TRANSVERSALITY IN HOLOMORPHIC DYNAMICS

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(Preliminary Version)

The space $\mathbf{Rat}_D$ of all degree D endomorphisms of projective space $\mathbb{P}^1$ is a smooth affine algebraic variety of dimension $2D+1$. Indeed, $f(z) = \frac{p(z)}{q(z)}$ for polynomials p and $q \neq 0$ of degree at most D: since p and q are defined up to a common nonzero factor, their coefficients provide an embedding of $\mathbf{Rat}_D$ into $\mathbb{P}^{2D+1}$. The group of projective transformations $\mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathbf{Rat}_D$ by conjugation, and the quotient Rat_D is an orbifold of dimension $2D-2$.

Our aim here is to develop a language for describing various dynamically meaningful subloci of $\mathbf{Rat}_D$ and Rat_D, and for proving their smoothness and transversality. For example, a rational map $f \in \mathbf{Rat}_D$ may have nontrivial *critical orbit relations* of the form $f^{\circ n}(c) = f^{\circ \hat{n}}(\hat{c})$ for some integers $n, \hat{n}$ and critical points $c, \hat{c}$. Other loci of particular interest are given by those maps $f \in \mathbf{Rat}_D$ which have a cycle of specified period and multiplier, or a parabolic cycle of specified period, multiplier and degeneracy, or which arise from such a map in a parabolic bifurcation with specified holomorphic index.

We work over $\mathbb{C}$ using transcendental methods. Nevertheless, various questions considered make sense over any algebraically closed field of characteristic 0, and so does much of the cohomological formalism to be employed.

It happens that these aspects of the local geometry of $\mathbf{Rat}_D$ are more readily investigated in (a hierarchy of) spaces $\mathrm{Def}^B_A(f)$ to be defined in §3. These deformation spaces are obtained from a functorial construction in Teichmüller theory. The relation between these spaces $\mathrm{Def}^B_A(f)$ and the parameter space $\mathbf{Rat}_D$ is somewhat comparable to the relation between Teichmüller space and moduli space in the classical theory of Riemann surfaces. The cotangent space to $\mathrm{Def}^B_A(f)$ is canonically isomorphic to a certain quotient of spaces of meromorphic quadratic differentials. We deduce smoothness of these deformation spaces, and transversality of critical orbit relation loci, from *Infinitesimal Thurston Rigidity* - the injectivity of the linear operator $\nabla_f = \mathrm{I} - f_*$ acting on meromorphic quadratic differentials with at worst simple poles - via Serre Duality and the Inverse Function Theorem. Following this paradigm, we deduce corresponding results for appropriate loci of maps with given multipliers, parabolic degeneracies, and holomorphic indices, from injectivity of ∇_f on appropriate spaces of meromorphic quadratic differentials with higher order poles.

The spaces $\mathrm{Def}^B_A(f)$ were defined and studied in [3] in the more general setting of (possibly transcendental) finite type maps. The underlying construction has conceptual roots in Thurston's Theorem [9] concerning the realization of combinatorially specified *postcritically finite* maps: in particular, that (with one well-understood set of exceptions, namely flexible Lattès examples) the corresponding subsets of moduli space are finite. The presentation of Douady-Hubbard [2], and of

McMullen [7] on related matters connected with the hyperbolization of 3-manifolds, profoundly influenced the standard of functoriality and the overall scope of this work. Other important precursors are Shishikura's seminal article [8] concerning the quasiconformal deformation of indifferent periodic points (and surgery on Herman rings), and the Eremenko-Lyubich construction [5] of deformation spaces for finite type entire maps. The injectivity of ∇_f for quadratic differentials with appropriate higher-order poles was obtained in [4], and applied there (see also §7) to give a new proof of a refined Fatou-Shishikura bound on the count of nonrepelling cycles. The present work essentially consists of the variational interpretation of that proof.

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Part 1. **The deformation space**

1. Teichmüller theory

In this paragraph, we recall basic notions in Teichmüller theory.

1.1. **Definitions.** Let S be a compact oriented surface, and $E \subset S$ be a finite subset such that $2\,\mathrm{genus}(S) + \mathrm{card}(E) \geq 3$ (in this paper, we will mainly be concerned by the case $\mathrm{genus}(S) = 0$ and $\mathrm{card}(E) \geq 3$).

A *complex structure* on S is a maximal atlas of orientation-preserving coordinates $\varphi : U \to \mathbb{C}$ with holomorphic transition maps. We denote by $\mathrm{Complex}(S)$ the set of all complex structures on S. An orientation-preserving homeomorphism $\psi : S \to S$ induces a map $\psi^* : \mathrm{Complex}(S) \to \mathrm{Complex}(S)$.

Definition 1. *The* Teichmüller space $\mathrm{Teich}(S,E)$ *is the quotient of* $\mathrm{Complex}(S)$ *by the equivalence relation* $\sim$ *where* $\mathfrak{s}_1 \sim \mathfrak{s}_2$ *if* $\mathfrak{s}_1 = \psi^*\mathfrak{s}_2$ *for some orientation-preserving homeomorphism* $\psi : S \to S$ *which is the identity on* E *and is isotopic to the identity relative to* E.

Note that S, equipped with a complex structure $\mathfrak{s} \in \mathrm{Complex}(S)$ is a *Riemann surface*. This yields the following description of the Teichmüller space: $\mathrm{Teich}(S,E)$ is the quotient of the space of pairs (X,φ), where X is a Riemann surface and $\varphi : S \to X$ is an orientation-preserving homeomorphism, modulo the equivalence relation $\sim$, whereby $(X_1,\varphi_1) \sim (X_2,\varphi_2)$ if there exists an analytic isomorphism $\alpha : X_1 \to X_2$ such that $\varphi_2|_E = (\alpha \circ \varphi_1)|_E$ and φ_2 is isotopic to $\alpha \circ \varphi_1$ relative to E.

Remark. The definition of $\mathrm{Teich}(S,E)$ is true only because S is compact and E is finite. The underlying reason is that a homeomorphism $\psi : S \to S$ fixing E can be uniformly approximated by a C^∞ diffeomorphism fixing E, and these are all isotopic relative to E when the approximation is close enough.

It is well-known that $\mathrm{Teich}(S,E)$ has a natural metric and complex manifold structure (see [6] for example). This complex structure may be defined as follows. Let $\mathfrak{s} \in \mathrm{Complex}(S)$ represent a point in $\mathrm{Teich}(S,E)$ and denote by X the Riemann surface S equipped with $\mathfrak{s}$. We shall use the notion of Beltrami form on the Riemann surface X.

1.2. **Beltrami forms.** An *infinitesimal Beltrami form* on X is an L^∞ section μ of the bundle $\overline{\mathrm{Hom}}(TX, TX)$ over X whose fiber above $x \in X$ is the space of $\mathbb{C}$-antilinear maps $T_xX \to T_xX$. Note that if ν is such a section, then $|\nu|$ is a real-valued function, so the notion of L^∞ section makes sense, and the space of infinitesimal Beltrami forms is a Banach space $\mathrm{bel}(X)$. In local coordinates, an infinitesimal Beltrami form may be written as $\nu(z)\dfrac{\mathrm{d}\bar z}{\mathrm{d}z}$. Then $\nu_x : T_xX \to T_xX$ is the $\mathbb{C}$-antilinear map $v \mapsto \nu(x) \cdot \bar v$.

Definition 2. *The space* $\mathrm{Bel}(X)$ *of Beltrami forms on the Riemann surface* X *is the unit ball of* $\mathrm{bel}(X)$*.*

Let μ be a Beltrami form on X. By the Measurable Riemann Mapping Theorem, we can define a complex structure $\mathfrak{s}_\mu$ on X whose coordinates are injective solutions h of

$$\overline{\partial} h = \partial h \circ \mu, \tag{1}$$

defined on open subsets of X with values in $\mathbb{C}$.

Remark. If h is of class C^1, then at each point $x \in X$, $\overline{\partial} h$ and ∂h are respectively a $\mathbb{C}$-antilinear map and a $\mathbb{C}$-linear map $T_xX \to \mathbb{C}$, so that Equation (1) makes sense. The Measurable Riemann Mapping Theorem says that if μ is a Beltrami form, then Equation (1) has solutions that are quasiconformal homeomorphisms, i.e., with distributional derivatives locally in L^2 and satisfying Equation (1) locally in L^2. The solutions are then unique up to postcomposition with an analytic function. In particular, the solutions which are injective on open subsets of X form an atlas.

The map $\mathrm{Bel}(X) \ni \mu \mapsto \mathfrak{s}_\mu \in \mathrm{Complex}(S)$ induces a map

$$\Pi : \mathrm{Bel}(X) \to \mathrm{Teich}(S, E).$$

There is a unique complex structure on $\mathrm{Teich}(S, E)$ such that the map Π is analytic. This mapping is a split submersion.

We still assume that $\mathfrak{s} \in \mathrm{Complex}(S)$ represents a point $\tau \in \mathrm{Teich}(S, E)$ and let X be the Riemann surface S equipped with the complex structure $\mathfrak{s}$. We will now describe the tangent and especially the cotangent space to $\mathrm{Teich}(S, E)$ at τ. The kernel of the derivative of $\Pi : \mathrm{Bel}(X) \to \mathrm{Teich}(S, E)$ at the zero Beltrami form $\mathbf{0}$ on X is the space of infinitesimal Beltrami forms that can be written $\bar\partial \xi$ for some continuous vector field ξ on X, with distributional derivative in L^∞ and vanishing on E. The tangent space $T_\tau \mathrm{Teich}(S, E)$ is canonically isomorphic to the quotient of $\mathrm{bel}(X)$ by this kernel. This is really a description of the Dolbeault cohomology group $H^1\big(\Theta_X(-E)\big)$. We will need higher cohomology groups later in the paper, but at the moment it is more convenient to define the cotangent space which is a simpler H^0 via Serre duality. This will require quadratic differentials.

1.3. **Quadratic differentials.** A holomorphic *quadratic differential* on a Riemann surface Y is an analytic section of the bundle $\Omega_Y \otimes \Omega_Y$ where Ω_Y is the cotangent bundle to Y. It will be very important to us to realize that if q is a quadratic differential on Y, then we can define a volume form $|q|$ on Y by its value

$$|q|_y(v, iv) = \big|q(v, v)\big|$$

on the basis (v, iv) of T_yY. The integrable holomorphic quadratic differentials, those for which

$$\|q\| = \int_Y |q| < \infty,$$

form a Banach space which we denote by $\mathcal{Q}(Y)$. In local coordinates, a quadratic differential may be written as $q(z)\mathrm{d}z^2$. Then,

$$\int_Y |q| = \int_Y |q(x+iy)|\mathrm{d}x\mathrm{d}y.$$

If $E \subset X$ is finite, we set

$$\mathcal{Q}(X,E) = \mathcal{Q}(X-E).$$

Requiring that q be holomorphic on $X - E$ and integrable means that q is meromorphic on X with at most simple poles on E.

If ν is an infinitesimal Beltrami form on Y and q is a quadratic differential on Y, then $q \wedge \nu$ is naturally a $(1,1)$-form defined by

$$T_yY \times T_yY \ni (v_1, v_2) \mapsto (q\wedge\nu)_y(v_1,v_2) = q_y\big(v_1, \nu_y(v_2)\big) - q_y\big(\nu_y(v_1), v_2\big) \in \mathbb{C}.$$

In particular, it can be integrated over Y. Moreover, the expression

$$\langle q, \nu\rangle = \frac{1}{2\pi\mathrm{i}}\int_Y q\wedge\nu$$

defines a bilinear pairing $\mathcal{Q}(Y) \times \mathrm{bel}(Y) \to \mathbb{C}$. The normalizing factor will simplify some later computations. In local coordinates, this may be understood as follows:

- an infinitesimal Beltrami form may be written as $\nu(z)\dfrac{\mathrm{d}\bar z}{\mathrm{d}z}$,
- a quadratic differential may be written as $q(z)\mathrm{d}z^2$,
- the product may be written as $q(z)\nu(z)\mathrm{d}z\wedge\mathrm{d}\bar z$,
- the surface integral for the pairing is given by

$$\frac{1}{2\pi\mathrm{i}}\int q(z)\nu(z)\ \mathrm{d}z\wedge\mathrm{d}\bar z = -\frac{1}{\pi}\int \nu(x+\mathrm{i}y)q(x+\mathrm{i}y)\ \mathrm{d}x\mathrm{d}y.$$

The pairing $\mathcal{Q}(X,E)\times\mathrm{bel}(X) \to \mathbb{C}$ vanishes on elements of $\mathrm{bel}(X)$ of the form $\bar\partial\xi$ for some continuous vector field ξ on X vanishing on E, i.e., on the kernel of

$$D_{\mathbf{0}}\Pi : \mathrm{bel}(X) \to T_\tau\mathrm{Teich}(S,E).$$

As such, it induces a pairing between $\mathcal{Q}(X,E)$ and $T_\tau\mathrm{Teich}(S,E)$. This pairing is in fact a duality. In particular, the cotangent space $T^\star_\tau\mathrm{Teich}(S,E)$ is canonically isomorphic to $\mathcal{Q}(X,E)$.

1.4. **The case of genus zero.** By the Uniformization Theorem, a Riemann surface of genus zero is isomorphic to $\mathbb{P}^1$. We shall consider the case $S = \mathbb{P}^1$ and let $\mathbf{0} \in \mathrm{Teich}(\mathbb{P}^1, E)$ be the basepoint represented by the standard complex structure on $\mathbb{P}^1$.

Denote by $-\mathbf{E}$ the divisor on $\mathbb{P}^1$ which consists of weighting point in E with weight -1. Let $\Theta_{\mathbb{P}^1}$ be the sheaf of holomorphic vector fields on $\mathbb{P}^1$ and $\Theta_{\mathbb{P}^1}(-\mathbf{E})$ be the sheaf of holomorphic vector fields on $\mathbb{P}^1$ which vanish on E. The short exact sequence of sheaves

$$0 \to \Theta_{\mathbb{P}^1}(-\mathbf{E}) \to \Theta_{\mathbb{P}^1} \to \Theta_{\mathbb{P}^1}/\Theta_{\mathbb{P}^1}(-\mathbf{E}) \to 0$$

induces an exact sequence

$$H^0\big(\Theta_{\mathbb{P}^1}(-\mathbf{E})\big) \to H^0(\Theta_{\mathbb{P}^1}) \to H^0\big(\Theta_{\mathbb{P}^1}/\Theta_{\mathbb{P}^1}(-\mathbf{E})\big) \to H^1\big(\Theta_{\mathbb{P}^1}(-\mathbf{E})\big) \to H^1(\Theta_{\mathbb{P}^1})$$

which we will identify as

$$0 \to \mathrm{aut}(\mathbb{P}^1) \to \bigoplus_{z\in E} T_z\mathbb{P}^1 \to T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1, E) \to 0. \tag{2}$$

Indeed,

- since E contains at least 3 points, we have $H^0\big(\Theta_{\mathbb{P}^1}(-\mathbf{E})\big) = 0$ (a holomorphic vector field on $\mathbb{P}^1$ with three zeroes vanishes);
- the space $H^0(\Theta_{\mathbb{P}^1})$ is the space $\mathrm{aut}(\mathbb{P}^1)$ of holomorphic vector fields on $\mathbb{P}^1$;
- since $\Theta_{\mathbb{P}^1}/\Theta_{\mathbb{P}^1}(-\mathbf{E})$ is a skyscraper sheaf supported on E with fiber $T_z\mathbb{P}^1$ at each $z \in E$, the space $H^0\big(\Theta_{\mathbb{P}^1}/\Theta_{\mathbb{P}^1}(-\mathbf{E})\big)$ is canonically identified to the space of vector fields on E, i.e. the direct sum $\bigoplus_{x\in E} T_z\mathbb{P}^1$;
- the space $H^1(\Theta_{\mathbb{P}^1})$ vanishes; the Riemann-Roch formula says
$$\dim H^0(\Theta_{\mathbb{P}^1}) - \dim H^1(\Theta_{\mathbb{P}^1}) = \deg \Theta_{\mathbb{P}^1} + 1 - \text{genus } \mathbb{P}^1 = 2 + 1 - 0 = 3,$$
but $\dim H^0(\Theta_{\mathbb{P}^1}) = 3$;
- finally, by Serre duality, $H^1\big(\Theta_{\mathbb{P}^1}(-\mathbf{E})\big)$ is the dual of $H^0\big(\Omega^{\otimes 2}_{\mathbb{P}^1}(\mathbf{E})\big)$, the space of meromorphic quadratic differentials on $\mathbb{P}^1$, holomorphic on $\mathbb{P}^1 - E$ with at worst simple poles on E; in other words, $H^1\big(\Theta_{\mathbb{P}^1}(-\mathbf{E})\big)$ is dual to $\mathcal{Q}(\mathbb{P}^1, E)$, hence canonically isomorphic to the tangent space $T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1, E)$.

The duality between $\mathcal{Q}(\mathbb{P}^1, E)$ and $H^1\big(\Theta_{\mathbb{P}^1}(-\mathbf{E})\big)$ may be understood as follows. Let θ be a vector field on E and let ϑ be a holomorphic extension to a neighborhood of E in $\mathbb{P}^1$. If $q \in \mathcal{Q}(\mathbb{P}^1, E)$, then $q\otimes\vartheta = q(\vartheta, \cdot)$ is a 1-form, defined and meromorphic in a neighborhood of E. It has a residue at each point $z \in E$ and since q has at most a simple pole at z, this residue only depends on the value of θ at z. We denote this residue as $\mathrm{Res}_z q \otimes \theta$.

Proposition 1. *The pairing between $\mathcal{Q}(\mathbb{P}^1, E)$ and $H^1\big(\Theta_{\mathbb{P}^1}(-\mathbf{E})\big)$ is induced by*

$$\mathcal{Q}(\mathbb{P}^1, E) \times \left(\bigoplus_{z\in E} T_z\mathbb{P}^1\right) \ni (q, \theta) \mapsto \langle q, \theta\rangle = \sum_{z\in E} \mathrm{Res}_z q \otimes \theta.$$

Proof. Let ϑ be a C^∞ extension of θ to $\mathbb{P}^1$, holomorphic in a neighborhood of E. The infinitesimal Beltrami form $\nu = \bar\partial\vartheta \in \mathrm{bel}(\mathbb{P}^1)$ represents an element of $T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1, E)$ which does not depend on the extension ϑ (since the difference of two extensions if a vector field that vanishes on E) and is canonically identified to the class of θ in $H^1\big(\Theta_{\mathbb{P}^1}(-\mathbf{E})\big)$. The proposition is therefore a consequence of the following lemma. □

Lemma 1. *Let $E \subset \mathbb{P}^1$ be finite, let ϑ be a smooth vector field on $\mathbb{P}^1$, holomorphic near E, let ν be the infinitesimal Beltrami differential $\bar\partial\vartheta$ and let q be a meromorphic quadratic differential on $\mathbb{P}^1$ which is holomorphic outside E (and may have multiple poles on E). Then,*

$$\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{P}^1} q \wedge \nu = \sum_{z\in E} \mathrm{Res}_z q \otimes \vartheta.$$

Remark. Note that ν vanishes near E, and so, the product $q \wedge \nu$ is a smooth area form on $\mathbb{P}^1$. In particular $\displaystyle\int_{\mathbb{P}^1} q \wedge \nu$ is well-defined.

