le \omega(x) \le w$ for all $x \in \mathbb{R}^n$.

The function ω is used to construct non-differentiable functions: $\omega'(x; e) = 1$ for $x \in G$ but from the more global point of view ω looks like having derivative zero.

Another use of ω is via the map $u : \mathbb{R}^n \to \mathbb{R}^n$, $u(x) = x - \omega(x)e$, whose distance from the identity is w. Then u maps G to a null set, which looks like a good start to answering Laczkovitch's problem (and for $n = 2$ it is).

Non-existence of weak derivatives

Let μ be a measure such that $\mu(S) > 0$ for some $S \in \mathcal{N}_{n,n-1}$. Find $E \subset S$ with $\mu(E) > 0$ of C-width zero for some cone C. Pick e in the interior of C and construct ω_j for $w = 1/j$. So we get a sequence of functions $\omega_j : \mathbb{R}^n \to \mathbb{R}$ with uniformly bounded Lipschitz constants, converging to $\omega := 0$ and such that $\omega_j'(x; e) \geq 0$ everywhere and $\omega_j'(x; e) = 1$ for $x \in E$.

A moment's reflection shows that ω_j' cannot converge to $0 = \omega'$ in any weak sense with respect to μ. And a straightforward argument can make them C^1.

Theorem. For a measure μ in $\mathbb{R}^n$, weak derivatives of Lipschitz functions may be defined iff μ is absolutely continuous with respect to $\mathcal{N}_{n,n-1}$, hence iff every Lipschitz function is differentiable μ a.e.

For $n = 2$ we know that the above holds iff μ is absolutely continuous with respect to the Lebesgue measure. This answers a problem due to G. Mokobodzki.

The case of uniformly purely unrectifiable sets

The construction of Lipschitz functions appropriately non-differentiable at every point of a given set $E \in \mathcal{N}_{n,m}$ is much harder than the construction of those that are not differentiable in the weak sense.

The construction may be somewhat simplified in the case of measures and for sets $E \in \mathcal{N}_{n,0}$.

In this case, for any unit vector e we can choose an open set $G \supset E$ with C-width null where $C \ni e$ is as close to the halfspace $\{x : \langle x, e\rangle \geq 0\}$ as we wish.

The function $\langle x, e\rangle - \omega(x)$ sees, from every point of G, some points in the direction e with slope almost one, but has local Lipschitz constant close to zero on G. This allows us to iterate the construction locally. Moving also the vectors e through a dense subset of the unit sphere, we get a function which is non-differentiable at any point of E in any direction.

Combinatorial connections

These were observed by J. Matoušek. He noted that part of Preiss's approach to Laczkovich's problem,

- *every planar set of positive Lebesgue measure can be mapped by a Lipschitz mapping onto a disk,*

is similar to the proof of the Erdős-Szekeres Theorem. And he recognised that this part may be replaced by its corollary

- *for any planar set Q having q^2 points there is a function $\psi : \mathbb{R} \to \mathbb{R}$ with Lip$(\psi) \leq 1$ such that one of the sets*

$$\{(x, y) \in Q : y = \psi(x)\} \quad \text{or} \quad \{(x, y) \in Q : x = \psi(y)\}$$

has at least q points.

Some of the results on which this talk is based exploited this connection and may be considered as a continuous analogy of the Erdős-Szekeres and/or Dilworth Theorems.

Combinatorial approach to the key problem

Matoušek also conjectured a variant of the Erdős-Szekeres Theorem that would solve Laczkovich's problem in all dimensions. This conjecture was disproved by Tardos.

There is, however, a version of Matoušek's conjecture that would imply a positive answer to our main problem (all Lebesgue null sets would belong to $\mathcal{N}_{n,n-1}$):

Conjecture. *For any set $Q \subset \mathbb{R}^n$ having q^n points there is a function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ with Lip$(\psi) \leq C_n$ and an orthonormal system of coordinates such that the set*

$$\{(x_1, \ldots, x_n) \in Q : x_n = \psi(x_1, \ldots, x_{n-1})\}$$

has at least $c_n q^{n-1}$ points.

This problem is open. We only know that, unlike in the plane, the coordinate systems cannot be restricted to permutations of the standard coordinate system.