New Variational Principles
and their Use in Proving Differentiability

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Differentiability problems we are interested in

Problem
Is is true that any three real-valued Lipschitz functions on an (infinite dimensional separable) Hilbert space have a common point of Fréchet differentiability?

Conjecture
Any countable collection of real-valued Lipschitz functions on a Banach space with separable dual has a common point of Fréchet differentiability.

Basic notions

Throughout this talk, functions will be real-valued.

A function $f \colon X \to \mathbb{R}$ is Lipschitz if there is $C < \infty$ so that

$$|f(x) - f(y)| \le C \operatorname{dist}(x, y) \text{ for all } x, y \in X$$

Definition
A function f on a Banach space X is said to be Fréchet differentiable at a point x_0 if there is $x^* \in X^*$ so that

$$f(x_0 + u) = f(x_0) + x^*(u) + o(\|u\|), \ u \to 0.$$

Definition
The directional derivative of f at x_0 in direction $u \in X$ is

$$f'(x_0; u) := \lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t}$$

provided that the limit exists.

Current status of the conjecture

Theorem (Preiss 1990)
Every Lipschitz function on a Banach space with separable dual is Fréchet differentiable at least at one point.

Theorem (Lindenstrauss, Preiss 1996)
On uniformly smooth spaces, every n Lipschitz functions have, for every $\varepsilon > 0$, a common point of ε-Fréchet differentiability.
(This results has been strengthened to asymptotically uniformly smooth spaces by Johnson, Lindenstrauss, Schechtman, Preiss 2002.)

Theorem (Lindenstrauss, Preiss 2003)
On "c_0-like spaces" the conjecture holds.

Theorem (Lindenstrauss, Tišer, Preiss 2008)
On spaces "almost as smooth as the Hilbert space," every pair of Lipschitz functions has a common point of Fréchet differentiability.

Ekeland's variational principle

Theorem (Ekeland's variational principle)

Let f be a bounded continuous function on a complete metric space M. Then for every $\varepsilon > 0$ there is $x \in M$ such that the function

$$y \to f(y) - \varepsilon d(x, y)$$

attains its maximum at $y = x$.

This is the simplest, yet very powerful, perturbational variational principle: after a small and easily controlled perturbation, the function that presumably didn't attain a maximum at all will suddenly attain it.

First versions of this statement were proved in a series of papers by Bishop and Phelps (and by Phelps alone) in 1961–63, but only in 1979 Ekeland recognised it as an abstract principle.

Ekeland's principle and differentiability

Fact. A continuous convex function f on a Banach space X is Fréchet differentiable at all points of a residual set provided that for every $\varepsilon > 0$ there is a dense set of points x at which

$$\limsup_{u \to 0} \big(f(x+u) + f(x-u) - 2f(x)\big)/\|u\| < \varepsilon. \qquad (\star)$$

Assumption on the Banach space: there is a C^1 bump function $b(x)$, that is $0 \le b(x) \le 1$, $b(x) = 0$ for $\|x\| \ge 1$, and $b(0) = 1$.

Use of Ekeland's principle to the function $x \to f(x) + Cb(x)$ (for a suitably large C) on the space $M = \{x : \|x\| \le 1\}$ gives $x \in M$ so that

$$y \to f(y) + Cb(y) - \varepsilon\|y - x\|$$

attains its maximum at $y = x$.

Conclusion. If C is large then $x \in \text{interior}(M)$ and $(\star)$ holds. An easy adjustment provides a dense set of such points. So f is Fréchet differentiable at all points of a residual set.

Smooth variational principles

Theorem (Borwein, Preiss 1987) *Let M be a complete metric space and $F_j : M \times M \to [0, \infty]$ be continuous, $F_j(x, x) = 0$ and, for some $r_j \searrow 0$,*

$$\inf_{d(x,y) > r_j} F_j(x, y) > 0.$$

Then for every continuous upper bounded function f on M one may find $x_j \in M$, $x_j \to x_\infty$ such that the function

$$h(x) := f(x) - \textstyle\sum_{j=0}^{\infty} F_j(x_j, x)$$

attains its maximum on at $x = x_\infty$.

A less specific, but often easier to use, variant of this principle is due to Deville, Godefroy, Zizler 1993.

With these principles, the convex function argument gives a point of differentiability directly, without the Baire category trick.

Variational method and Lipschitz functions

Observation

Suppose that for the given Lipschitz function f on a Hilbert space H we find a point $x \in H$ and a unit vector u such that $f'(x; u)$ exists and equals the Lipschitz constant of f.
Then f is Fréchet differentiable at x.

Better Observation

Suppose that there are a point $x \in H$ and a unit vector u such that $f'(x; u)$ exists and equals the limit of Lipschitz constants of f on balls around x.
Then f is Fréchet differentiable at x.

Similar, but more complicated, variants of these observations hold in every Banach space with separable dual.
Unfortunately, x and u with these properties need not exist even if $H = \mathbb{R}$.

Better variational reasons behind differentiability

Even Better Observation

Suppose that f is a Lipschitz function and $\Phi : X \times X \to \mathbb{R}$ is a locally uniformly continuous function having continuous derivative with respect to the second variable.
Then f is Fréchet differentiable at a point x whenever we may find a vector u such that the function

$$(y, v) \to f'(y; v) - \Phi(y, v)$$

attains its maximum at (x, u).

The main difficulty with the use of smooth variational principles to find such Φ and (x, u) is that the function whose maximum we are seeking is not defined on a complete metric space (it is not defined on all of $X \times X$) and, even if it is defined everywhere, it is not continuous (not even semi-continuous).