Proof. Let $U \subset \mathbb{P}^1$ be the complement of pairwise sufficiently small disjoint disks around points of E, so that ϑ is holomorphic (and thus ν vanishes) outside U. Since q is meromorphic

$$\bar\partial(q \otimes \vartheta) = -q \wedge \bar\partial\vartheta.$$

According to Stokes's Theorem and to the Residue Theorem:

$$\int_U q \wedge \nu = \int_U q \wedge \bar\partial\vartheta \underset{\text{Stokes}}{=} -\oint_{\partial U} q \otimes \vartheta = 2\pi\mathrm{i} \sum_{z \in E} \mathrm{Res}_z q \otimes \vartheta$$

(the change of sign in the last equality is due to the fact that ∂U is oriented as the boundary of U, not as the boundary of the small disks around the points of E). $\square$

2. Maps between Teichmüller spaces

2.1. **A forgetful map.** If $A \subseteq B \subset S$ are finite subsets, there is a canonical analytic submersion

$$\varpi_{A,B} : \mathrm{Teich}(S,B) \to \mathrm{Teich}(S,A)$$

given by forgetting the points in $B - A$. If a complex structure $\mathfrak{s} \in \mathrm{Complex}(S)$ represents a point $\tau \in \mathrm{Teich}(S,B)$ and X is the Riemann surface S equipped with $\mathfrak{s}$, then $\varpi_{A,B}(\tau)$ can be represented by the same complex structure $\mathfrak{s}$ (the equivalence relation is now only isotopy relative to A). The coderivative of the analytic map $\varpi_{A,B}$ at τ is the inclusion

$$D^\bigstar_\tau \varpi_{A,B} = \iota_{A,B} : \mathcal{Q}(X,A) \hookrightarrow \mathcal{Q}(X,B).$$

We shall use the notation ϖ instead of $\varpi_{A,B}$ when there is no possible confusion.

2.2. **A pullback map.** The following definition of a pullback map on Teichmüller spaces is due to Thurston (see [2]). Let $f : S_1 \to S_2$ be an orientation-preserving ramified covering. It is not difficult to see that f induces a map

$$f^* : \mathrm{Complex}(S_2) \to \mathrm{Complex}(S_1).$$

Indeed, let $\mathfrak{s}$ be a complex structure on S_2. Clearly, if $s \in S_1$ is not a critical point of f and if $\varphi \in \mathfrak{s}$ is a local coordinate on S_2 near $f(s)$, then $\varphi \circ f$ is a local coordinate on S_1 near s. If $s \in S_1$ is a critical point of f with local degree d_s and if $\varphi \in \mathfrak{s}$ is a local coordinate on S_2 near $f(s)$ with $\varphi\big(f(s)\big) = 0$, then there is a local coordinate ψ on S_1 near s such that $\psi^{d_s} = \varphi \circ f$ near s. These local coordinates fit together to define a complex structure $f^*(\mathfrak{s})$ on S_1.

In view of the homotopy-lifting property, if

- $B \subset S_2$ is a finite subset containing the critical values of f, and
- $A \subseteq f^{-1}(B) \subset S_1$,

then $f^* : \mathrm{Complex}(S_2) \to \mathrm{Complex}(S_1)$ descends to a well-defined map $\sigma_{f,A,B}$ between the corresponding Teichmüller spaces:

$$\begin{array}{ccc} \mathrm{Complex}(S_2) & \xrightarrow{f^*} & \mathrm{Complex}(S_1) \\ \downarrow & & \downarrow \\ \mathrm{Teich}(S_2,B) & \xrightarrow{\sigma_{f,A,B}} & \mathrm{Teich}(S_1,A). \end{array}$$

Indeed, let $\varphi_t : S_2 \to S_2$ be an isotopy to the identity relative to B. Since B contains the critical values of f, there is an isotopy $\psi_t : S_1 \to S_1$ to the identity relative to $f^{-1}(B)$ such that $f \circ \psi_t = \varphi_t \circ f$. And since $A \subseteq f^{-1}(B)$ this is an isotopy to the identity relative to A. The map $\sigma_{f,A,B}$ is known as a *pullback map* induced by f.

We shall use the notation σ_f instead of $\sigma_{f,A,B}$ when there is no possible confusion.

Lemma 2. *The map* $\sigma_f : \mathrm{Teich}(S_2, B) \to \mathrm{Teich}(S_1, A)$ *is analytic.*

Proof. Let $\mathfrak{s} \in \mathrm{Complex}(S_2)$ be any complex structure on S_2. Denote by X_2 the Riemann surface S_2 equipped with the complex structure $\mathfrak{s}$ and by X_1 the Riemann surface S_1 equipped with the complex structure $f^*\mathfrak{s}$. Then, $f : X_1 \to X_2$ is an analytic map.

The pullback map on Beltrami forms

$$\mathrm{Bel}(X_2) \ni \mu \mapsto f^*\mu \in \mathrm{Bel}(X_1)$$

is $\mathbb{C}$-linear, in particular analytic. It projects to σ_f which is therefore analytic. □

We now wish to understand the coderivative of the pullback map σ_f. This will require defining a pushforward operator on quadratic differentials.

Definition 3. *If* $f : X \to Y$ *is a finite cover of Riemann surfaces, and* q *is a holomorphic quadratic differential on* X*, then*

$$f_* q = \sum_g g^* q$$

where the sum ranges over the inverse branches of f*.*

Note that if q is integrable on X, then

$$\int_Y |f_* q| = \int_Y \left| \sum_g g^* q \right| \le \int_Y \sum_g |g^* q| = \int_X |q|. \tag{3}$$

As a consequence, if $f : X \to Y$ is a ramified cover, $\mathcal{S}_f$ is the set of critical values and q is integrable on X, then the holomorphic quadratic differential $f_* q$ on $Y - \mathcal{S}_f$ is integrable, hence meromorphic on Y with at worst simple poles on $\mathcal{S}_f$.

Let $\mathfrak{s} \in \mathrm{Complex}(S_2)$ represent a point $\tau \in \mathrm{Teich}(S_2, B)$. Recall that the cotangent space to $\mathrm{Teich}(S_2, B)$ is $\mathcal{Q}(X_2, B)$ where X_2 is the Riemann surface S_2 equipped with the complex structure $\mathfrak{s}$ and the cotangent space to $\mathrm{Teich}(S_1, A)$ is $\mathcal{Q}(X_1, A)$ where X_1 is the Riemann surface S_1 equipped with the complex structure $f^*\mathfrak{s}$.

Proposition 2. *The coderivative of* σ_f *at* τ *is the linear map*

$$D_\tau^\bigstar \sigma_f = f_* : \mathcal{Q}(X_1, A) \to \mathcal{Q}(X_2, B).$$

It was necessary for B to contain the critical values of f for this map to be well defined.

Proof. The pullback map $f^* : \mathrm{Bel}(X_2) \to \mathrm{Bel}(X_1)$ on Beltrami forms induces the pullback map $\sigma_f : \mathrm{Teich}(S_2, B) \to \mathrm{Teich}(S_1, A)$. Its derivative is the pullback map $f^* : \mathrm{bel}(X_2) \to \mathrm{bel}(X_1)$ on infinitesimal Beltrami forms and

$$\langle q, f^*\nu \rangle = \frac{1}{2\pi \mathrm{i}} \int_{X_1} q \wedge f^*\nu = \frac{1}{2\pi \mathrm{i}} \int_{X_2} f_* q \wedge \nu = \langle f_* q, \nu \rangle.$$ □

Equation (3) above shows that $f_* : \mathcal{Q}(X_1, A) \to \mathcal{Q}(X_2, B)$ is a weak contraction, i.e., $\|f_* q\| \le \|q\|$. In fact, it is not difficult to prove that

$$\|f_* q\| = \|q\| \quad \Longleftrightarrow \quad f^* f_* q = D \cdot q, \tag{4}$$

where D is the degree of f.

3. The deformation space

From now on, we assume that S is of genus 0, i.e., a topological sphere. By the Uniformization Theorem, for every complex structure $\mathfrak{s}$ on S, the Riemann surface S equipped with $\mathfrak{s}$ is isomorphic to the Riemann sphere $\mathbb{P}^1$. Thus, we may regard Teichmüller spaces modeled on S as equivalence classes of homeomorphisms $\varphi : S \to \mathbb{P}^1$.

We assume that $f : S \to S$ is an orientation-preserving ramified covering of degree $D \ge 2$. The domain and range of f are the same surface S, which allow us to regard $f : S \to S$ as a dynamical system.

We denote by $\mathcal{C}_f$ the set of critical points of f and by $\mathcal{S}_f$ the set of critical values of f. We assume that

- $B \subset S$ is a finite set containing $\mathcal{S}_f$ and
- $A \subseteq B \cap f^{-1}(B)$ contains at least 3 points.

Let $\varpi : \mathrm{Teich}(S, B) \to \mathrm{Teich}(S, A)$ and $\sigma_f : \mathrm{Teich}(S, B) \to \mathrm{Teich}(S, A)$ be the forgetful and pullback maps defined above.

Definition 4. *The deformation space* $\mathrm{Def}_A^B(f)$ *is the analytic set*

$$\mathrm{Def}_A^B(f) = \bigl\{\tau \in \mathrm{Teich}(S, B)\ ;\ \varpi(\tau) = \sigma_f(\tau)\bigr\}.$$

Note that if $\varphi : S \to \mathbb{P}^1$ represents a point $\tau \in \mathrm{Def}_A^B(f)$, then, there is a unique $\psi : S \to \mathbb{P}^1$ representing $\varpi(\tau) = \sigma_f(\tau)$, coinciding with φ on A, such that the map $F = \varphi \circ f \circ \psi^{-1}$ is a rational map of degree D; we have the following commutative diagram:

$$\begin{array}{ccc} (S, A) & \xrightarrow{\ \psi\ } & \bigl(\mathbb{P}^1, \psi(A)\bigr) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ (S, B) & \xrightarrow[\ \varphi\ ]{} & \bigl(\mathbb{P}^1, \varphi(B)\bigr) \end{array} \quad \text{with} \quad \begin{array}{l} \varphi|_A = \psi|_A \text{ and} \\ \varphi \text{ isotopic to } \psi \text{ relative to } A. \end{array} \tag{5}$$

Any point of $\mathrm{Def}_A^B(f)$ is represented by triple (φ, ψ, F) as in this diagram.

For $a \in A$ we have

$$F\bigl(\varphi(a)\bigr) = F\bigl(\psi(a)\bigr) = \varphi\bigl(f(a)\bigr).$$

In particular, φ sends cycles of f contained in A to cycles of F. More generally, we have the following result.

Lemma 3. *Let* (φ, ψ, F) *represent a point in* $\mathrm{Def}_A^B(f)$.

- *If c is a critical point of f, then $\psi(c)$ is a critical point of F.*
- *If v is a critical value of f, then $\varphi(v)$ is a critical value of F.*
- *If $x \in f^{-1}(B)$ and $y \in B$ are such that $f^{\circ n}(x) = y$ for some integer $n \ge 1$ and $f^{\circ k}(x) \in A$ for $k \in [1, n-1]$, then $F^{\circ n}\bigl(\psi(x)\bigr) = \varphi(y)$.*

Proof. Let c be a critical point of f. Then, c is a critical point of $\varphi\circ f = F\circ\psi$. Since f is a homeomorphism, $\psi(c)$ is a critical point of F.

Let v be a critical value of f. Then, $v = f(c)$ for some critical point c. Thus, $\varphi(v) = F\bigl(\psi(c)\bigr)$ is a critical value of F associated to the critical point $\psi(c)$.

Let $x\in f^{-1}(B)$ and $y\in B$ be such that $f^{\circ n}(x) = y$ for some integer $n\geq 1$ and $f^{\circ k}(x)\in A$ for $k\in[1,n-1]$. For $k=1$, we have that

$$F^{\circ 1}\circ\psi(x) = \varphi\circ f^{\circ 1}(x)$$

and we see by induction that for $k\in[2,n]$

$$\begin{aligned} F^{\circ k}\circ\psi(x) = F\circ F^{\circ k-1}\circ\psi(x) &= F\circ\varphi\circ f^{\circ k-1}(x)\\ &= F\circ\psi\circ f^{\circ k-1}(x)\\ &= \varphi\circ f\circ f^{\circ k-1}(x) = \varphi\circ f^{\circ k}(x). \end{aligned}$$

In particular, for $k=n$, we have that $F^{\circ n}\bigl(\psi(x)\bigr) = \varphi\bigl(f^{\circ n}(x)\bigr) = \varphi(y)$ as required. □

Remark. When f is postcritically finite and $A = B$ is the postcritical set of f, then the pullback map $\sigma_f : \mathrm{Teich}(S,A)\to\mathrm{Teich}(S,A)$ is an endomorphism and $\mathrm{Def}_A^B(f)$ is the set of its fixed points. If f is not a flexible Lattès example (see below), then $\mathrm{Def}_A^B(f)$ is either empty or consists of a single point. Characterizing the cases where it is nonempty is precisely the object of [2].

Recall that flexible Lattès examples are obtained as follows: let $\Gamma\subset\mathbb{C}$ be a discrete subgroup isomorphic to $\mathbb{Z}^2$, and $\tilde\Gamma\subset\mathrm{Aut}(\mathbb{C})$ be the group generated by translations by elements of Γ and $z\mapsto -z$, so that $\mathbb{C}/\tilde\Gamma$ is isomorphic to $\mathbb{P}^1$, using for instance the Weierstrass $\wp$-function. Then for any (rational) integer n with $|n|\geq 2$ and any element $m\in\frac{1}{2}\Gamma$, the map $z\mapsto nz+m$ induces a rational function $L:\mathbb{C}/\tilde\Gamma\to\mathbb{C}/\tilde\Gamma$ of degree n^2. Any rational map which is analytically conjugate to such a map L is called a flexible Lattès example. We use the term flexible because Γ depends on one parameter, providing a 1-parameter family of rational functions with topologically conjugate dynamics.

We say that $f : S\to S$ is a topological flexible Lattès example if f is postcritically finite with postcritical set $\mathcal{P}_f$ and if there are homeomorphisms $\varphi : S\to\mathbb{P}^1$ and $\psi : S\to\mathbb{P}^1$ such that

- ψ and φ agree on $\mathcal{P}_f$,
- ψ is isotopic to φ relative to $\mathcal{P}_f$ and
- $\varphi\circ f\circ\psi^{-1} : \mathbb{P}^1\to\mathbb{P}^1$ is a flexible Lattès example.

Recall that we use the notation $\nabla_F = \mathrm{I} - F_*$. In the following theorem, we regard it as the linear map

$$\nabla_F : \mathcal{Q}\bigl(\mathbb{P}^1,\psi(A)\bigr)\to\mathcal{Q}\bigl(\mathbb{P}^1,\varphi(B)\bigr).$$

Theorem 1. *Assume f is not a topological flexible Lattès example or A does not contain the postcritical set of f. Then, the space $\mathrm{Def}_A^B(f)$ is either empty, or a smooth analytic submanifold of $\mathrm{Teich}(S,B)$ of dimension $\mathrm{card}(B-A)$ whose tangent space at a point τ is the kernel of the linear map*

$$D_\tau\varpi - D_\tau\sigma_f : T_\tau\mathrm{Teich}(S,B)\to T_{\varpi(\tau)}\mathrm{Teich}(S,A).$$

If (φ,ψ,F) *represents a point* $\tau\in \mathrm{Def}_A^B(f)$ *as in (5), then we have the canonical identification*

$$T^\bigstar_\tau \mathrm{Def}_A^B(f)\simeq \mathcal{Q}\bigl(\mathbb{P}^1,\varphi(B)\bigr)/\nabla_F\mathcal{Q}\bigl(\mathbb{P}^1,\psi(A)\bigr).$$

Proof. We shall prove this theorem by using the Implicit Function Theorem. Let (φ,ψ,F) represent a point $\tau\in\mathrm{Def}_A^B(f)$ as in (5). Recall that

$$D^\bigstar_\tau\varpi = \mathrm{I}:\mathcal{Q}\bigl(\mathbb{P}^1,\psi(A)\bigr)\to\mathcal{Q}\bigl(\mathbb{P}^1,\varphi(B)\bigr)$$

and

$$D^\bigstar_\tau\sigma_f = F_*:\mathcal{Q}\bigl(\mathbb{P}^1,\psi(A)\bigr)\to\mathcal{Q}\bigl(\mathbb{P}^1,\varphi(B)\bigr);$$

When F is not a flexible Lattès example or $\psi(A)$ does not contain the postcritical set of F, then the linear map

$$D^\bigstar_\tau\varpi - D^\bigstar_\tau\sigma_f = \mathrm{I}-F_* = \nabla_F:\mathcal{Q}\bigl(\mathbb{P}^1,\psi(A)\bigr)\to\mathcal{Q}\bigl(\mathbb{P}^1,\varphi(B)\bigr)$$

is injective (Proposition 3 below). So, its transpose is surjective. According to the Implicit Function Theorem, the deformation space $\mathrm{Def}_A^B(f)$ is smooth at τ and its cotangent space is the quotient space:

$$T^\bigstar_\tau\mathrm{Def}_A^B(f) = \mathcal{Q}\bigl(\mathbb{P}^1,\varphi(B)\bigr)/\nabla_F\bigl(\mathcal{Q}\bigl(\mathbb{P}^1,\psi(A)\bigr)\bigr).$$

Since ∇_F is injective on $\mathcal{Q}\bigl(\mathbb{P}^1,\psi(A)\bigr)$, the dimension of $T^\bigstar_\tau\mathrm{Def}_A^B(f)$ is

$$\dim\bigl(\mathcal{Q}\bigl(\mathbb{P}^1,\varphi(B)\bigr)\bigr)-\dim\bigl(\mathcal{Q}\bigl(\mathbb{P}^1,\psi(A)\bigr)\bigr)=\mathrm{card}(B-A).$$ □

Proposition 3. *Let* $F\in\mathbf{Rat}_D$ *with* $D\geq 2$ *and* $A\subset\mathbb{P}^1$ *be a finite set. Assume* F *is not a flexible Lattès example or* A *does not contain the postcritical set of* F*. Then,* ∇_F *is injective on* $\mathcal{Q}(\mathbb{P}^1,A)$*.*

Proof. The proof of this algebraic statement is transcendental. Let $q\in\mathcal{Q}(\mathbb{P}^1,A)$ be in the kernel of ∇_F. Then, $F_*q=q$, and thus $\|F_*q\|=\|q\|$. According to (4), it follows that

$$F^*q = F^*F_*q = D\cdot q.$$

In particular, the set $Z\subseteq A$ of poles of q satisfies $F(Z)\subseteq Z$ and $F^{-1}(Z)\subseteq Z\cup\mathcal{C}_F$, with $\mathcal{C}_F$ the set of critical points of F. Then

$$\mathrm{card}(Z)+(2D-2)\geq\mathrm{card}(Z)+\mathrm{card}(\mathcal{C}_F)\geq\mathrm{card}\bigl(F^{-1}(Z)\bigr)\geq D\cdot\mathrm{card}(Z)-(2D-2).$$

This implies $\mathrm{card}(Z)\leq 4$. Hence

- either $q=0$
- or $\mathrm{card}(Z)=4$, all critical points of F are simple, and $F(\mathcal{C}_F)\cap\mathcal{C}_F=\emptyset$.

In the latter case, we have a Lattès example: the orbifold of F is 4 points marked 2, so the orbifold universal covering space is a copy of $\mathbb{C}$, with covering group Γ generated by a lattice and $z\mapsto -z$. The quadratic differential q lifts to $a\,\mathrm{d}z^2$ for some $a\in\mathbb{C}$. Further F is a covering map of orbifolds. Thus, it lifts to a covering map $\widetilde F:\mathbb{C}\to\mathbb{C}$, hence an affine map $z\mapsto\alpha z+\beta$, with $\widetilde F\Gamma\widetilde F^{-1}\subset\Gamma$. By considering areas, we can see that $|\alpha|^2=D$. The condition $F_*q=q$ implies that the multiplier α of the affine map $\widetilde F$ is real so that α is also an integer n. The condition $\widetilde F\Gamma\widetilde F^{-1}\subset\Gamma$ then implies that $\beta\in\frac{1}{2}\Gamma$. □

4. From the deformation space to the moduli space of rational maps

From now on, we will work under the assumption that $\mathrm{Def}_A^B(f)$ is non empty. Recall that every point $\tau \in \mathrm{Def}_A^B(f)$ can be represented by a triple (φ, ψ, F) as in diagram (5) with $F : \mathbb{P}^1 \to \mathbb{P}^1$ a rational map. The map $\varphi \mapsto F$ is well defined from the space of homeomorphisms $S \to \mathbb{P}^1$ to the space of rational maps $\mathbf{Rat}_D$ and we shall now see that it induces a map

$$\underline{\Phi} : \mathrm{Def}_A^B(f) \to \mathrm{Rat}_D.$$

Proposition 4. *Let (φ_0, ψ_0, F_0) and (φ_1, ψ_1, F_1) represent the same point τ in $\mathrm{Def}_A^B(f)$. Then, F_0 and F_1 are conjugate by the (unique) Möbius transformation M which agrees with $\varphi_1 \circ \varphi_0^{-1}$ on $\varphi_0(B)$, i.e., $F_1 \circ M = M \circ F_0$.*

Proof. Replacing φ_0 by $M \circ \varphi_0$ if necessary, it is enough to consider the case where φ_0 and φ_1 agree on B and φ_0 is isotopic to φ_1 relative to B. We must prove that in this case, $F_0 = F_1$. Let $(\varphi_t)_{t\in[0,1]}$ be an isotopy relative to B between φ_0 and φ_1 and let $(\psi_t)_{t\in[0,1]}$ and $(F_t)_{t\in[0,1]}$ be the corresponding maps. Then, the set of critical values of F_t does not depend on t. It follows that the identity map lifts to a family of analytic maps which are Möbius transformations $(M_t)_{t\in[0,1]}$ such that $M_0 = \mathrm{I}$ and $F_t \circ M_t = F_0$ for all $t \in [0, 1]$. Note that M_t has to agree with ψ_t on A, thus fixes A. Since $|A| \geq 3$, we see that $M_t = \mathrm{I}$, and thus $F_t = F_0$, for all $t \in [0, 1]$. □

Proposition 4 defines a map $\underline{\Phi} : \mathrm{Def}_A^B(f) \to \mathrm{Rat}_D$. We will now show that this map is analytic. It isn't quite clear what this means since Rat_D is not a manifold. Proposition 5 is clearly a reasonable interpretation.

Proposition 5. *There exist analytic mappings $\Phi : \mathrm{Def}_A^B(f) \to \mathbf{Rat}_D$ that induce $\underline{\Phi} : \mathrm{Def}_A^B(f) \to \mathrm{Rat}_D$.*

Proof. Let a_1, a_2, a_3 be three distinct points in A and z_1, z_2, z_3 be three distinct points in $\mathbb{P}^1$. Let $\mathfrak{s} \in \mathrm{Complex}(S)$ be a complex structure on S, X_2 be the Riemann surface S equipped with the complex structure $\mathfrak{s}$ and X_1 be the Riemann surface S equipped with the complex structure $f^*(\mathfrak{s})$. Note that $f : X_1 \to X_2$ is analytic.

The Uniformization Theorem and the Measurable Riemann Mapping Theorem imply that for any Beltrami form $\mu \in \mathrm{Bel}(X_2)$, there is a quasiconformal homeomorphism $\varphi^\mu : X_2 \to \mathbb{P}^1$ satisfying the Beltrami equation $\bar\partial\varphi^\mu = \partial\varphi^\mu \circ \mu$, and this homeomorphism is uniquely determined by requiring $\varphi^\mu(a_i) = z_i$ for $i = 1, 2, 3$. Similarly, there is a unique $\psi^\mu : X_1 \to \mathbb{P}^1$ satisfying the Beltrami equation $\bar\partial\psi^\mu = \partial\psi^\mu \circ (f^*\mu)$, normalized by $\psi^\mu(a_i) = z_i$ for $i = 1, 2, 3$.

The map $F^\mu = \varphi^\mu \circ f \circ (\psi^\mu)^{-1} : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map and we must show that the map

$$\mathrm{Bel}(X_2) \ni \mu \mapsto F^\mu \in \mathbf{Rat}_D$$

is analytic. The reason why this is not completely obvious is that, even though φ^μ and ψ^μ depend analytically on μ, $(\psi^\mu)^{-1}$ does not necessarily depend analytically on μ.

So, assume μ_t is an analytic family of Beltrami forms and set

$$\varphi_t := \varphi^{\mu_t}, \quad \psi_t := \psi^{\mu_t} \quad \text{and} \quad F_t := F^{\mu_t}.$$

Differentiating $F_t\circ\psi_t(z)=\varphi_t\circ f(z)$ with respect to $\bar t$ in the sense of distributions yields

$$\frac{\partial F_t}{\partial \bar t}\Big|_{\psi_t(z)}+\frac{\partial F_t}{\partial z}\Big|_{\psi_t(z)}\cdot\underbrace{\frac{\partial \psi_t}{\partial \bar t}\Big|_z}_{=0}+\underbrace{\frac{\partial F_t}{\partial \bar z}\Big|_{\psi_t(z)}}_{=0}\cdot\frac{\partial \overline{\psi_t}}{\partial \bar t}\Big|_z=\underbrace{\frac{\partial \varphi_t}{\partial \bar t}\Big|_{f(z)}}_{=0}. \qquad \square$$

5. Cross-ratios and orbit relations

Up to now, we worked with a branched mapping $f:S\to S$ so as to avoid choosing a basepoint and having to show that the construction was independent of the basepoint. From here on, we choose a basepoint. This amounts to choosing a rational map $f:\mathbb{P}^1\to\mathbb{P}^1$, a subset $B\subset\mathbb{P}^1$ containing the critical values of f, and a set $A\subseteq B\cap f^{-1}(B)$ containing at least three points. We shall use the notation $\mathbf{0}$ to denote indifferently the basepoint of $\mathrm{Teich}(\mathbb{P}^1,A)$ or $\mathrm{Teich}(\mathbb{P}^1,B)$ given by the standard complex structure on $\mathbb{P}^1$.

Various choices of A will be interesting but obviously, the largest possible choice is

$$A'=B\cap f^{-1}(B).$$

A tangent vector $v\in T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1,B)$ may be represented by an analytic curve of homeomorphisms $(\varphi_t:\mathbb{P}^1\to\mathbb{P}^1)_{t\in\mathbb{D}}$ such that $\nu=\bar\partial\dot\varphi$ is an infinitesimal Beltrami differential whose class in $T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1,B)$ is v. As t varies, the points $\varphi_t(b)$, $b\in B$, move. Movements of individual points do not mean much, since φ_t is only defined up to postcomposition by a Möbius transformation. But cross-ratios of quadruples of points are meaningful on $\mathrm{Teich}(\mathbb{P}^1,B)$. In fact, more functions like the above are meaningful, for example the cross-ratios of quadruple of points in $f_t^{-1}\big(\varphi_t(B)\big)$. Even if for $t=0$ some points of $f^{-1}\big(B\big)$ coincide with points of B, there is no reason to expect that they coincide for $t\neq 0$ and these cross-ratios can be used to measure how fast these points move apart. Appropriate derivatives of these cross-ratios are elements of

$$T^\bigstar_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1,B)\simeq\mathcal{Q}(\mathbb{P}^1,B).$$

5.1. Cross-ratios.

Definition 5. *Given 4 distinct points z_1,z_2,z_3,z_4 in $\mathbb{P}^1$, we denote by $\vec z$ the quadruple $\vec z=(z_1,z_2,z_3,z_4)$ and we let $[z_1:z_2:z_3:z_4]$ be the cross-ratio*

$$[z_1:z_2:z_3:z_4]=M(z_4)$$

where $M:\mathbb{P}^1\to\mathbb{P}^1$ is the Möbius transformation which sends z_1,z_2,z_3 to respectively $0,\infty,1$.

Suppose $\mathrm{card}(A')\geq 4$ and let a_1, a_2, a_3 and a_4 be 4 distinct points in A'. Denote by $\hat\sigma_f$ the pullback map

$$\hat\sigma_f=\sigma_{f,f^{-1}(B),B}:\mathrm{Teich}(\mathbb{P}^1,B)\to\mathrm{Teich}\big(\mathbb{P}^1,f^{-1}(B)\big).$$

Let $\varphi:\mathbb{P}^1\to\mathbb{P}^1$ represent a point $\tau\in\mathrm{Teich}(\mathbb{P}^1,B)$ and $\psi:\mathbb{P}^1\to\mathbb{P}^1$ represent the point $\hat\sigma_f(\tau)$ in $\mathrm{Teich}\big(\mathbb{P}^1,f^{-1}(B)\big)$ (since we do not require $\tau\in\mathrm{Def}_A^B(f)$, the representative ψ of $\hat\sigma_f(\tau)$ is not uniquely determined by φ). Since the points a_1,a_2,a_3,a_4 are in B, the cross-ratio

$$\big[\varphi(a_1):\varphi(a_2):\varphi(a_3):\varphi(a_4)\big]$$

depends only on τ, not on φ. Since the points a_1, a_2, a_3, a_4 are in $f^{-1}(B)$, the cross-ratio

$$\big[\psi(a_1) : \psi(a_2) : \psi(a_3) : \psi(a_4)\big]$$

depends only on $\hat\sigma_f(\tau)$, thus only on τ.

Definition 6. *Given a_1, a_2, a_3, a_4 in A', we denote by $\kappa_{\vec a} : \mathrm{Teich}(\mathbb{P}^1, B) \to \mathbb{C} - \{0\}$ the function defined by*

$$\kappa_{\vec a}(\tau) = \frac{\big[\psi(a_1) : \psi(a_2) : \psi(a_3) : \psi(a_4)\big]}{\big[\varphi(a_1) : \varphi(a_2) : \varphi(a_3) : \varphi(a_4)\big]}$$

where φ represents $\tau \in \mathrm{Teich}(\mathbb{P}^1, B)$ and ψ represents $\hat\sigma_f(\tau) \in \mathrm{Teich}\big(\mathbb{P}^1, f^{-1}(B)\big)$.

Note that if a_1, a_2, a_3, a_4 belong to $A \subseteq A'$, then $\kappa_{\vec a} \equiv 1$ on $\mathrm{Def}_A^B(f)$. In fact, we shall see later that if we denote by $a_1, \ldots, a_N$ the points of A, then the deformation space $\mathrm{Def}_A^B(f)$ is locally defined by the equations:

$$\kappa_{a_1,a_2,a_3,a_j}(\tau) = 1, \quad j \in [4, N].$$

5.2. **Logarithmic derivatives of κ's.** Note that if $q \in \mathcal{Q}(\mathbb{P}^1, A')$, then the poles of q and f_*q are simple and belong to B, so

$$\nabla_f q \in \mathcal{Q}(\mathbb{P}^1, B) \simeq T_{\mathbf{0}}^{\star}\mathrm{Teich}(\mathbb{P}^1, B).$$

Let $\vec a$ be a quadruple of distinct points in A'. We shall now identify the derivative $D_{\mathbf{0}} \log \kappa_{\vec a}$ as $\nabla_f q$ for a particular quadratic differential $q \in \mathcal{Q}(\mathbb{P}^1, A')$.

Definition 7. *Given 4 distinct points z_1, z_2, z_3, z_4 in $\mathbb{P}^1$, we let $q_{\vec z}$ be the quadratic differential defined by*

$$q_{\vec z} = \omega_{z_1,z_2} \otimes \omega_{z_3,z_4}$$

where $\omega_{x,y}$ is the meromorphic 1-form on $\mathbb{P}^1$ which has simple poles at x and y with residue 1 at x and residue -1 at y.

When x, y, z_1, z_2, z_3, z_4 are in $\mathbb{C}$,

$$\omega_{x,y} = \frac{x - y}{(z - x)(z - y)}\, \mathrm{d}z$$

and

$$q_{\vec z} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)}\, \mathrm{d}z^2.$$

Proposition 6. *Let $\vec a$ be a quadruple of distinct points in A'. The logarithmic derivative $D_{\mathbf{0}} \log \kappa_{\vec a}$ is $\nabla_f q_{\vec a} \in T_{\mathbf{0}}^{\star}\mathrm{Teich}(\mathbb{P}^1, B)$.*

Proof. Let $\mathrm{CR} : \mathrm{Teich}(\mathbb{P}^1, B) \to \mathbb{C} - \{0, 1\}$ be the function defined by

$$\mathrm{CR}\big([\varphi]\big) = \big[\varphi(a_1) : \varphi(a_2) : \varphi(a_3) : \varphi(a_4)\big].$$

The following result is well-known in Teichmüller theory.

Lemma 4. *The logarithmic derivative $D_{\mathbf{0}} \log \mathrm{CR}$ is $-q_{\vec a} \in T_{\mathbf{0}}^{\star}\mathrm{Teich}(\mathbb{P}^1, B)$.*

Proof. Let $\nu \in \mathrm{bel}(\mathbb{P}^1)$ be an infinitesimal Beltrami differential representing a tangent vector $v \in T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1, B)$ and vanishing in a neighborhood of B. For $t \in \mathbb{C}$ sufficiently close to 0, set $\mu_t = t \cdot \nu \in \mathrm{Bel}(\mathbb{P}^1)$ and let φ_t be the solution of the Beltrami equation $\bar\partial \varphi_t = \partial \varphi_t \circ \mu_t$ which fixes a_1, a_2 and a_3. Let θ be the restriction of $\dot\varphi$ on B. This is a vector field on B representing the tangent vector $v \in T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1, B)$.

Since φ_t fixes a_1, a_2 and a_3, the vector field θ vanishes at those points. Let M be the Möbius transformation which sends a_1, a_2, a_3 to respectively $0, \infty, 1$.

On the one hand, $q_{\vec a} \otimes \theta$ is the 1-form $\omega_{a_1,a_2}(\theta) \cdot \omega_{a_3,a_4}$ whose residue at a_4 is

$$\mathrm{Res}_{a_4} q_{\vec a} \otimes \theta = \omega_{a_1,a_2}\big(a_4; \theta(a_4)\big) \cdot \mathrm{Res}_{a_4}\omega_{a_3,a_4} = -\omega_{a_1,a_2}\big(a_4;\theta(a_4)\big).$$

On the other hand, we have $\mathrm{CR}\big([\varphi_t]\big) = M\big(\varphi_t(a_4)\big)$ and so

$$D_{\mathbf{0}} \log \mathrm{CR}(v) = \frac{\mathrm{d}M}{M}\big(a_4; \dot\varphi(a_4)\big) = \omega_{a_1,a_2}\big(a_4;\theta(a_4)\big).$$

Indeed, the 1-form

$$\frac{\mathrm{d}M}{M} = M^*\left(\frac{\mathrm{d}z}{z}\right) = M^*(\omega_{0,\infty})$$

is meromorphic with simple poles at a_1 and a_2, with residue 1 at a_1 and with residue -1 at a_2; thus it is ω_{a_1,a_2}.

The result follows since

$$\langle q_{\vec a}, \theta\rangle = \sum_{b\in B} \mathrm{Res}_b q_{\vec a} \otimes \theta = \mathrm{Res}_{a_4} q_{\vec a} \otimes \theta = -D_{\mathbf{0}} \log \mathrm{CR}(v). \qquad \square$$

By an abuse of notations, denote by $\mathrm{CR} : \mathrm{Teich}\big(\mathbb{P}^1, f^{-1}(B)\big) \to \mathbb{C} - \{0,1\}$ the function defined by

$$\mathrm{CR}\big([\psi]\big) = \big[\psi(a_1) : \psi(a_2) : \psi(a_3) : \psi(a_4)\big].$$

Then, $\kappa_{\vec a} = \dfrac{\mathrm{CR}\circ \hat\sigma_f}{\mathrm{CR}}$ and so,

$$D_{\mathbf{0}} \log \kappa_{\vec a} = D_{\mathbf{0}} \log \mathrm{CR} \circ D_{\mathbf{0}}\hat\sigma_f - D_{\mathbf{0}} \log \mathrm{CR}.$$

Since $D_{\mathbf{0}} \log \mathrm{CR} = -q_{\vec a}$ and since $D_{\mathbf{0}}^\bigstar \hat\sigma_f = f_*$, we see that

$$D_{\mathbf{0}} \log \kappa_{\vec a} = f_*(-q_{\vec a}) - (-q_{\vec a}) = \nabla_f q_{\vec a}. \qquad \square$$

5.3. **Independence of κ's.** The deformation space $\mathrm{Def}_A^B(f) \subseteq \mathrm{Teich}(\mathbb{P}^1, B)$ was defined by the equation $\sigma_f = \varpi$. We will now analyse this equation in terms of the κ's. This is a consequence of our first transversality result.

Theorem 2. *Let $B \subset \mathbb{P}^1$ be a finite set containing the critical values of a rational map $f \in \mathbf{Rat}_D$. If $B \cap f^{-1}(B) = \{a_1, \ldots, a_M\}$ with $M \geq 4$ and if f is not a flexible Lattès example, then the map*

$$\vec\kappa = \begin{pmatrix} \kappa_{a_1,a_2,a_3,a_4} \\ \kappa_{a_1,a_2,a_3,a_5} \\ \vdots \\ \kappa_{a_1,a_2,a_3,a_M}\end{pmatrix} : \mathrm{Teich}(\mathbb{P}^1, B) \to \big(\mathbb{C} - \{0\}\big)^{M-3}$$

is a submersion at $\mathbf{0}$.

Proof. We must show that the derivative $D_{\mathbf{0}}\vec\kappa : T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1, B) \to \mathbb{C}^{M-3}$ is surjective. The quadratic differentials q_{a_1,a_2,a_3,a_j}, $j \in [4, M]$, are linearly independent. They belong to $\mathcal{Q}(\mathbb{P}^1, A')$. Since f is not a flexible Lattès example, ∇_f is injective on $\mathcal{Q}(\mathbb{P}^1, A')$. Thus, the logarithmic derivatives

$$D_{\mathbf{0}} \log \kappa_{a_1,a_2,a_3,a_j} = \nabla_f q_{a_1,a_2,a_3,a_j}, \quad j \in [4, M]$$

are linearly independent. So, the derivatives $D_{\mathbf{0}}\kappa_{a_1,a_2,a_3,a_j}$, $j \in [4, M]$, generate a vector space of dimension $M-3$ and the rank of $D_{\mathbf{0}}\vec{\kappa} : T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1, B) \to \mathbb{C}^{M-3}$ is maximal. □

In particular, the sets

$$X_j = \big\{\tau \in \mathrm{Teich}(\mathbb{P}^1, B) \ : \ \kappa_{a_1,a_2,a_3,a_j}(\tau) = 1\big\}, \quad j \in [4, M]$$

are smooth and transverse analytic submanifolds of $\mathrm{Teich}(\mathbb{P}^1, B)$ of codimension 1. If $A = \{a_1, \ldots, a_N\}$ for some $N \in [4, M]$, then $\mathrm{Def}_A^B(f) \subseteq X_4 \cap \cdots \cap X_N$. Both are smooth analytic submanifolds of $\mathrm{Teich}(\mathbb{P}^1, B)$ of codimension $\mathrm{card}(A) - 3 = N - 3$, thus they locally coincide. In other words, $\mathrm{Def}_A^B(f)$ is locally defined in $\mathrm{Teich}(\mathbb{P}^1, B)$ by the system of equations:

$$\begin{cases} \kappa_{a_1,a_2,a_3,a_4}(\tau) = 1 \\ \vdots \\ \kappa_{a_1,a_2,a_3,a_M}(\tau) = 1. \end{cases}$$

However, there might be points outside $\mathrm{Def}_A^B(f)$ satisfying those equations.

5.4. **Transversality for critical orbit relations.** We will now illustrate those results with an example.