Handling discontinuous functions

Even though the function $(y, v) \to f'(y; v)$ is discontinuous, if a sequence (y_k, v_k) is constructed recursively by requiring that y_{k+1} is very close to y_k, then $f'(y_k; v_k) \to f'(y; v)$.

A nice way of expressing the recursive procedure is to state it as a continuity with respect to two metrics: the usual distance d and the distance d_0 of the first projections.

Definition. Suppose that (M, d) is a metric space and d_0 a continuous pseudometric on M. We say that a function $f : M \to \mathbb{R}$ is (d, d_0)-continuous if there are functions $\delta_j(x_0, \dots, x_j) : M^{j+1} \to (0, \infty)$ such that

$$\lim_{j \to \infty} f(x_j) = f(x)$$

whenever $x_j \in M$ d-converge to x and

$$d_0(x_j, x_{j+1}) \le \delta_j(x_0, \dots, x_j) \text{ for each } j = 0, 1, \dots.$$

Difficulties with incomplete domains

In addition to discontinuity, another serious problem in applying variational principles is that the function $(y, v) \to f'(y; v)$ is not defined everywhere.

In some cases, we can weaken completeness in a similar way we weakened continuity, but in general this remains an unsurmountable obstacle to direct use of variational principles to prove differentiability of Lipschitz functions.

Definition. A space (M, d, d_0), where d_0 is a pseudometric continuous with respect to the metric d, is (d, d_0)-complete if there are functions $\delta_j(x_0, \dots, x_j) : M^{j+1} \to (0, \infty)$ such that every d-Cauchy sequence $(x_j)_{j=0}^{\infty}$ converges provided that

$$d_0(x_j, x_{j+1}) \le \delta_j(x_0, \dots, x_j) \text{ for each } j = 0, 1, \dots.$$

A variational principle

Let (M, d) be a metric space and $F_j : M \times M \to [0, \infty]$ be d-lower semi-continuous in the second variable with $F_j(x, x) = 0$ for all $x \in M$ and, for some $r_j \searrow 0$,

$$\inf_{d(x,y) > r_j} F_j(x, y) > 0.$$

Suppose further that d_0 is a continuous pseudometric on M such that M is (d, d_0)-complete and that $f : M \to \mathbb{R}$ is (d, d_0)-continuous and bounded from above.
Then one may find a sequence $x_j \in M$ converging in the metric d to some $x_\infty \in M$ and a d_0 continuous function $\phi : M \to \mathbb{R}$ such that the function

$$h(x) := f(x) - \phi(x) - \textstyle\sum_{j=0}^{\infty} F_j(x_j, x)$$

attains its maximum on M at $x = x_\infty$.

The case of directionally differentiable functions

As a rather straightforward application of our variational principle we get a special case of the Fréchet differentiability result for Lipschitz function.

Application

If a Lipschitz function on a Banach space with separable dual has directional derivative at all points and directions, then it has points of Fréchet differentiability.

A slightly weaker statement was the first result on existence of Fréchet derivatives of Lipschitz functions in infinitely dimensional spaces (Preiss, 1984).

In spite of several known proofs of this “simple” special case, it is still not known whether three Lipschitz functions on a Hilbert space, all having directional derivatives at all points and directions, have a common point of Fréchet differentiability!

Differentiability of distance functions

Suppose that E is a closed subset of a Banach space X. How small is the set P of points of E at which the distance function

$$x \to \mathrm{dist}(x, E)$$

is not Fréchet differentiable? And how small is the σ-ideal generated by these sets?
The sets from this σ-ideal are (obviously) meager and (by definition) they are exactly the σ-porous sets.
To prove the Conjecture, we have to find points outside such sets at which other functions have (more than) directional derivatives.
For this (and other) reasons, we define Γ_n-null sets as those sets that are null on “many” n dimensional surfaces and investigate in which spaces σ-porous sets are Γ_n-null.

Definition of Γ_n-null sets

Definition

A Borel set $E \subset X$ is Γ_n-null if

$$\{\gamma \in \Gamma_n(X) : |\gamma^{-1}(E)| > 0\}$$

is a first category subset of $\Gamma_n(X) := C^1([0,1]^n, X)$.

Notice that the Baire category theorem shows that Γ_n-null sets form a nontrivial σ-ideal of Borel subsets of X.

The definition makes sense also for $n = \infty$, in which case we leave out the index ∞. In this case, $\Gamma(X)$ is not a Banach space but only a Fréchet space.

Key Facts. If every σ porous set in X is Γ_n-null, then any n Lipschitz functions have, for every $\varepsilon > 0$, plenty of points of ε-Fréchet differentiability. If $n = \infty$, the Conjecture holds for X.

Variational approach to smallness of σ-porous sets

To prove that a porous set P is Γ_n-null, we consider $\Gamma_n(X)$ with two metrics: d induced by the C^1-norm and d_0 induced by the maximum norm.

Arguing by contradiction, we manage to find somewhere d-dense, d_0-G_δ subset M of $\Gamma_n(X)$ on which $|\gamma^{-1}(P)| > c > 0$.

This M is (d, d_0)-complete and the function $f(\gamma) := |\gamma^{-1}(\overline{P})|$ is (d, d_0)-continuous. We intend to perturb it so that it attains a minimum and use this to find a contradiction.

However, finding perturbation functions for which this plan works is a challenge. For Hilbert spaces we can do it only if $n \leq 2$; and for ℓ_p spaces only if $n \leq p$. Hence in these cases, we prove that σ-porous sets are Γ_n-null.

Our failure to establish the result in Hilbert spaces for $n \geq 3$ and in ℓ_p for $n > p$ has a simple explanation:
In these cases there are σ-porous sets whose complement is null on all nondegenerated surfaces from $\Gamma_n(X)$.