Let Λ be a complex analytic manifold. Let $(F_\lambda)_{\lambda\in\Lambda}$ be an analytic family of rational maps parametrized by Λ. Let $v_1 : \Lambda \to \mathbb{P}^1$ and $v_2 : \Lambda \to \mathbb{P}^1$ be two maps such that $v_1(\lambda)$ and $v_2(\lambda)$ are critical values of F_λ. Suppose that there exist integers $n_1, n_2 \geq 1$ such that

$$F_{\lambda_0}^{\circ n_1}\big(v_1(\lambda_0)\big) = F_{\lambda_0}^{\circ n_2}\big(v_2(\lambda_0)\big).$$

Such an equation is called a critical orbit relation for F_{λ_0} and the subset

$$\Big\{\lambda \in \Lambda \ : \ F_\lambda^{\circ n_1}\big(v_1(\lambda)\big) = F_\lambda^{\circ n_2}\big(v_2(\lambda)\big)\Big\}$$

is the locus where this relation is preserved. We will give a setting giving equations for the locus where appropriate critical orbit relations are preserved, allowing us to tell when such loci for various critical orbit relations are transverse.

We say that a critical value $v \in \mathcal{P}_f$ is *latest* if for all $n \geq 1$, $f^{\circ n}(v) \notin \mathcal{S}_f$. If $v \in \mathcal{P}_f$ is not latest, there is a least integer $n \geq 1$ such that $f^{\circ n}(v) \in \mathcal{S}_f$. This critical value is called the *successor* of v.

The postcritical set may be decomposed as $\mathcal{P}_f = \mathcal{P}_f^0 \sqcup \mathcal{P}_f^\infty$ with

$$\mathcal{P}_f^0 = \big\{z \in \mathcal{P}_f \mid O(z) \text{ is finite}\big\} \quad \text{and} \quad \mathcal{P}_f^\infty = \big\{z \in \mathcal{P}_f \mid O(z) \text{ is infinite}\big\}$$

(points in $\mathcal{P}_f^0$ are periodic or preperiodic). We may define a partial order on $\mathcal{P}_f^\infty$ by

$$z_1 \preceq z_2 \quad \Longleftrightarrow \quad z_2 \in O(z_1).$$

We say that two points z_1 and z_2 are *orbit-equivalent* if their orbits intersect. If $z_1 \in \mathcal{P}_f^\infty$ and $z_2 \in \mathcal{P}_f^\infty$ are orbit-equivalent, there are least integers $n_1 \geq 0$ and $n_2 \geq 0$ such that $f^{\circ n_1}(z_1) = f^{\circ n_2}(z_2) \in \mathcal{P}_f^\infty$. This point is called the *least upper bound* of z_1 and z_2. More generally, if $E \subset \mathcal{P}_f^\infty$ is a finite set contained in a single orbit-equivalence class, then there is a least $w \in \mathcal{P}_f^\infty$ such that $z \preceq w$ for all $z \in E$. We denote it by $\sup(E)$.

Definition 8. *We denote by $\delta(f)$ the number of orbit-equivalence classes in $\mathcal{P}_f^\infty$.*

Set $\delta = \delta(f)$ and let $P_1, \ldots, P_\delta$ be the orbit-equivalence classes in $\mathcal{P}_f^\infty$. For each, $j \in [1,\delta]$, set

$$b_j = \sup(P_j \cap \mathcal{S}_f).$$

Each P_j contains at least one latest critical value. For each $j \in [1,\delta]$, let $v_j \in P_j$ be a latest critical value. Set

$$\mathcal{S}'_f = \mathcal{S}_f - \{v_j\}_{j\in[1,\delta]}.$$

Note that if P_j contains a single critical value, then this is $v_j = b_j$. In particular,

$$\mathcal{S}'_f \subseteq B \cap f^{-1}(B).$$

Critical values in $\mathcal{S}'_f$ are either periodic, or preperiodic, or orbit-equivalent to one of the critical values v_j, $j \in [1,\delta]$.

We may now define finite sets A and B in $\mathbb{P}^1$ as follows. Set

$$B_j = \{z \in \mathcal{P}_f^\infty \mid z \preceq b_j\}.$$

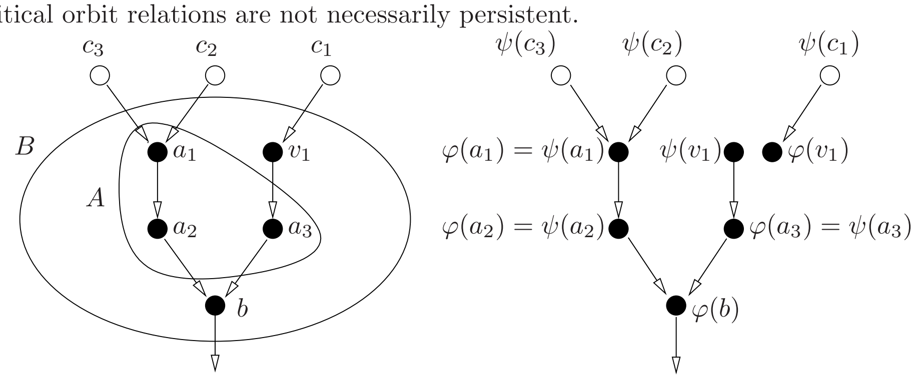
Let $B \subset \mathbb{P}^1$ be the union of $\mathcal{P}_f^0 \cup \displaystyle\bigcup_{j=1}^{\delta} B_j$ and finitely many cycles of f, so that $\operatorname{card}(B) \geq \operatorname{card}(\mathcal{S}_f) + 3$. This last requirement may be achieved by adjoining to B a repelling cycle of period ≥ 3. Set

$$A = B \cap f^{-1}(B) - \mathcal{S}'_f.$$

We obtain A by removing from B the points b_j and the critical values which are not one of the v_j. If $v_j = b_j$, then it is not in A. We have

$$\operatorname{card}(A) = \operatorname{card}\big(B \cap f^{-1}(B)\big) - \operatorname{card}(\mathcal{S}'_f) = \operatorname{card}(B) - \operatorname{card}(\mathcal{S}_f) \geq 3.$$

If f is not a flexible Lattès example, then $\operatorname{Def}_A^B(f)$ is a smooth analytic submanifold of $\operatorname{Teich}(\mathbb{P}^1, B)$ of dimension $\operatorname{card}(\mathcal{S}_f)$. The map $\Phi : \operatorname{Def}_A^B(f) \to \operatorname{Rat}_D$ parametrizes a family of equivalence classes of rational maps which have the same number of critical points and the same number of critical values as f. However, the critical orbit relations are not necessarily persistent.



If $v \in \mathcal{S}'_f$, then $v \in B \cap f^{-1}(B) - A$. If (φ, ψ, F) represents a point $\tau \in \operatorname{Def}_A^B(f)$, then $\varphi(v)$ is a critical value of F but *a priori* $\psi(v)$ is not a critical value of f. However, according to lemma 3, we have the following characterization of $\psi(v)$ in terms of the dynamics of F:

- If $v \in \mathcal{S}'_f$ is not latest and if $v' \in \mathcal{S}_f$ is the successor of v for the dynamics of f, then $\psi(v)$ is in the backward orbit of $\varphi(v')$ for the dynamics of F.
- If $v \in \mathcal{S}'_f$ is latest, then

– either $v \in \mathcal{S}'_f \cap \mathcal{P}^0_f$ and $\psi(v)$ is preperiodic.
– or $v \in \mathcal{S}'_f \cap P_j$ for some $j \in [1, \delta]$ and $\psi(v)$ is orbit-equivalent to $\varphi(v_j)$.

In particular, we see that for $v \in \mathcal{S}'_f$, we have $\varphi(v) = \psi(v)$ if and only if some critical orbit relation is preserved. Denote by X_v the locus in $\mathrm{Def}^B_A(f)$ where this critical orbit relation is preserved. More precisely, let a_1, a_2 and a_3 be three distinct points in A. Then $\varphi(v) = \psi(v)$ if and only if $\kappa_{a_1,a_2,a_3,v}(\tau) = 1$. For $v \in \mathcal{S}'_f$, set

$$X_v = \big\{\tau \in \mathrm{Def}^B_A(f) \ : \ \kappa_{a_1,a_2,a_3,v}(\tau) = 1\big\}.$$

As a corollary of theorem 2, we see that near the basepoint $\mathbf{0}$, the loci $(X_v)_{v\in\mathcal{S}'_f}$ are smooth and transverse analytic submanifolds of $\mathrm{Def}^B_A(f)$.

6. Cycles and their local invariants

Let $x \in \mathbb{P}^1$ be a periodic point of f with period $p \geq 1$ and orbit contained in A. Denote by

$$\langle x\rangle = \big\{x, f(x), \ldots, f^{\circ(p-1)}(x)\big\}$$

the cycle containing x. Note that if (φ, ψ, F) represents a point $\tau \in \mathrm{Def}^B_A(f)$, then $\big\langle\varphi(x)\big\rangle$ is a cycle of F. There are formal invariants attached to this cycle, such as the multiplier $\rho_{\langle x\rangle}(\tau)$. The function

$$\rho_{\langle x\rangle} : \mathrm{Def}^B_A(f) \to \mathbb{C}$$

is holomorphic and we will identify its logarithmic derivative at points where $\rho_{\langle x\rangle}$ does not vanish. In particular, we will see that when $\rho \in \overline{\mathbb{D}} - \{0\}$, the locus where $\rho_{\langle x\rangle}$ takes the value ρ is a smooth analytic submanifold of $\mathrm{Def}^B_A(f)$ of codimension 1.

When $\big\langle\varphi(x)\big\rangle$ is a parabolic cycle of F, there are other invariants such as the parabolic multiplicity or the holomorphic index. We will also study those invariants.

6.1. Formal invariants of a cycle.

Let us recall the following classical definitions. The derivative $D_x f^{\circ p} : T_x\mathbb{P}^1 \to T_x\mathbb{P}^1$ is a linear map whose single eigenvalue is called the *multiplier* ρ of $\langle x\rangle$ as a cycle of f. This multiplier only depends on the cycle, not on the point of the cycle. The cycle $\langle x\rangle$ is

- superattracting if $\rho = 0$,
- attracting if $0 < |\rho| < 1$,
- repelling if $|\rho| > 1$,
- irrationally indifferent if $|\rho| = 1$ and ρ is not root of unity and
- parabolic if ρ is a root of unity.

The *holomorphic index* of f along $\langle x\rangle$ is the residue

$$\iota = \mathrm{Res}_x \frac{\mathrm{d}\zeta}{\zeta - \zeta\circ f^{\circ p}}$$

where ζ is a local coordinate at x. It is remarkable that this residue does not depend on the choice of local coordinate ζ. If $\rho \neq 1$, then

$$\iota = \frac{1}{1-\rho}.$$

When $\rho = \mathrm{e}^{2\pi\mathrm{i}r/s}$ is a s-th root of unity, there are

- a unique integer $m \geq 1$ called the *parabolic multiplicity* of $f^{\circ p}$ at x
- a unique real number $\beta \in \mathbb{C}$ called the *résidu itératif* of f at x and

- a (non unique) local coordinate ζ vanishing at x

such that the expression of f is

$$\zeta \mapsto \rho\zeta\left(1 + \ \zeta^{ms} + \left(\frac{ms+1}{2} - \beta\right)\zeta^{2ms}\right) + \mathcal{O}(\zeta^{2ms+2}).$$

Such a coordinate ζ is called a *preferred coordinate for f at x*. The résidu itératif of f at x is related to the holomorphic index ι of $f^{\circ s}$ at x by

$$\iota = \frac{ms+1}{2} - \frac{\beta}{s}$$

(see for example [1]).

Let us now assume that $x \in U$ is a periodic point of f of period p and let $\langle x\rangle$ be the cycle containing x. The formal invariants of the cycle are by definition the formal invariant of $f^{\circ p}$ at any point of the cycle (they do not depend on the point of the cycle).

6.2. **Families of germs with a common cycle.** Let $(f_t : U \to \mathbb{P}^1)$ be an analytic family of holomorphic maps, parametrized by a neighborhood of 0 in $\mathbb{C}$. Assume the maps f_t have a common cycle $\langle x\rangle \subset U$. We wish to study how the formal invariants of f_t along $\langle x\rangle$ vary as t goes away from 0.

For this purpose, we first need to encode the variation of the maps f_t. If $\langle x\rangle$ is not a superattracting cycle of f, then f is locally invertible along $\langle x\rangle$ and for t sufficiently close to 0, the maps $\chi_t = f^{-1} \circ f_t$ are defined near $\langle x\rangle$. Then, we have that

$$f_t = f \circ \chi_t$$

near $\langle x\rangle$. Note that $\chi = \mathrm{I}$ and so, $\theta = \dot\chi$ is a germ of holomorphic vector field along $\langle x\rangle$. Further χ_t fixes $\langle x\rangle$ and so, the vector field θ vanishes along $\langle x\rangle$.

The formal invariants at a fixed point are invariant under holomorphic conjugacy. It will be important to understand how the definition of θ given above depends on the choice of coordinates. So, suppose $(\hat f_t = f \circ \hat\chi_t)$ and (h_t) are analytic families of holomorphic maps satisfying:

$$\chi = h = \mathrm{I} \quad \text{and} \quad \hat f_t \circ h_t = h_t \circ f_t.$$

Then the holomorphic invariants of the families (f_t) and $(\hat f_t)$ along $\langle x\rangle$ are the same. In addition, the vector field $\hat\theta$ corresponding to the family $(\hat f_t)$ satisfies

$$Df \circ (\hat\theta + \dot h) = \dot h \circ f + Df \circ \theta.$$

As a consequence,

$$\theta - \hat\theta = \dot h - f^*\dot h.$$

In order to study the variation of the formal invariants of $f \circ \chi_t$ along $\langle x\rangle$, it will therefore be convenient to consider the class $[\theta]_{\langle x\rangle}$ of θ within the space of holomorphic vector vanishing along $\langle x\rangle$, modulo those which are of the form $\eta - f^*\eta$ for some holomorphic vector field η vanishing along $\langle x\rangle$.

Definition 9. *For $z \in \mathbb{P}^1$, we denote by $\mathcal{V}_z$ the space of germs of holomorphic vector fields at x which vanish at z. If $E \subset \mathbb{P}^1$ is finite, we set $\mathcal{V}_E = \bigoplus_{z\in E} \mathcal{V}_z$.*

If C is a union of non-superattracting cycles of a map f defined and holomorphic near C, we set $\mathcal{V}_C(f) = \mathcal{V}_C/\Delta_f\mathcal{V}_C$ with $\Delta_f = \mathrm{I} - f^* : \mathcal{V}_C \to \mathcal{V}_C$.

Lemma 5. *If $f_t = f\circ\chi_t$ with $\chi = \mathrm{I}$, then for all $n\geq 1$*

$$f_t^{\circ n} = f^{\circ n}\circ\chi_t^n \quad \textit{with} \quad \chi^n = \mathrm{I} \quad \textit{and} \quad \dot\chi^n = \sum_{k=0}^{n-1}(f^{\circ k})^*\dot\chi.$$

Proof. We will argue by induction on $n\geq 1$. For $n=1$, this is trivially true. By definition,

$$f_t^{\circ(n+1)} = f\circ\chi_t\circ f_t^{\circ n}.$$

If the inductive hypothesis holds for n,

$$\chi_t\circ f_t^{\circ n}(z) = f_t^{\circ n} + t\cdot\theta\circ f_t^{\circ n} + o(t) = f^{\circ n} + t\cdot Df^{\circ n}\circ\dot\chi^n + t\cdot\theta\circ f^{\circ n} + o(t)$$

and so

$$\begin{aligned} f_t^{\circ(n+1)} &= f^{\circ(n+1)} + t\cdot Df\big(Df^{\circ n}\circ\dot\chi^n + \theta\circ f^{\circ n}\big) + o(t)\\ &= f^{\circ(n+1)} + t\cdot Df^{\circ(n+1)}\circ\big(\dot\chi^n + (f^{\circ n})^*\theta\big) + o(t).\end{aligned}$$

This shows that, as required,

$$\dot\chi^{n+1} = \dot\chi^n + (f^{\circ n})^*\theta.$$

□

If x is periodic of period p, it may therefore be relevant to consider the linear map $\triangleright_x : \mathcal{V}_{\langle x\rangle}\to\mathcal{V}_x$ defined by

$$\triangleright_x\theta = \sum_{k=0}^{p-1}(f^{\circ k})^*\theta. \tag{6}$$

If $\eta\in\mathcal{V}_{\langle x\rangle}$ and $\theta = \eta - f^*\eta$, then for all $n\geq 1$, $\triangleright_x\theta = \triangleright_x\eta - f^*\big(\triangleright_x\eta\big)$. It follows that $\triangleright_x$ induces a linear map

$$\triangleright_x : \mathcal{V}_{\langle x\rangle}(f)\to\mathcal{V}_x(f^{\circ p}).$$

6.3. **Invariant divergences.**

Definition 10. *Two meromorphic quadratic differentials q_1 and q_2 at $z\in\mathbb{P}^1$ have the same* divergence *at z if q_1-q_2 has at most a simple pole at z. We denote by $\mathcal{D}_z$ the vector space of divergences $[q]_z$ at z. If $E\subset\mathbb{P}^1$ is finite, we denote by $\mathcal{D}_E$ the vector space $\mathcal{D}_E = \bigoplus_{z\in E}\mathcal{D}_z$. An element $[q]_E$ of $\mathcal{D}_E$ is called a* divergence on E.

In other words, a divergence at z is a polar part of degree ≤ -2 of meromorphic quadratic differentials at z.

The pushforward operator f_* induces a linear map $f_*:\mathcal{D}_C\to\mathcal{D}_C$. The operator ∇_f induces a linear map $\nabla_f:\mathcal{D}_C\to\mathcal{D}_C$ whose kernel $\mathcal{D}_C(f)$ has been characterized in [4].

Proposition 7.
- *The space $\mathcal{D}_C(f)$ is computed cycle by cycle:*

$$\mathcal{D}_C(f) = \bigoplus_{\langle x\rangle\subseteq C}\mathcal{D}_{\langle x\rangle}(f).$$

Let $x\in\mathbb{P}^1$ be a periodic point of f of period p.

- *The projection* $\mathcal{D}_{\langle x\rangle} \to \mathcal{D}_x$ *restricts to an isomorphism* $\mathcal{D}_{\langle x\rangle}(f) \to \mathcal{D}_x(f^{\circ p})$ *whose inverse is*

$$\triangleleft_x : \mathcal{D}_x \ni [q]_x \mapsto \bigoplus_{k=0}^{p-1} \bigl[f_*^{\circ k} q\bigr]_{f^{\circ k}(x)}.$$

- *If* $\langle x\rangle$ *is superattracting, then* $\mathcal{D}_x(f^{\circ p}) = 0$*.*
- *If* $\langle x\rangle$ *is attracting, repelling or irrationally indifferent, then* $\mathcal{D}_x(f^{\circ p})$ *is the vector space spanned by* $\left[\dfrac{\mathrm{d}\zeta^2}{\zeta^2}\right]_x$ *for any local coordinate* ζ *vanishing at* x*.*
- *If* $\langle x\rangle$ *is parabolic with multiplier* $\mathrm{e}^{2\pi \mathrm{i} r/s}$*, parabolic multiplicity* m *and résidu itératif* β*, then* $\mathcal{D}_x(f^{\circ p})$ *is the vector space spanned by*

$$\left[\frac{\mathrm{d}\zeta^2}{\zeta^2}\right]_x, \ldots, \left[\frac{\mathrm{d}\zeta^2}{\zeta^{sk+2}}\right]_x, \ldots, \left[\frac{\mathrm{d}\zeta^2}{\zeta^{(m-1)s+2}}\right]_x \quad \textit{and by} \quad \left[\frac{\mathrm{d}\zeta^2}{(\zeta^{ms+1} - \beta\zeta^{2ms+1})^2}\right]_x$$

for any preferred coordinate ζ *for* $f^{\circ p}$ *at* x*.*

Assume ζ_1 and ζ_2 are two coordinates vanishing at x. Then, the quadratic differentials $\mathrm{d}\zeta_1^2/\zeta_1^2$ and $\mathrm{d}\zeta_2^2/\zeta_2^2$ have the same divergence at x. It follows that when x is not superattracting, the invariant divergence

$$[q_{\rtimes}]_{\langle x\rangle} = \triangleleft_x \left[\frac{\mathrm{d}\zeta^2}{\zeta^2}\right]_x$$

is natural (ζ is a coordinate vanishing at x).

Let $\langle x\rangle$ be parabolic with multiplier $\mathrm{e}^{2\pi \mathrm{i} r/s}$, parabolic multiplicity m and résidu itératif β. Let ζ be a preferred coordinate for $f^{\circ p}$ at x. On the one hand, the family of invariant divergences

$$[q_k]_{\langle x\rangle} = \triangleleft_x \left[\frac{\mathrm{d}\zeta^2}{\zeta^{(k-1)s+2}}\right]_x \quad \text{with} \quad 2 \le k \le m$$

are not quite natural: if ζ_1 and ζ_2 are two preferred coordinates for $f^{\circ p}$ at x, then

$$\zeta_2 = \lambda\zeta_1\bigl(1 + \mathcal{O}(\zeta_1^{ms})\bigr) \quad \text{with} \quad \lambda^{ms} = 1$$

and

$$\left[\frac{\mathrm{d}\zeta_1^2}{\zeta_1^{(k-1)s+2}}\right]_{\langle x\rangle} = \lambda^{(k-1)s} \left[\frac{\mathrm{d}\zeta_2^2}{\zeta_2^{(k-1)s+2}}\right]_{\langle x\rangle}.$$

On the other hand, the invariant divergence

$$[q_f]_{\langle x\rangle} = \frac{1}{s} \cdot \triangleleft_x \left[\frac{\mathrm{d}\zeta^2}{(\zeta^{ms+1} - \beta\zeta^{2ms+1})^2}\right]_x$$

is natural (it does not depend on the choice of preferred coordinate).

Finally, for $k \in [1, m]$, we let $V^k_{\langle x\rangle} \subset \mathcal{D}_C(f)$ be the vector space

$$V^k_{\langle x\rangle} = \mathrm{Vect}\bigl([q_{\rtimes}]_{\langle x\rangle}, [q_2]_{\langle x\rangle}, \ldots, [q_k]_{\langle x\rangle}\bigr).$$

6.4. **The variation of invariants.** The reason why the kernel $\mathcal{D}_{\langle x\rangle}(f)$ is important for us is that it is possible to pair an invariant divergence $[q]_{\langle x\rangle} \in \mathcal{D}_{\langle x\rangle}(f)$ with a class of vector field $[\theta]_{\langle x\rangle} \in \mathcal{V}_{\langle x\rangle}(f)$:

$$\big\langle [q]_{\langle x\rangle}, [\theta]_{\langle x\rangle}\big\rangle_{\langle x\rangle} = \sum_{y\in\langle x\rangle} \mathrm{Res}_y q\otimes\theta.$$

Remark. With an abuse of notation, we shall write $\big\langle [q],[\theta]\big\rangle_{\langle x\rangle}$ for the pairing of an invariant divergence $[q]_{\langle x\rangle} \in \mathcal{D}_{\langle x\rangle}(f)$ and a class of vector fields $[\theta]_{\langle x\rangle} \in \mathcal{V}_{\langle x\rangle}(f)$. Similarly, if x has period p, we shall use the notation $\big\langle [q],[\theta]\big\rangle_x$ for the pairing of an invariant divergence $[q]_x \in \mathcal{D}_x(f^{\circ p})$ and a class of vector fields $[\theta]_x \in \mathcal{V}_x(f^{\circ p})$.

Note that the result does not depend on the representatives of $[q]_{\langle x\rangle}$ or $[\theta]_{\langle x\rangle}$. Indeed, if $q_2 - q_1$ has at most a simple pole at y, then $\mathrm{Res}_y q_1\otimes\theta = \mathrm{Res}_y q_2\otimes\theta$ since θ vanishes at y. And if $\theta_2-\theta_1 = \eta - f^*\eta$ for some holomorphic vector field η vanishing along $\langle x\rangle$, then

$$\sum_{y\in\langle x\rangle}\mathrm{Res}_y q\otimes(\theta_2-\theta_1) = \sum_{y\in\langle x\rangle}\mathrm{Res}_y q\otimes(\eta - f^*\eta) = \sum_{y\in\langle x\rangle}\mathrm{Res}_y (q - f_*q)\otimes\eta.$$

This is 0 since $\nabla_f[q]_{\langle x\rangle} = 0$, thus $q - f_*q$ has at most a simple pole along $\langle x\rangle$, and since η vanishes along $\langle x\rangle$.

In addition, for all $[q]_x \in \mathcal{D}_x(f^{\circ p})$ and all $[\theta]_{\langle x\rangle} \in \mathcal{V}_{\langle x\rangle}(f)$, we have that

$$\big\langle \triangleleft_x[q],[\theta]\big\rangle_{\langle x\rangle} = \big\langle [q],\triangleright_x[\theta]\big\rangle_x. \tag{7}$$

Indeed, it follows from the change of variable formula that for all $k\ge 1$,

$$\mathrm{Res}_x q\otimes(f^{\circ k})^*\theta = \mathrm{Res}_{f^{\circ k}(x)} f_*^{\circ k} q\otimes\theta.$$

We may now give a dynamical interpretation of the divergences $[q_{\rtimes}]_{\langle x\rangle} \in \mathcal{D}_{\langle x\rangle}(f)$. Again, let $f_t = f\circ\chi_t$ be a analytic family of maps defined and holomorphic near a common cycle $\langle x\rangle$ of period p and set $\theta = \dot\chi \in \mathcal{V}_{\langle x\rangle}$.

Lemma 6. *Let ρ_t be the multiplier of $\langle x\rangle$ as a cycle of f_t. Then*

$$\frac{\dot\rho}{\rho} = \big\langle [q_{\rtimes}],[\theta]\big\rangle_{\langle x\rangle}.$$

Proof. Note that ρ_t is the multiplier of $f_t^{\circ p}$ at x. On the one hand, if ζ is a coordinate vanishing at x

$$\zeta\circ f_t^{\circ p} = \rho_t\zeta + o(\zeta) = \rho\zeta + t\dot\rho\zeta + o(t) + o(\zeta).$$

On the other hand, according to lemma 5,

$$f_t^{\circ p} = f^{\circ p} + t\cdot Df^{\circ p}\circ\triangleright_x\theta + o(t).$$

It follows that

$$\zeta\circ f_t^{\circ p} = \zeta\circ f^{\circ p} + t\cdot \mathrm{d}\zeta\big(D(f^{\circ p})\circ\triangleright_x\theta\big) + o(t) = \rho\zeta + t\dot\rho\cdot\mathrm{d}\zeta(\theta_p) + o(t) + o(\zeta).$$

This shows that

$$\mathrm{d}\zeta(\triangleright_x\theta) = \frac{\dot\rho}{\rho}\zeta + o(\zeta)\quad\text{and so,}\quad \big\langle [q_{\rtimes}],\triangleright_x[\theta]\big\rangle_x = \mathrm{Res}_x\left(\frac{d\zeta^2}{\zeta^2}\otimes\triangleright_x\theta\right) = \frac{\dot\rho}{\rho}.$$

The result now follows from equation (7). □

Let us now assume that $\langle x\rangle$ is a parabolic cycle with multiplier $\mathrm{e}^{2\pi\mathrm{i}r/s}$. We will give an interpretation of the divergence $[q_f]_{\langle x\rangle}$. Note that x is a multiple fixed point of $f^{\circ sp}$ with multiplicity $ms+1$. Thus, if t is sufficiently close to 0, the map $f_t^{\circ sp}$ has $ms+1$ fixed points close to x, counting multiplicity. We shall denote by ι_t be the sum of holomorphic indices of $f_t^{\circ sp}$ at those fixed points close to x. If $\mathcal{C}$ is a small circle around x, then for t sufficiently close to 0, we have that

$$\iota_t = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}} \frac{\mathrm{d}\zeta}{\zeta - \zeta\circ f_t^{\circ sp}}.$$

Lemma 7. *We have that*

$$\dot\iota = \big\langle [q_f],[\theta]\big\rangle_{\langle x\rangle}.$$

Proof. According to lemma 5,

$$f_t^{\circ sp} = f^{\circ p} + t\cdot Df^{\circ sp}\circ\theta_{sp} + o(t) \quad \text{with} \quad \theta_{sp} = \sum_{k=0}^{sp-1} (f^{\circ k})^*\theta.$$

It follows that

$$\zeta\circ f_t^{\circ sp} = \zeta\circ f^{\circ sp} + t\ \mathrm{d}\zeta\big(D(f^{\circ sp})\circ\theta_{sp}\big) + o(t).$$

Thus,

$$\dot\iota = \frac{1}{2\pi\mathrm{i}}\int_C \frac{\mathrm{d}\zeta\big(D(f^{\circ sp})\circ\theta_{sp}\big)}{(\zeta-\zeta\circ f^{\circ sp})^2}\mathrm{d}\zeta.$$

An elementary computation shows that if ζ is a preferred coordinate for $f^{\circ p}$ at x, then

$$\zeta\circ f^{\circ sp} = \left(1 + s\zeta^{ms} + s^2\left(\frac{ms+1}{2} - \frac{\beta}{s}\right)\zeta^{2ms}\right)\zeta + \mathcal{O}(\zeta^{2ms+2}).$$

So,

$$\begin{aligned}\frac{\mathrm{d}\zeta\big(D(f^{\circ sp})\circ\theta_{sp}\big)}{(\zeta-\zeta\circ f^{\circ sp})^2} &= \frac{\big(1+s(ms+1)\zeta^{ms}\big)\ \mathrm{d}\zeta(\theta_{sp})}{\zeta^2\big(s\zeta^{ms} + s^2\big(\frac{ms+1}{2}-\frac{\beta}{s}\big)\zeta^{2ms}\big)^2} + \mathcal{O}\left(\frac{1}{\zeta}\right)\\ &= \frac{\mathrm{d}\zeta(\theta_{sp})}{s^2(\zeta^{ms}-\beta\zeta^{2ms})^2} + \mathcal{O}\left(\frac{1}{\zeta}\right).\end{aligned}$$

Thus,

$$\dot\iota = \mathrm{Res}_x\left(\frac{\mathrm{d}\zeta^2}{s^2(\zeta^{ms}-\beta\zeta^{2ms})^2}\otimes\theta_{sp}\right) = \frac{1}{s}\big\langle[q_f],[\theta_{sp}]\big\rangle_x.$$

We have that

$$\theta_{sp} = \triangleright_x\theta + (f^{\circ p})^*\triangleright_x\theta + \cdots + (f^{\circ(s-1)p})^*\triangleright_x\theta.$$

In addition, $\big[(f^{\circ p})^*\triangleright_x\theta\big]_x = [\triangleright_x\theta]_x$ and so $[\theta_{sp}]_x = s\cdot[\triangleright_x\theta]_x$. Therefore

$$\dot\iota = \big\langle[q_f],[\triangleright_x\theta]\big\rangle_x.$$

The result now follows from equation (7). □

6.5. A first application of the Snake Lemma.

Definition 11. *Two quadratic differentials q_1 and q_2 meromorphic and integrable in a neighborhood of $z \in \mathbb{P}^1$ have the same* simple polar part *if $q_1 - q_2$ is analytic in a neighborhood of z. We denote by $\mathcal{E}_z$ the vector space of simple polar parts at z. If $E \subset \mathbb{P}^1$ is finite, we denote by $\mathcal{E}_E$ the vector space $\mathcal{E}_E = \bigoplus_{z \in E} \mathcal{E}_z$.*

Note that $\mathcal{E}_z$ is canonically isomorphic to the cotangent space $T_z^\star \mathbb{P}^1$.

Definition 12. *If $E \subset \mathbb{P}^1$ is a finite set containing C, we denote by $\widehat{\mathcal{Q}}_C(\mathbb{P}^1, E)$ the vector space of meromorphic quadratic differentials whose poles are contained in E and have at most simple poles in $E - C$.*

Assume $E \subset F$ are finite subsets of $\mathbb{P}^1$ containing C with $\operatorname{card}(E) \geq 3$. Let $\mathcal{Q}_{\mathbb{P}^1}(\mathbf{E})$ be the sheaf of meromorphic quadratic differentials having at worst simple poles on E. Let $\widehat{\mathcal{Q}}_{\mathbb{P}^1}(\mathbf{F}, \mathbf{C})$ be the sheaf of meromorphic quadratic differentials having at most simple poles on $F - C$ and arbitrary poles on C. The short exact sequence of sheaves

$$0 \to \mathcal{Q}_{\mathbb{P}^1}(\mathbf{E}) \to \widehat{\mathcal{Q}}_{\mathbb{P}^1}(\mathbf{F}, \mathbf{C}) \to \mathcal{Q}_{\mathbb{P}^1}(\mathbf{F}, \mathbf{C}) / \mathcal{Q}_{\mathbb{P}^1}(\mathbf{E}) \to 0$$

leads to a long exact sequence

$$0 \to \mathcal{Q}(\mathbb{P}^1, E) \to \widehat{\mathcal{Q}}_C(\mathbb{P}^1, F) \to \mathcal{D}_C \oplus \mathcal{E}_{F-E} \to H^1\big(\mathbb{P}^1, \mathcal{Q}_{\mathbb{P}^1}(E)\big) \to \cdots$$

By Serre duality, $H^1\big(\mathbb{P}^1, \mathcal{Q}_{\mathbb{P}^1}(\mathbf{E})\big)$ is dual to $H^0\big(\mathbb{P}^1, \Theta_{\mathbb{P}^1}(-\mathbf{E})\big)$ which is zero since $\operatorname{card}(E) \geq 3$, and so, a vector field vanishing on E is 0. In particular, we have the canonical identification

$$\widehat{\mathcal{Q}}_C(\mathbb{P}^1, F) / \mathcal{Q}(\mathbb{P}^1, E) \simeq \mathcal{D}_C \oplus \mathcal{E}_{F-E}.$$

Thus, the following diagram commutes:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{Q}(\mathbb{P}^1, A) & \xrightarrow{\nabla_f} & \mathcal{Q}(\mathbb{P}^1, B) & \longrightarrow & T_{\mathbf{0}}^\star \mathrm{Def}_A^B(f) \\
 & & \downarrow & & \downarrow & & \\
\mathcal{K}(f) & \longrightarrow & \widehat{\mathcal{Q}}_C(\mathbb{P}^1, A') & \xrightarrow{\nabla_f} & \widehat{\mathcal{Q}}_C(\mathbb{P}^1, B) & & \\
 & & \downarrow & & \downarrow & & \\
\mathcal{D}_C(f) \oplus \mathcal{E}_{A'-A} & \longrightarrow & \mathcal{D}_C \oplus \mathcal{E}_{A'-A} & \xrightarrow{\nabla_f} & \mathcal{D}_C & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{K}(f)$ is the kernel of the linear map $\nabla_f : \widehat{\mathcal{Q}}_C(\mathbb{P}^1, A') \to \widehat{\mathcal{Q}}_C(\mathbb{P}^1, B)$.

According to the Snake Lemma, there is a linear map $\blacktriangledown_f : \mathcal{D}_C(f) \to T_{\mathbf{0}}^\star \mathrm{Def}_A^B(f)$ such that the following sequence is exact:

$$0 \to \mathcal{K}(f) \to \mathcal{D}_C(f) \xrightarrow{\blacktriangledown_f} T_{\mathbf{0}}^\star \mathrm{Def}_A^B(f).$$

For cycles $\langle x\rangle \subseteq A$, we shall give a dynamical interpretation of the forms

$$\blacktriangledown_f[q_k]_{\langle x\rangle} \in T^\bigstar_{\mathbf{0}}\mathrm{Def}^B_A(f),\ 1\le k\le m \quad\text{and}\quad \blacktriangledown_f[q_f]_{\langle x\rangle} \in T^\bigstar_{\mathbf{0}}\mathrm{Def}^B_A(f).$$

6.6. **Setting up the deformation of cycles.** Let $v\in T_{\mathbf{0}}\mathrm{Def}^B_A(f)$ be represented by an analytic curve $\big(\tau_t\in\mathrm{Def}^B_A(f)\big)$ with $\dot\tau = v$. Let (φ_t,ψ_t,f_t) be a triple representing τ_t such that $\varphi = \mathrm{I}:\mathbb{P}^1\to\mathbb{P}^1$ and φ_t is smooth on $\mathbb{P}^1$, holomorphic on U and depends analytically on t. Note that $\psi = \mathrm{I}:\mathbb{P}^1\to\mathbb{P}^1$, ψ_t is holomorphic in $f^{-1}(U)$ and the maps ψ_t and f_t depend analytically on t.

Let $x\in C$ be a periodic point of f of period p. As t varies away from 0, the point $x_t=\varphi_t(x)=\psi_t(x)$ is a periodic point of f_t of period p and the cycle $\langle x_t\rangle$ moves. It will be convenient to keep the cycle fixed by conjugating by φ_t: set

$$\chi_t = \psi_t^{-1}\circ\varphi_t \quad\text{and}\quad g_t = \varphi_t^{-1}\circ f_t\circ\varphi_t = f\circ\chi_t.$$

Then, $\langle x\rangle$ is a persistent cycle of g_t. Note that g_t is not globally holomorphic on $\mathbb{P}^1$, but it is holomorphic on $U\cap f^{-1}(U)\supset\langle x\rangle$.

Any smooth 1-parameter family of diffeomorphisms passing through the identity is to order 1 adding a vector field:

$$\varphi_t = \mathrm{I}+t\dot\varphi+o(t),\quad \psi_t=\mathrm{I}+t\dot\psi+o(t)\quad\text{and}\quad \chi_t=\mathrm{I}+t\dot\chi+o(t)$$

with

$$\dot\chi=\theta_v=\dot\varphi-\dot\psi.$$

According to section 6.2, the class $[\theta_v]_C$ of θ_v in $\mathcal{V}_C$ only depends on $v\in T_{\mathbf{0}}\mathrm{Def}^B_A(f)$, not on the choice of homeomorphisms φ_t representing τ_t. It encodes the variation of g_t.

We shall denote by $\blacktriangledown^\bigstar_f : T_{\mathbf{0}}\mathrm{Def}^B_A(f)\to\mathcal{V}_C$ the linear map

$$\blacktriangledown^\bigstar_f : v\mapsto [\theta_v]_C.$$

The notation is justified by the following proposition.

Proposition 8. *For all $[q]_C\in\mathcal{D}_C(f)$ and $v\in T^\bigstar_{\mathbf{0}}\mathrm{Def}^B_A(f)$, we have the following duality:*

$$\big\langle \blacktriangledown_f[q]_C, v\big\rangle = \big\langle [q]_C, \blacktriangledown^\bigstar_f v\big\rangle_C.$$

Proof. Fix $[q]_C\in\mathcal{D}_C(f)$ and $v\in T_{\mathbf{0}}\mathrm{Def}^B_A(f)$. Let $q\in\widehat{\mathcal{Q}}_C(\mathbb{P}^1,A)$ be a meromorphic quadratic differential representing $[q]_C$. Let (τ_t) be a curve in $\mathrm{Def}^B_A(f)$ with $\tau_0=\mathbf{0}$ and $\dot\tau=v$. Let (φ_t,ψ_t,f_t) represent τ_t as above. Set $\theta_v=\dot\varphi-\dot\psi$. We must show that for all

$$\big\langle\blacktriangledown_f[q]_C,v\big\rangle = \sum_{x\in C}\mathrm{Res}_x q\otimes\theta_v.$$

The infinitesimal Beltrami form $\nu=\bar\partial\dot\varphi$ represents $v\in T_{\mathbf{0}}\mathrm{Def}^B_A(f)$. Thus,

$$\big\langle\blacktriangledown_f[q]_C,v\big\rangle = \big\langle\nabla_f q,\nu\big\rangle = \langle q,\nu-f^*\nu\rangle.$$

Note that $\bar\partial\dot\psi=f^*\nu$ and $\bar\partial\theta_v=\nu-f^*\nu$. So,

$$\langle q,\nu-f^*\nu\rangle=\langle q,\theta_v\rangle.$$

Since φ_t agrees with ψ_t on A, we have that $\dot\varphi=\dot\psi$ on A and thus, θ_v vanishes on A. As a consequence,

$$\langle q,\theta_v\rangle=\sum_{x\in C}\mathrm{Res}_x q\otimes\theta_v.$$

□

6.7. **Differentiating the formal invariants.** Let $\langle x\rangle$ be a cycle of f contained in A. If (φ,ψ,F) represents a point $\tau\in\mathrm{Def}_A^B(f)$, then $\varphi(x)=\psi(x)$ is a periodic point of F and its multiplier only depends on τ since another choice of representative $\hat\varphi$ of τ yields a Möbius conjugate rational map $\hat F$.

Definition 13. *If $\langle x\rangle$ is a cycle of f contained in A, we denote by*

$$\rho_{\langle x\rangle}:\mathrm{Def}_A^B(f)\to\mathbb{C}$$

the function that sends a point $\tau\in\mathrm{Def}_A^B(f)$ represented by a triple (φ,ψ,F) to the multiplier of the cycle $\big\langle\varphi(x)\big\rangle$ of F.

The function $\rho_{\langle x\rangle}$ is globally defined on $\mathrm{Def}_A^B(f)$ (but it is *a priori* not defined on $\mathrm{Teich}(\mathbb{P}^1,B)$). Lemma 6 allows us to give the following interpretation of $\blacktriangledown_f[q_{\rtimes}]_{\langle x\rangle}$.

Proposition 9. *If $\langle x\rangle$ is a cycle of f contained in A, then the logarithmic derivative $D_{\mathbf{0}}\log\rho_{\langle x\rangle}$ is $\blacktriangledown_f[q_{\rtimes}]_{\langle x\rangle}$.*

Now, assume $\langle x\rangle$ is a parabolic cycle of f contained in A. Let p be the period of x, $\mathrm{e}^{2\pi\mathrm{i}r/s}$ the multiplier of $f^{\circ p}$ at x, m the parabolic multiplicity of $f^{\circ p}$ at x, β the résidu itératif of $f^{\circ p}$ at x and ι the holomorphic index of $f^{\circ sp}$ at x. Then,

$$\iota=\frac{ms+1}{2}-\frac{\beta}{s}.$$

If $U\subset\mathbb{P}^1$ is a sufficiently small disk containing x, then x is the only fixed point of $f^{\circ sp}$ in U and the holomorphic index ι is given by the integral

$$\frac{1}{2\pi\mathrm{i}}\oint_{\partial U}\frac{\mathrm{d}\zeta}{\zeta-\zeta\circ f^{\circ sp}}$$

where ζ is a local coordinate in a neighborhood of $\overline{U}$. If $F\in\mathbf{Rat}_D$ is sufficiently close to f, the rational map $F^{\circ sp}$ has $ms+1$ fixed points in U, counting multiplicities and the integral

$$\mathrm{I}(F)=\frac{1}{2\pi\mathrm{i}}\oint_{\partial U}\frac{\mathrm{d}\zeta}{\zeta-\zeta\circ F^{\circ sp}}$$

is the sum of holomorphic indices of $F^{\circ sp}$ at those fixed points. This defines a germ of analytic map $\mathrm{I}:(\mathbf{Rat}_D,f)\to\mathbb{C}$. Note that I is constant along the orbits under conjugacy by Möbius transformations.

Definition 14. *We denote by $\iota_{\langle x\rangle}:\big(\mathrm{Def}_A^B(f),\mathbf{0}\big)\to\mathbb{C}$ the germ defined by*

$$\iota_{\langle x\rangle}=\mathrm{I}\circ\Phi$$

where $\Phi:\mathrm{Def}_A^B(f)\to\mathbf{Rat}_D$ is any lift of $\underline{\Phi}:\mathrm{Def}_A^B(f)\to\mathrm{Rat}_D$ sending $\mathbf{0}$ to f.

The function $\iota_{\langle x\rangle}$ is not globally defined on $\mathrm{Def}_A^B(f)$: it is only a germ of analytic map at $\mathbf{0}$. Lemma 7 enables us to give the following interpretation of $\blacktriangledown_f[q_f]_{\langle x\rangle}$.

Proposition 10. *If $\langle x\rangle$ is a parabolic cycle of f contained in A with multiplier $\mathrm{e}^{2\pi\mathrm{i}r/s}$, then the derivative $D_{\mathbf{0}}\iota_{\langle x\rangle}$ is $\blacktriangledown_f[q_f]_{\langle x\rangle}$.*

7. Injectivity of $\blacktriangledown_f$

7.1. **Extract from** [4]**.** Before going further, we will need results which have been proved in [4] and may be summarized as follows.

If q is meromorphic along $\langle x\rangle$ with divergences in $\mathcal{D}_{\langle x\rangle}(f)$ then the limit

$$\lim_{U\searrow\langle x\rangle}\left(\int_{f(U)-U}|q|-\int_{U-f(U)}|q|\right)$$

exists and is finite. So, the following definition makes sense.

Definition 15. *If* $[q]_{\langle x\rangle}\in\mathcal{D}_C(f)$*, set*

$$\mathrm{Res}_{\langle x\rangle}(f:q)=\frac{1}{2\pi}\lim_{U\searrow\langle x\rangle}\left(\int_{f(U)-U}|q|-\int_{U-f(U)}|q|\right)$$

and

$$\mathrm{Res}(f:q)=\sum_{\langle x\rangle\in C}\mathrm{Res}_{\langle x\rangle}(f:q).$$

On the one hand, the residue $\mathrm{Res}_{\langle x\rangle}(f:q)$ may be computed in terms of the local invariants of the cycle $\langle x\rangle$ as follows.

Proposition 11. *If $\langle x\rangle$ is attracting, repelling or irrationally indifferent with multiplier ρ, then*

$$[q]_{\langle x\rangle}=\alpha\cdot[q_{\rtimes}]_{\langle x\rangle}\quad\Longrightarrow\quad\mathrm{Res}_{\langle x\rangle}(f:q)=|\alpha|\cdot\log|\rho|.$$

If $\langle x\rangle$ is parabolic with parabolic multiplicity m and résidu itératif β, then

$$[q]_{\langle x\rangle}\in\alpha\cdot[q_f]_{\langle x\rangle}+V^{m-1}_{\langle x\rangle}\quad\Longrightarrow\quad\mathrm{Res}_{\langle x\rangle}(f:q)=|\alpha|\cdot\mathrm{Re}\,(\beta).$$

On the other hand, the following result is a consequence of a generalization of the Thurston Contraction Principle.

Proposition 12. *Let $E\subset\mathbb{P}^1$ be a finite set containing C. If f is not a flexible Lattès example or if E does not contain the postcritical set of f, then*

$$\big(q\in\widehat{\mathcal{Q}}_C(\mathbb{P}^1,E)\text{ and }\nabla_f q=0\big)\quad\Longrightarrow\quad\mathrm{Res}(f:q)>0.$$

It is therefore natural to consider the following set.

Definition 16. *Let $\mathcal{D}^\flat_C(f)$ be the set of invariant divergences $[q]_{\langle x\rangle}\in\mathcal{D}_C(f)$ for which* $\mathrm{Res}(f:q)\le 0$*.*

Proposition 11 implies that the set $\mathcal{D}^\flat_C(f)$ is a vector space. More precisely, $\mathcal{D}^\flat_C(f)=\bigoplus_{\langle x\rangle\subseteq C}\mathcal{D}^\flat_{\langle x\rangle}(f)$ where

- if $\langle x\rangle$ is superattracting or repelling, then $\mathcal{D}^\flat_{\langle x\rangle}(f)=0$;
- if $\langle x\rangle$ is attracting or irrationally indifferent, then $\mathcal{D}^\flat_{\langle x\rangle}(f)=V^1_{\langle x\rangle}$;
- if $\langle x\rangle$ is parabolic with parabolic multiplicity m and résidu itératif β,
  - if $\mathrm{Re}\,(\beta)>0$, then $\mathcal{D}^\flat_{\langle x\rangle}(f)=V^m_{\langle x\rangle}$;
  - if $\mathrm{Re}\,(\beta)\le 0$, then $\mathcal{D}^\flat_{\langle x\rangle}(f)=V^m_{\langle x\rangle}\oplus\mathrm{Vect}(q_f)$.

Proposition 12 implies that $\blacktriangledown_f$ is injective on $\mathcal{D}^\flat_C(f)$. In fact, it implies more.

As observed in the previous section, if there are points in $B \cap f^{-1}(B) - A$, then the map $\nabla_f : \mathcal{Q}\big(\mathbb{P}^1, B \cap f^{-1}(B)\big) \to \mathcal{Q}(\mathbb{P}^1, B)$ induces a non trivial linear map

$$\blacktriangledown_f : \mathcal{E} \to T^\bigstar_{\mathbf{0}} \mathrm{Def}^B_A(f).$$

Proposition 13. *The map* $\blacktriangledown_f : \mathcal{D}^\flat_C(f) \oplus \mathcal{E} \to T^\bigstar_{\mathbf{0}} \mathrm{Def}^B_A(f)$ *is injective.*

7.2. **Transversality for multipliers of cycles.** We shall now present a second transversality result.

Let $f \in \mathbf{Rat}_D$ be a rational map having $k \geq 1$ disjoint cycles $\langle x_j \rangle, \ldots, \langle x_k \rangle$ with multipliers $\rho_1, \ldots, \rho_k$ in $\overline{\mathbb{D}} - \{0\}$. Assume further that those cycles do not contain critical values of f. Let $C \subset \mathbb{P}^1$ be the union of the cycles $\langle x_j \rangle$, $j \in [1, k]$. Let B be the union of $\mathcal{S}_f$, C and if necessary, a repelling cycle of f which does not intersect $\mathcal{S}_f$, so that $\mathrm{card}(B) \geq \mathrm{card}(\mathcal{S}_f) + 3$. Set $A = B - \mathcal{S}_f$ and note that

$$\mathrm{card}(A) = \mathrm{card}(B) - \mathrm{card}(\mathcal{S}_f) \geq 3.$$

Since f has at least one nonrepelling cycle, f is not a Lattès example and thus, $\mathrm{Def}^B_A(f)$ is a smooth analytic submanifold of $\mathrm{Teich}(\mathbb{P}^1, B)$ of dimension $\mathrm{card}(\mathcal{S}_f)$. Let $\vec{\varrho} : \mathrm{Def}^B_A(f) \to \mathbb{C}^k$ be the analytic map defined by

$$\vec{\varrho} = (\rho_{\langle x_1 \rangle}, \ldots, \rho_{\langle x_k \rangle}).$$

Proposition 14. *The linear map* $D_{\mathbf{0}} \vec{\varrho} : T_{\mathbf{0}} \mathrm{Def}^B_A(f) \to \mathbb{C}^k$ *is surjective.*

Proof. The divergences

$$[q_{\rtimes}]_{\langle x_j \rangle}, \quad j \in [1, k]$$

are linearly independent. Thanks to the injectivity of $\blacktriangledown_f$ on $\mathcal{D}^\flat_C(f)$, the logarithmic derivatives

$$D_{\mathbf{0}} \log \rho_{\langle x_j \rangle} = \blacktriangledown_f [q_{\rtimes}]_{\langle x_j \rangle}$$

are linearly independent. This shows that $D_{\mathbf{0}} \vec{\varrho}$ is surjective since it has maximal rank k. □

7.3. **Preserving the multiplicities of parabolic cycles.** Let $\langle x \rangle$ be a parabolic cycle of f contained in A, with multiplier $\mathrm{e}^{2\pi \mathrm{i} r/s}$ parabolic multiplicity m. As τ varies in $\mathrm{Def}^B_A(f)$, the multiplier of the corresponding cycle changes. There is a locus $Z^1_{\langle x \rangle} \subset \mathrm{Def}^B_A(f)$ where the multiplier of the cycle remains unchanged:

$$Z^1_{\langle x \rangle} = \big\{ \tau \in \mathrm{Def}^B_A(f) \ : \ \rho_{\langle x \rangle}(\tau) = \mathrm{e}^{2\pi \mathrm{i} r/s} \big\}.$$

Since the function $\rho_{\langle x \rangle}$ is analytic, $Z^1_{\langle x \rangle}$ is an analytic subset of $\mathrm{Def}^B_A(f)$.

Proposition 15. *The locus* $Z^1_{\langle x \rangle}$ *is a smooth* $\mathbb{C}$*-analytic submanifold of* $\mathrm{Def}^B_A(f)$ *of codimension* 1*. Its tangent space at* $\mathbf{0}$ *is the orthogonal of* $\blacktriangledown_f V^1_{\langle x \rangle}$*.*

Proof. Changing our basepoint in $Z^1_{\langle x \rangle}$ if necessary, it is enough to prove that $Z^1_{\langle x \rangle}$ is smooth at $\mathbf{0}$ and identify its tangent space at $\mathbf{0}$. The result follows from the Implicit Function Theorem. Indeed, we have seen that $D_{\mathbf{0}} \log \rho_{\langle x \rangle} = \blacktriangledown_f [q_{\rtimes}]_{\langle x \rangle}$ and according to proposition 13, this logarithmic derivative is non zero. □

If $\tau \in Z^1_{\langle x\rangle}$ is represented by a triple (φ,ψ,F), we let $m_{\langle x\rangle}(\tau)$ be the parabolic multiplicity of the cycle $\big\langle\varphi(x)\big\rangle$ of F. For $1 \le k \le m-1$, we define

$$Z^k_{\langle x\rangle} = \big\{\tau \in Z^1_{\langle x\rangle} \ : \ m_{\langle x\rangle}(\tau) \ge k\big\}.$$

Then,

$$Z^1_{\langle x\rangle} \supset Z^2_{\langle x\rangle} \supset \cdots \supset Z^m_{\langle x\rangle}.$$

We shall now prove that those are smooth $\mathbb{C}$-analytic submanifold of $\mathrm{Def}^B_A(f)$ of codimension k and identify their cotangent spaces.

Note that for all $k \in [1,m]$, the dimension of $V^k_{\langle x\rangle}$ is k, and since $\blacktriangledown_f$ is injective on $V^k_{\langle x\rangle} \subseteq \mathcal{D}^\flat_C(f)$, the dimension of $\blacktriangledown_f V^k_{\langle x\rangle}$ is also k.

Proposition 16. *For all $k \in [1,m]$, the locus $Z^k_{\langle x\rangle}$ is a smooth $\mathbb{C}$-analytic submanifold of $\mathrm{Def}^B_A(f)$ of codimension k. Its tangent space at $\mathbf{0}$ is the orthogonal of $\blacktriangledown_f V^k_{\langle x\rangle}$.*

Proof. As we will see, the result follows from the Implicit Function Theorem and the injectivity of $\blacktriangledown_f$ on $V^k_{\langle x\rangle}$. Changing our basepoint in $Z^1_{\langle x\rangle}$ if necessary, it is enough to prove that $Z^k_{\langle x\rangle}$ is smooth at $\mathbf{0}$ and show that its tangent space at $\mathbf{0}$ is the orthogonal of $\blacktriangledown_f V^k_{\langle x\rangle}$.

The proof goes by induction on $k \ge 1$. For $k=1$, this is given by proposition 15. So, let us assume $Z^k_{\langle x\rangle}$ is smooth of codimension k, and that its tangent space at $\mathbf{0}$ is the orthogonal of $\blacktriangledown_f V^k_{\langle x\rangle}$.

If $\mathbf{U}$ is a sufficiently small neighborhood of $\mathbf{0}$ in $\mathrm{Def}^B_A(f)$, then all $\tau \in \mathbf{U}$ may be represented by a triple $(\varphi_\tau,\psi_\tau,f_\tau)$ such that

- φ_τ and ψ_τ are smooth on $\mathbb{P}^1$ and holomorphic near A;
- $\varphi_{\mathbf{0}} = \psi_{\mathbf{0}} = \mathrm{I}$;
- φ_τ, ψ_τ and f_τ depend analytically on τ.

Set

$$\chi_\tau = \psi_\tau^{-1} \circ \varphi_\tau \quad \text{and} \quad g_\tau = f \circ \chi_\tau.$$

Let ζ be a preferred coordinate for $f^{\circ p}$ at x. If $\tau \in Z^k_{\langle x\rangle} \cap \mathbf{U}$, then x is a multiple fixed point of $g_\tau^{\circ sp}$ with multiplicity $\ge sk+1$. So,

$$\zeta \circ g_\tau^{\circ sp} = \zeta + \gamma_\tau \cdot \zeta^{sk+1} + \mathcal{O}(\zeta^{sk+2}) \quad \text{with} \quad \gamma_\tau \in \mathbb{C}.$$

Since the map $\tau \mapsto g_\tau^{\circ sp}$ is analytic, the map

$$\gamma : Z^k_{\langle x\rangle} \cap \mathbf{U} \ni \tau \mapsto \gamma_\tau \in \mathbb{C}$$

is analytic. In addition,

$$Z^{k+1}_{\langle x\rangle} \cap \mathbf{U} = \big\{\tau \in Z^k_{\langle x\rangle} \cap \mathbf{U} \ : \ \gamma_\tau = 0\big\}.$$

This shows that Z^{k+1}_τ is analytic at $\mathbf{0}$. In order to prove that Z^{k+1}_τ is smooth at $\mathbf{0}$, we will determine the derivative $D_{\mathbf{0}}\gamma$.

Lemma 8. *The derivative $D_{\mathbf{0}}\gamma : T_{\mathbf{0}} Z^k_{\langle x\rangle} \to \mathbb{C}$ is $s \cdot \blacktriangledown_f \circ [q_{k+1}]_{\langle x\rangle}$.*

Proof. Let v be a tangent vector in $T_{\mathbf{0}} Z^k_{\langle x\rangle}$ and let $(\tau_t)_{t\in U}$ be an analytic curve in $Z^k_{\langle x\rangle} \cap \mathbf{U}$ with $\tau_0 = \mathbf{0}$ and $\dot\tau = v$. With an abuse of notations, write $f_t, \chi_t, \gamma_t, \ldots$

in place of $f_{\tau_t}, \chi_{\tau_t}, \gamma_{\tau_t}, \ldots$ Set $\theta = \dot\chi$, so that $[\theta]_{\langle x\rangle} = \blacktriangledown_f^\star(v)$. If U is a sufficiently small disk containing x, we have that

$$\gamma_t = \mathrm{Res}_x \frac{\zeta\circ g_t^{\circ sp}}{\zeta^{sk+2}}\mathrm{d}\zeta = \frac{1}{2\pi i}\oint_{\partial U} \frac{\zeta\circ g_t^{\circ sp}}{\zeta^{sk+2}}\,\mathrm{d}\zeta.$$

As in the proof of lemma 7, it follows that

$$\dot\gamma = \frac{1}{2\pi \mathrm{i}}\oint_{\partial U}\frac{\mathrm{d}\zeta\big(D(f^{\circ sp})\circ\theta_{sp}\big)}{\zeta^{sk+2}}\,\mathrm{d}\zeta \quad \text{with} \quad \theta_{sp} = \sum_{k=0}^{sp-1}(f^{\circ k})^*\theta.$$

Now, Since ζ is a preferred coordinate at x, since $\mathrm{d}\zeta(\theta_{sp})$ is a holomorphic function vanishing at x and since $ms - sk - 2 \geq s - 2 \geq -1$, we have that

$$\frac{\mathrm{d}\zeta\big(D(f^{\circ sp})\circ\theta_{sp}\big)}{\zeta^{sk+2}} = \frac{\mathrm{d}\zeta(\theta_{sp})}{\zeta^{sk+2}} + \mathcal{O}\left(\frac{\zeta^{ms+1}}{\zeta^{sk+2}}\right) = \frac{\mathrm{d}\zeta(\theta_{sp})}{\zeta^{sk+2}} + \mathcal{O}(1).$$

As in the proof of lemma 7, $[\theta_{sp}]_x = s\cdot[\triangleright_x\theta]_x$. So, according to equation (7),

$$\begin{aligned} D_{\mathbf{0}}\gamma(v) = \mathrm{Res}_x\left(\frac{\mathrm{d}\zeta^2}{\zeta^{sk+2}}\otimes\theta_{sp}\right) &= s\cdot\left\langle\left[\frac{\mathrm{d}\zeta^2}{\zeta^{sk+2}}\right], [\triangleright_x\theta]\right\rangle_x \\ &= s\cdot\big\langle[q_{k+1}], \blacktriangledown_f^\star(v)\big\rangle_{\langle x\rangle} = s\cdot\big\langle\blacktriangledown_f[q_{k+1}], v\big\rangle_{\langle x\rangle}. \quad \square \end{aligned}$$

According to proposition 13, the derivative $D_{\mathbf{0}}\gamma : T_{\mathbf{0}}Z^{k-1}_{\langle x\rangle} \to \mathbb{C}$ is non zero. Otherwise, $\blacktriangledown_f[q_{k+1}]_{\langle x\rangle}$ would belong to the orthogonal of $T_{\mathbf{0}}Z^k_{\langle x\rangle}$, i.e., we would have that $\blacktriangledown_f[q_{k+1}]_{\langle x\rangle} \in \blacktriangledown_f V^k_{\langle x\rangle}$, which is not possible since $[q_{k+1}]_{\langle x\rangle} \in V^{k+1}_{\langle x\rangle} - V^k_{\langle x\rangle}$ and $\blacktriangledown_f$ is injective on $V^{k+1}_{\langle x\rangle}$.

Thus, we may use the Implicit Function Theorem to conclude that $Z^{k+1}_{\langle x\rangle}$ is smooth at $\mathbf{0}$ and that the tangent space $T_{\mathbf{0}}Z^{k+1}_{\langle x\rangle}$ is the orthogonal of $\blacktriangledown_f V^{k+1}_{\langle x\rangle}$. $\square$

Part 2. The moduli space of rational maps

We will now explain how transversality results obtained in $\mathrm{Def}_A^B(f)$ might be transferred to the spaces $\mathbf{Rat}_D$ or Rat_D. Since Rat_D is *a priori* not a smooth manifold, it may be convenient to work in $\mathbf{Rat}_D$.

8. The tangent spaces to $\mathbf{Rat}_D$ and Rat_D

8.1. The tangent spaces to $\mathbf{Rat}_D$. Let $\mathcal{C}_f$ be the set of critical points of f (there are $2D-2$ such points, counting multiplicities). The tangent space to $\mathbf{Rat}_D$ at f is the space $T_f\mathbf{Rat}_D$ of global holomorphic sections of the vector bundle

$$f^\star(T\mathbb{P}^1) = \big\{(z,v)\in\mathbb{P}^1\times T\mathbb{P}^1\ ; v\in T_{f(z)}\mathbb{P}^1\big\}.$$

Given $\xi \in T_f\mathbf{Rat}_D$, there is a unique meromorphic vector field η_ξ on $\mathbb{P}^1$, with poles contained in $\mathcal{C}_f$, such that $Df\circ\eta_\xi = -\xi$. Indeed, if z is not a critical point of f, then $D_zf : T_z\mathbb{P}^1 \to T_{f(z)}\mathbb{P}^1$ is an isomorphism, whence we may define

$$\eta_\xi(z) = D_zf^{-1}\big(-\xi(z)\big)$$

which in coordinates consists of dividing by $f'(z)$. The order of the pole at $c\in\mathcal{C}_f$ is at most the multiplicity of c as a critical point of f. In other words, η_ξ is an element of $H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)\big)$ where $\mathbf{C}_f$ is the critical divisor of f (the weight of each critical point is the local degree minus 1) and $\Theta_{\mathbb{P}^1}$ is the tangent sheaf to $\mathbb{P}^1$. The linear map

$$T_f\mathbf{Rat}_D \ni \xi \mapsto \eta_\xi \in H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)\big),$$

is clearly injective since $\eta_\xi = 0$ if and only if $\xi = 0$. It is an isomorphism since the dimensions of $T_f\mathbf{Rat}_D$ and $H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)\big)$ are both equal to $2D+1$.

8.2. **The tangent space to** Rat_D**.** The group $\mathrm{Aut}(\mathbb{P}^1)$ of Möbius transformations acts on $\mathbf{Rat}_D$ by conjugacy. The tangent space to $\mathrm{Aut}(\mathbb{P}^1)$ at the identity is the Lie algebra $\mathrm{aut}(\mathbb{P}^1)$ of globally holomorphic vector fields on $\mathbb{P}^1$.

The derivative of the map

$$\mathrm{Aut}(\mathbb{P}^1) \ni M \mapsto M \circ f \circ M^{-1} \in \mathbf{Rat}_D$$

is the linear map

$$\mathrm{aut}(\mathbb{P}^1) \ni \theta \mapsto \theta \circ f - Df \circ \theta \in T_f\mathbf{Rat}_D.$$

With the identification $T_f\mathbf{Rat}_D \simeq H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)\big)$, this derivative is identified to the linear map

$$\Delta_f = \mathrm{I} - f^* : \mathrm{aut}(\mathbb{P}^1) \to H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)\big).$$

Lemma 9. *The linear map Δ_f is injective on* $\mathrm{aut}(\mathbb{P}^1)$.

Proof. Assume θ is a holomorphic vector field on $\mathbb{P}^1$ such that $f^*\theta = \theta$. Then, $f^*\theta$ does not have any pole. It follows that θ vanishes at the critical values of f and thus, $\theta = f^*\theta$ vanishes at the critical points of f. Since a holomorphic vector field on $\mathbb{P}^1$ has at most 2 zeros, either $\theta = 0$ or f is a bicritical map for which the set of critical values coincides with the set of critical points. In the latter case, f is a power map: we may work in a coordinate on $\mathbb{P}^1$ in which f takes the form $z \mapsto z^k$ with $k = \pm d$. In this coordinate, θ vanishes at 0 and ∞, and thus is of the form $\lambda z \dfrac{\partial}{\partial z}$ for some $\lambda \in \mathbb{C}$. But we then have $f^*\theta = \dfrac{1}{k}\theta$, which yields $\theta = 0$ since $k \neq 1$. □

Denote by $\mathcal{O}(f)$ the set of rational maps which are conjugate to f by a Möbius transformation. It is a smooth submanifold of $\mathbf{Rat}_D$ of complex dimension 3, isomorphic to $\mathrm{Aut}(\mathbb{P}^1)/\Gamma$, where $\Gamma \subset \mathrm{Aut}(\mathbb{P}^1)$ is the finite subgroup of Möbius transformations which commute with f. The tangent space $T_f\mathcal{O}(f)$ is the image of $\Delta_f : \mathrm{aut}(\mathbb{P}^1) \to H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)\big)$.

Definition 17. *We denote $\underline{T_f\mathbf{Rat}_D}$ the quotient space*

$$\underline{T_f\mathbf{Rat}_D} = H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)\big)/\Delta_f\big(\mathrm{aut}(\mathbb{P}^1)\big).$$

The space $\underline{T_f\mathbf{Rat}_D}$ is canonically isomorphic to $T_{[f]}\mathrm{Rat}_D$ at smooth points of Rat_D and is the appropriate orbifold tangent space at singular points.

Definition 18. *Let X be a complex manifold. If $L : X \to \mathbf{Rat}_D$ is an analytic map sending $x \in X$ to $f \in \mathbf{Rat}_D$, we define $\underline{D_xL} : T_xX \to \underline{T_f\mathbf{Rat}_D}$ by*

$$\forall v \in T_xX, \quad \underline{D_xL}(v) = \big[D_xL(v)\big] \in \underline{T_f\mathbf{Rat}_D}.$$

Lemma 10. *Let $L : X \to \mathbf{Rat}_D$ be an analytic map sending $x \in X$ to $f \in \mathbf{Rat}_D$.*

- *The linear map* $\underline{D_xL} : T_xX \to \underline{T_f\mathbf{Rat}_D}$ *is surjective if and only if the image of* D_xL *is transverse to* $T_f\mathcal{O}(f)$*:*

$$T_f\mathcal{O}(f) + \mathrm{Im}(D_xL) = T_f\mathbf{Rat}_D.$$

- *The linear map* $\underline{D_xL} : T_xX \to \underline{T_f\mathbf{Rat}_D}$ *is injective if and only if*

$$T_f\mathcal{O}(f) \cap \mathrm{Im}(D_xL) = \{0\}.$$

8.3. **Examples.**

8.3.1. Consider the analytic map $L : \mathbb{C} \to \mathrm{Rat}_2$ which sends $c \in \mathbb{C}$ to the quadratic polynomial $P_c(z) = z^2 + c$. We claim that $\underline{D_cL}$ is injective for every $c \in \mathbb{C}$, or equivalently that the tangent vector $\xi \in T_{P_c}\mathrm{Rat}_2$ defined by :

$$\xi(z) = \frac{\partial P_c(z)}{\partial c} \in T_{P_c(z)}\mathbb{P}^1$$

does not belong to $T_{P_c}\mathcal{O}(P_c)$. Note that

$$\xi(z) = \frac{\partial}{\partial z} \quad \text{whence} \quad \eta_\xi = -\frac{1}{2z}\frac{\partial}{\partial z}.$$

In particular, η_ξ has 3 zeros at infinity and a simple pole at 0. Assume there is a holomorphic vector field $\theta \in \mathrm{aut}(\mathbb{P}^1)$ such that $\eta_\xi = \theta - P_c^*\theta$. Since η_ξ vanishes at infinity, $P_c^*\theta$ cannot have a pole at infinity, thus θ vanishes at ∞. An elementary computation shows that if θ has a simple zero at infinity, then $\Delta_{P_c}(\theta)$ also has a simple zero at infinity. Thus,

$$\theta(z) = a\frac{\partial}{\partial z} \quad \text{with} \quad a \in \mathbb{C}$$

and

$$\Delta_{P_c}(\theta)(z) = \Big(a - \frac{a}{2z}\Big)\frac{\partial}{\partial z} = -\frac{1}{2z}\frac{\partial}{\partial z}.$$

This is not possible. Thus, η_ξ does not belong to $\Delta_{P_c}\big(\mathrm{aut}(\mathbb{P}^1)\big)$.

8.3.2. Consider the analytic map $L : \mathbb{C} \to \mathrm{Rat}_2$ which sends $\lambda \in \mathbb{C}$ to the quadratic polynomial $Q_\lambda(z) = \lambda z + z^2$. We claim that $\underline{D_\lambda L}$ is not injective for $\lambda = 1$. Indeed, its image is the vector space spanned by $\xi \in T_{Q_1}\mathrm{Rat}_2$, where

$$\xi(z) = \frac{\partial Q_\lambda(z)}{\partial \lambda} \in T_{Q_1(z)}\mathbb{P}^1.$$

We have

$$\eta_\xi = -\frac{z}{1+2z}\frac{\partial}{\partial z} = \Delta_{Q_1}(\theta) \quad \text{with} \quad \theta = -\frac{1}{2}\frac{\partial}{\partial z} \in \mathrm{aut}(\mathbb{P}^1).$$

9. The derivative of $\underline{\Phi}$

We will now study the derivative $D_{\mathbf{0}}\underline{\Phi}$. Since Rat_D is not a manifold, we need to explain what we mean by the derivative $D_{\mathbf{0}}\underline{\Phi}$. In particular, we need to give some explanations regarding the tangent space to Rat_D at $[f]$.

9.1. **Exact sequences.** Denote by $-\mathbf{A}$ the divisor on $\mathbb{P}^1$ which consists of weighting points in A with weight -1. Denote by $-\mathbf{B}$ the divisor on $\mathbb{P}^1$ which consists of weighting points in B with weight -1.

As mentioned earlier, the short exact sequence of sheaves

$$0 \to \Theta_{\mathbb{P}^1}(-\mathbf{B}) \to \Theta_{\mathbb{P}^1} \to \Theta_{\mathbb{P}^1}/\Theta_{\mathbb{P}^1}(-\mathbf{B}) \to 0$$

induces an exact sequence

$$H^0\big(\Theta_{\mathbb{P}^1}(-\mathbf{B})\big) \to H^0(\Theta_{\mathbb{P}^1}) \to H^0\big(\Theta_{\mathbb{P}^1}/\Theta_{\mathbb{P}^1}(-\mathbf{B})\big) \to H^1\big(\Theta_{\mathbb{P}^1}(-\mathbf{B})\big) \to H^1(\Theta_{\mathbb{P}^1})$$

which we identify as

$$0 \to \mathrm{aut}(\mathbb{P}^1) \to \bigoplus_{b\in B} T_b\mathbb{P}^1 \to T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1, B) \to 0. \tag{8}$$

In addition, the short exact sequence of sheaves

$$0 \to \Theta_{\mathbb{P}^1}(-\mathbf{A}) \to \Theta_{\mathbb{P}^1}(\mathbf{C}_f) \to \Theta_{\mathbb{P}^1}(\mathbf{C}_f)/\Theta_{\mathbb{P}^1}(-\mathbf{A}) \to 0$$

leads to a long exact sequence

$$\begin{aligned} &H^0(\Theta_{\mathbb{P}^1}(-\mathbf{A})) \to H^0(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)) \to H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)/\Theta_{\mathbb{P}^1}(-\mathbf{A})\big) \to \\ \to\; &H^1(\Theta_{\mathbb{P}^1}(-\mathbf{A})) \to H^1(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)) \end{aligned}$$

which we will identify as

$$0 \to H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)\big) \to \bigoplus_{a\in A\cup\mathcal{C}_f} S_a \to T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1, A) \to 0,$$

where S_a be the vector space defined by

- if $a \notin \mathcal{C}_f$, then $S_a = T_a\mathbb{P}^1$,
- if $a \in \mathcal{C}_f$ is a critical point of multiplicity m_a, then S_a is the space of polar parts of meromorphic vector fields at a of degree k with
  - $-m_a \le k \le -1$ if $a \notin A$ and
  - $-m_a \le k \le 0$ if $a \in A$.

Indeed,

- as above, $H^0(\Theta_{\mathbb{P}^1}(-\mathbf{A})) = 0$ since A contains at least 3 points;
- since $\Theta_{\mathbb{P}^1}(\mathbf{C}_f)/\Theta_{\mathbb{P}^1}(-\mathbf{A})$ is a skyscraper sheaf supported on $A \cup \mathcal{C}_f$ with fiber S_a at each $a \in A\cup\mathcal{C}_f$, the space $H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)/\Theta_{\mathbb{P}^1}(-\mathbf{A})\big)$ is canonically identified to the direct sum
$$\bigoplus_{a\in A\cup\mathcal{C}_f} S_a;$$
- as above, $H^1\big(\Theta_{\mathbb{P}^1}(-\mathbf{A})\big) \simeq T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1, A)$.

9.2. **A commutative diagram.** We will now write a commutative diagram. Applying the Snake Lemma to this diagram yields a map which, as we shall see later, is the derivative $\underline{D_{\mathbf{0}}\Phi}$. In particular, we will use this construction to identify the kernel and the cokernel of $\underline{D_{\mathbf{0}}\Phi}$.

Proposition 17. *We have the following commutative diagram*

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
0 & \longrightarrow & \mathrm{aut}(\mathbb{P}^1) & \xrightarrow{\Delta_f} & H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)\big) & \longrightarrow & \underline{T_f\mathbf{Rat}_D} \\
 & & \downarrow & & \downarrow & & \\
\mathcal{K} & \longrightarrow & \bigoplus_{b\in B} T_b\mathbb{P}^1 & \xrightarrow{\Delta_f} & \bigoplus_{a\in A\cup\mathcal{C}_f} S_a & \longrightarrow & \mathcal{S} \\
 & & \downarrow & & \downarrow & & \\
T_{\mathbf{0}}\mathrm{Def}_A^B(f) & \longrightarrow & T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1,B) & \xrightarrow{D_{\mathbf{0}}\varpi - D_{\mathbf{0}}\sigma_f} & T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1,A) & \longrightarrow & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

Proof. First, we have to define the middle arrow $\bigoplus_{b\in B} T_b\mathbb{P}^1 \xrightarrow{\Delta_f} \bigoplus_{a\in A\cup\mathcal{C}_f} S_a$. Assume θ is a vector field, defined and holomorphic in a neighborhood of B. If θ vanishes on B, then $\theta - f^*\theta$ vanishes on A. It follows that the polar parts of degree ≤ 0 of $\theta - f^*\theta$ only depends on the restriction of θ to B. The middle arrow is defined as follows: given a vector field θ on B, let ϑ be a holomorphic extension to a neighborhood of B and let $\Delta_f(\theta)$ be the appropriate restriction of the polar parts of $\vartheta - f^*(\vartheta)$ at points of A. The result will not depend on the choice of extension ϑ.

With this definition, it is clear that the top square

$$\begin{array}{ccc}
\mathrm{aut}(\mathbb{P}^1) & \xrightarrow{\Delta_f} & H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)\big) \\
\downarrow & & \downarrow \\
\bigoplus_{b\in B} T_b\mathbb{P}^1 & \xrightarrow{\Delta_f} & \bigoplus_{a\in A\cup\mathcal{C}_f} S_a
\end{array}$$

commutes.

To show that the bottom square

$$\begin{array}{ccc}
\bigoplus_{b\in B} T_b\mathbb{P}^1 & \xrightarrow{\Delta_f} & \bigoplus_{a\in A\cup\mathcal{C}_f} S_a \\
\downarrow & & \downarrow \\
T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1,B) & \xrightarrow{D_{\mathbf{0}}\varpi - D_{\mathbf{0}}\sigma_f} & T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1,A)
\end{array}$$

commutes, let θ be a vector field on B and let ϑ be a C^∞ extension to $\mathbb{P}^1$ which is holomorphic in a neighborhood of B. The infinitesimal Beltrami form $\nu = \bar\partial\vartheta$ represents an element of $T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1, B)$ which does not depend on the extension ϑ (since the difference of two extensions if a vector field that vanishes on B). The image of θ under the map $\bigoplus_{b\in B} T_b\mathbb{P}^1 \to T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1, B)$ is the class of ν. Going through the bottom left corner of the diagram, we end up with $\nu - f^*\nu$. Going through the top right corner of the diagram, we end up with $\bar\partial(\vartheta - f^*\vartheta)$. The commutativity of the diagram follows from

$$\bar\partial\vartheta - f^*(\bar\partial\vartheta) = \bar\partial(\vartheta - f^*\vartheta). \qquad \square$$

9.3. **The derivative $\underline{D_{\mathbf{0}}\Phi}$.** According to the Snake Lemma, there is a map

$$\blacktriangle_f : T_{\mathbf{0}}\mathrm{Def}_A^B(f) \to \underline{T_f\mathbf{Rat}_D}$$

so that the following sequence is exact

$$0 \longrightarrow \mathcal{K} \longrightarrow T_{\mathbf{0}}\mathrm{Def}_A^B(f) \xrightarrow{\blacktriangle_f} \underline{T_f\mathbf{Rat}_D} \longrightarrow \mathcal{S} \longrightarrow 0.$$

Theorem 3. *The derivative $D_{\mathbf{0}}\Phi$ is :*

$$\underline{D_{\mathbf{0}}\Phi} = \blacktriangle_f : T_{\mathbf{0}}\mathrm{Def}_A^B(f) \to \underline{T_f\mathbf{Rat}_D}.$$

Proof. Assume $\nu_2 \in \mathrm{bel}(\mathbb{P}^1)$ represents a tangent vector v to $T_{\mathbf{0}}\mathrm{Def}_A^B(f)$, i.e.,

$$D_0\Pi(\nu_2) = v.$$

Without loss of generality, we may assume that ν_2 is of class C^∞ and vanishes on a neighborhood of B. Then, we can find an analytic family of Beltrami forms $(\mu_{2,t})_{t\in\mathbb{D}}$, such that

- $\mu_{2,0} = 0$,
- $\mu_{2,t} = 0$ vanishes in a neighborhood of B for all $t \in \mathbb{D}$,
- $\tau_t = \Pi(\mu_{2,t})$ belongs to $\mathrm{Def}_A^B(f)$ and
- $\dot\mu_2 = \nu_2$.

Set $\mu_{1,t} = f^*\mu_{2,t}$. Then, $\dot\mu_1 = \nu_1 = f^*\nu_2 \in \mathrm{bel}(\mathbb{P}^1)$ represents the tangent vector

$$D_{\mathbf{0}}\varpi(v) = D_{\mathbf{0}}\sigma_f(v) \in T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1, A).$$

Let $\psi_t : \mathbb{P}^1 \to \mathbb{P}^1$ and $\varphi_t : \mathbb{P}^1 \to \mathbb{P}^1$ be quasiconformal homeomorphisms depending analytically on $t \in \mathbb{D}$, such that

- $\psi_0 = \mathrm{I} : \mathbb{P}^1 \to \mathbb{P}^1$, $\varphi_0 = \mathrm{I} : \mathbb{P}^1 \to \mathbb{P}^1$,
- φ_t and ψ_t coincide on A,
- $\bar\partial\psi_t = \partial\psi_t \circ \mu_{1,t}$ and $\bar\partial\varphi_t = \partial\varphi_t \circ \mu_{2,t}$.

Since φ_t depends analytically on $t \in \mathbb{D}$, its derivative $\dot\varphi$ with respect to t at $t = 0$ is a continuous vector field on $\mathbb{P}^1$ with distributional derivatives in L^∞. Since $\mu_{2,t}$ vanishes near B, the maps φ_t are holomorphic near B and so,

$$\dot\varphi = \lim_{t\to 0} \frac{\varphi_t - \varphi}{t}$$

is holomorphic near B. Similarly, $\dot\psi$ is continuous with distributional derivatives in L^∞ and holomorphic near A.

Set $f_t = \varphi_t \circ f \circ \psi_t^{-1}$. Then, by construction,

$$\underline{\Phi}(\tau_t) = [f_t] \in \mathrm{Rat}_D.$$

So, the classes of $D_{\mathbf{0}}\Phi(v)$ and $\dot f$ in the quotient space $\underline{T_f\mathbf{Rat}_D}$ are equal. Note that

$$\dot f(z) = \dot\varphi \circ f(z) - D_z f \circ \dot\psi(z)$$

and its image by the isomorphism $T_f\mathbf{Rat}_D \to H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)\big)$ is the meromorphic vector field

$$\dot\psi - f^*\dot\varphi \in H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)\big).$$

Thus, we must show that

$$[\dot\psi - f^*\dot\varphi] = \blacktriangle_f(v) \in \underline{T_f\mathbf{Rat}_D}.$$

For this, we will follow the construction of $\blacktriangle_f$ given by the snake lemma.

(1) The vector $v \in T_{\mathbf{0}}\mathrm{Def}_A^B(f)$ is a vector of $T_{\mathbf{0}}\mathrm{Teich}(\mathbb{P}^1, B)$.
(2) It has a preimage in $\bigoplus_{b\in B} T_b\mathbb{P}^1$ which is simply the restriction to B of the vector field $\dot\varphi$ (which is holomorphic near B). Its image ζ by Δ_f in $\bigoplus_{a\in A\cup\mathcal{C}_f} S_a$ consists of the appropriate polar parts along $A\cup\mathcal{C}_f$ of $\dot\varphi - f^*\dot\varphi$.
(3) Since φ_t coincide with ψ_t on A, ζ consists also of the appropriate polar parts along $A\cup\mathcal{C}_f$ of $\dot\psi - f^*\dot\varphi \in H^0\big(\Theta_{\mathbb{P}^1}(\mathbf{C}_f)\big)$.
(4) Thus, $\blacktriangle_f(v)$ is the class of $\dot\psi - f^*\dot\varphi$ in $\underline{T_f\mathbf{Rat}_D}$ as required.

□

9.4. **The kernel of $\underline{D_{\mathbf{0}}\Phi}$.** The kernel $\mathcal{K}$ of $\underline{D_{\mathbf{0}}\Phi}$ is the kernel of the map

$$\Delta_f : \bigoplus_{b\in B} T_b\mathbb{P}^1 \to \bigoplus_{a\in A\cup\mathcal{C}_f} S_a.$$

We may understand this as follows: an element of $\bigoplus_{b\in B} T_b\mathbb{P}^1$ is a vector field θ on B. This vector field belongs to the kernel of Δ_f if and only if

- θ vanishes on the set of critical values of f, so that pullback any holomorphic extension of θ in a neighborhood of a critical value $v = f(c)$, we do not create a pole at c, but rather a zero and
- for any $a \in A$, we have $f^*\theta = \theta$ at $\psi(a)$ (which in particular implies that $\theta(c) = 0$ for all critical points c of f contained in A).

Example. Consider the quadratic polynomial $f(z) = z^2 + 1/4$. Set

$$A = \{0, 1/2, \infty\} \quad \text{and} \quad B = \{0, 1/4, 1/2, \infty\}$$

(note that $f(1/2) = 1/2$ and $f'(1/2) = 1$). Then $\mathcal{K}$ is the line defined by the equations:

$$\theta(0) = \theta(1/4) = \theta(\infty) = 0.$$

There is no condition on the vector $\theta(1/2)$.

In fact, when A contains a cycle of f with multiplier 1, the map $\underline{D_{\mathbf{0}}\Phi}$ is not injective. Equivalently, if $\Phi : \mathrm{Def}_A^B(f) \to \mathbf{Rat}_D$ induces $\underline{\Phi} : \mathrm{Def}_A^B(f) \to \mathrm{Rat}_D$, the tangent space $T_{\mathbf{0}}\mathrm{Def}_A^B(f)$ contains a vector whose image by $D_{\mathbf{0}}\Phi$ is tangent to the orbit $\mathcal{O}(f)$ of f under conjugacy by Möbius transformations.

Definition 19. *We say that two points $b_1 \in B$ and $b_2 \in B$ are (f, A)-equivalent if there are integers $j_1 \ge 0$ and $j_2 \ge 0$ such that*

- $f^{\circ j_1}(b_1) = f^{\circ j_2}(b_2)$,
- $f^{\circ j}(b_1) \in A$ *for all* $j \in [0, j_1 - 1]$ *and* $f^{\circ j}(b_2) \in A$ *for all* $j \in [0, j_2 - 1]$.

Proposition 18. *Let $\mathcal{K}$ be the kernel of the map $\blacktriangle_f : T_{\mathbf{0}}\mathrm{Def}_A^B(f) \to \underline{T_f\mathbf{Rat}_D}$. The dimension of $\mathcal{K}$ is the number of (f, A)-equivalence classes which do not contain a critical value of f and do not contain a cycle of f with multiplier $\neq 1$.*

Proof. An element of $\mathcal{K}$ is a vector field θ on B. Let N be the number of $\big(f,A\big)$-equivalence classes which do not contain a critical value of f and do not contain a cycle of f whose multiplier is $\neq 1$. Let $b_i \in B$, $i \in [1,N]$, be representatives of those N classes chosen as follows:

- if the class does not contain a cycle of f, we let b_i be the unique point which is not in A;
- if the class contains a cycle of f, we let b_i be any point of the cycle.

Then, every point in the class is an iterated preimage of b_i. The proposition is an immediate consequence of the following lemma. □

Lemma 11. *The restriction from* $\mathcal{K} \subset \bigoplus_{b\in B} T_b\mathbb{P}^1$ *to* $\bigoplus_{i=1}^{N} T_{b_i}\mathbb{P}^1$ *is an isomorphism.*

Proof. Assume $\theta \in \mathcal{K}$. Recall that θ vanishes at the critical values of f and $f^*\big(\theta(a)\big) = \theta(a)$ for all $a \in A$ (which means $\theta(a) = 0$ if a is a critical point of f). Thus,

- θ vanishes at all point of $\big(f,A\big)$-equivalence classes containing a critical value of f;
- θ vanishes at all point of $\big(f,A\big)$-equivalence classes containing a cycle of f with multiplier $\neq 1$; indeed, if a belongs to a periodic cycle of f of period p contained in A, then
$$(f^{\circ p})^*\big(\theta(a)\big) = \theta(a),$$
so, if the multiplier of the cycle is $\neq 1$, then $\theta(a) = 0$;
- if θ vanishes at a point b_i, $i \in [1,N]$, then it vanishes at any point b in the $\big(f,A\big)$-equivalence class of b_i since $\theta(b) = (f^{\circ j})^*\big(\theta(b_i)\big)$ for some $j \geq 0$.

Thus, the restriction from $\mathcal{K}$ to $\bigoplus_{i=1}^{N} T_{b_i}\mathbb{P}^1$ is injective.

For $i \in [1,N]$, let θ_i be any vector in $T_{b_i}\mathbb{P}^1$. If $b \in B$ is not $\big(f,A\big)$-equivalent to one of the b_i, we set $\theta(b) = 0$. Otherwise, there is an integer j such that $f^{\circ j}(b) = b_i$ and we set $\theta(b) = (f^{\circ j})^*\big(\theta(b_i)\big)$. We are allowed to pullback since the class of b_i does not contain a critical value of f, and so, $f^{\circ j}$ does not have a critical point at b. The result does not depend on j since when b_i is periodic for f, its multiplier is 1. The resulting vector field on B belongs to $\mathcal{K}$. Thus, the restriction from $\mathcal{K}$ to $\bigoplus_{i=1}^{N} T_{b_i}\mathbb{P}^1$ is surjective. □

9.5. **Injectivity of $\underline{D_{\mathbf{0}}\Phi}$.** In order to transfer transversality results from $\mathrm{Def}_A^B(f)$ to the space of rational maps, it may be useful to give additional informations regarding the kernel of $\blacktriangle_f$.

We will now work under the hypothesis that every (f,A)-equivalence class contains a critical value or a cycle of f. We have seen that a cycle of f with multiplier 1, contained in A and whose (f,A)-equivalence class do not contain a critical value of f contributes to one dimension in the kernel of $\blacktriangle_f = \underline{D_{\mathbf{0}}\Phi}$. Let $\langle x_1\rangle, \ldots, \langle x_k\rangle$ be the cycles in A which have multiplier 1 and are not in the (f,A)-equivalence class of a critical value of f. Let p_j be the period of $\langle x_j\rangle$ and let m_j be its parabolic multiplicity. Finally, set

$$T_j^{m_j-1} = (\blacktriangledown_f V_{\langle x_j\rangle}^{m_j-1})^\star \quad \text{and} \quad T_j^{m_j} = (\blacktriangledown_f V_{\langle x_j\rangle}^{m_j})^\star$$

with the convention that $T_j^0 = T_{\mathbf{0}}\mathrm{Def}_A^B(f)$.

Proposition 19. *The linear map $\blacktriangle_f$ is injective on $T_1^{m_j} \cap \cdots \cap T_k^{m_k}$ and its kernel is contained in $T_1^{m_j-1} + \cdots + T_k^{m_k-1}$.*

Proof. Let θ be a vector field on B representing a vector field $v \in \mathrm{Ker}(\blacktriangledown_f)$. Let ϑ be an extension of θ, holomorphic near B and C^∞ on $\mathbb{P}^1$. Since $v \in \mathrm{Ker}(\blacktriangledown_f)$, we have that $\theta = f^*\theta$ on A, in particular along the cycles $\langle x_j\rangle$. Thus, without loss of generality, we may assume that if $y \in \langle x_j\rangle$ with $f^{\circ n}(y) = x_j$ for some $n \in [1, p_j-1]$, then $\vartheta = (f^{\circ n})^*\vartheta$ near y. If $y \in \langle x_j\rangle$, then

- $f^*\vartheta = \vartheta$ is $y \neq x_j$ and
- $f^*\vartheta = (f^{\circ p_j})^*\vartheta$ is $y = x_j$.

Let ζ be a preferred coordinate for $f^{\circ p_j}$ at x_j:

$$\zeta \circ f^{\circ p_j} = \zeta + \zeta^{m_j+1} + \mathcal{O}(\zeta^{m_j+2}).$$

An elementary computation shows that

$$(f^{\circ p_j})^*\vartheta = \vartheta \cdot \big(1 - (m_j+1)\zeta^{m_j} + \mathcal{O}(\zeta^{m_j+1})\big).$$

Thus, $\vartheta - f^*\vartheta$ vanishes with order at least m_j at x_j, and with order exactly m_j if $\theta(x_j) \neq 0$.

The infinitesimal Beltrami differential $\nu = \bar\partial\theta$ represents v. Since ϑ vanishes at the critical values, $f^*\vartheta$ is C^∞ on $\mathbb{P}^1$. In addition, $\bar\partial(f^*\vartheta) = f^*\bar\partial\theta = f^*\nu$. Thus, for all $q \in \widehat{\mathcal{Q}}_C(P, A)$,

$$\langle q, f^*\vartheta\rangle = \langle q, f^*\nu\rangle = \langle f_*q, \nu\rangle = \langle f_*q, \vartheta\rangle.$$

Since $\vartheta - f^*\vartheta$ vanishes on A, for all $q \in \widehat{\mathcal{Q}}_C(\mathbb{P}^1, A)$ with divergences along the cycles $\langle x_j\rangle$,

$$\begin{aligned}\langle \blacktriangledown_f q, \theta\rangle = \langle q, \vartheta - f^*\vartheta\rangle &= \sum_{x\in A} \mathrm{Res}_x q \otimes (\vartheta - f^*\vartheta)\\ &= \sum_{j=1}^k \mathrm{Res}_{x_j} q \otimes (\vartheta - f^*\vartheta).\end{aligned}$$

The proposition now follows easily.

- Since $\vartheta - f^*\vartheta$ has a zero of order at least m_j at x_j, for all $q \in \widehat{\mathcal{Q}}_C(P, A)$ with divergences in $V_{\langle x_1\rangle}^{m_1-1} \cap \cdots \cap V_{\langle x_k\rangle}^{m_k-1}$, we have

$$\langle \blacktriangledown_f q, \theta\rangle = \sum_{j=1}^k \mathrm{Res}_{x_j} q \otimes (\vartheta - f^*\vartheta) = 0.$$

  Thus,

$$\theta \in \big(\blacktriangledown_f V_{\langle x_1\rangle}^{m_1-1} \cap \cdots \cap \blacktriangledown_f V_{\langle x_k\rangle}^{m_k-1}\big)^\star = T_1^{m_j-1} + \cdots + T_k^{m_k-1}.$$

- If $\theta \in T_j^{m_j}$ and if ζ is a preferred coordinate for $f^{\circ p_j}$ at x_j, then

$$\operatorname{Res}_{x_j} \frac{\mathrm{d}\zeta}{\zeta^{m_j+1}} \cdot (\vartheta - f^*\vartheta) = 0$$

  and so, $\theta(x_j) = 0$. If this holds for all $j \in [1,k]$, then necessarily θ vanishes along the cycles $\langle x_j \rangle$, thus on B. This proves the injectivity of $\blacktriangle_f$ on $T_1^{m_j} \cap \cdots \cap T_k^{m_k}$.

□

10. Transversality in $\mathbf{Rat}_D$ or $\mathbf{Poly}_D$

We will now illustrate how one may transfer transversality results in $\mathrm{Def}_A^B(f)$ to transversality results in $\mathbf{Rat}_D$ or in the space $\mathbf{Poly}_D$ of polynomials of degree D. This is only one example among the many possible applications which are left to the reader.

Proposition 20. *Let f be a monic centered polynomial (respectively a rational map fixing $0, 1, \infty$) of degree D, having $D-1$ (respectively $2D-2$) cycles with multipliers in $\overline{\mathbb{D}} - \{0,1\}$. Then, we can locally parameterize the space of monic centered polynomials (respectively the space of rational maps fixing $0, 1, \infty$) of degree D by the mutlipliers of the corresponding cycles.*

Proof. If f is a monic centered polynomial, we let A be the union of $\{\infty\}$ and the $D-1$ cycles of f which have multipliers in $\overline{\mathbb{D}} - \{0,1\}$. If f is a rational map fixing $0, 1, \infty$, we let A be the union of $\{0, 1, \infty\}$ and the $2D-2$ cycles of f which have multipliers in $\overline{\mathbb{D}} - \{0,1\}$. In both cases, we let B we the union of A and the set of critical values of f. According to the Fatou-Shishikura inequality, f has no critical orbit relation, so that the cardinal of $B - A$ is $D-1$ in the polynomial case, and $2D-2$ in the rational case. In addition, f has no cycle with multiplier 1.

Let $\Phi : \mathrm{Def}_A^B(f) \to \mathbf{Rat}_D$ be an analytic map lifting $\underline{\Phi} : \mathrm{Def}_A^B(f) \to \mathrm{Rat}_D$. Recall that such a Φ may be obtained by choosing three points a_1, a_2, a_3 in A, three points z_1, z_2, z_3 in $\mathbb{P}^1$, and then define $\Phi(\tau) = F$ where (φ, ψ, F) is a triple representing τ with $\varphi(a_i) = z_i$ for $i = 1, 2, 3$. If f is a rational map fixing $0, 1, \infty$, we may require that $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(\infty) = \infty$, in which case Φ will take its values in the space of rational maps fixing $0, 1, \infty$. If f is a monic centered polynomial, we may assume that $\varphi(\infty) = \infty$ in which case Φ will take its values in the space of polynomials of degree D. The coefficients of $\Phi(\tau)$ depend analytically on $\tau \in \mathrm{Def}_A^B(f)$. It follows that there is an analytic family of affine maps A_τ parameterized by a simply connected neigborhood of $\mathbf{0}$ in $\mathrm{Def}_A^B(f)$, such that $A_\tau \circ \Phi(\tau) \circ A_\tau^{-1}$ is a monic centered polynomial depending analytically on τ. So, without loss of generality, we may assume that Φ is defined in a neighborhood of $\mathbf{0}$ and takes values in the space of monic centered polynomials.

Note that every (f, A)-equivalence class either is reduced to a critical values of f, or is reduced to a cycle of f whose multiplier is not 1. Thus, the map $\blacktriangle_f : T_{\mathbf{0}}\mathrm{Def}_A^B(f) \to \underline{T_f \mathbf{Rat}_D}$ is injective. It follows that $D_{\mathbf{0}}\Phi$ is injective and so, Φ is an immersion at $\mathbf{0}$. Since the dimension of the space of monic centered polynomials is $D-1$ and the dimension of the space of rational maps fixing $0, 1, \infty$ is $2D-2$, i.e., that of $\mathrm{Def}_A^B(f)$, the map Φ is a local isomorphism. Thus, the result follows from Proposition 14. □

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