

Bifurcation measure and postcritically finite rational maps

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Bassanelli and Berteloot [BB07] have defined a bifurcation measure μ_{bif} on the moduli space \mathcal{M}_d of rational maps of degree $d \geq 2$. They have proved that it is a positive measure of finite mass and that its support is contained in the closure of the set \mathcal{Z}_d of conjugacy classes of rational maps of degree d having $2d - 2$ indifferent cycles.

Denote by \mathcal{X}_d the set of conjugacy classes of strictly postcritically finite rational maps of degree d which are not flexible Lattès examples. We prove that $\text{Supp}(\mu_{\text{bif}}) = \overline{\mathcal{X}_d} = \overline{\mathcal{Z}_d}$. Our proof is based on a transversality result due to the second author.

A similar result was obtained with different techniques by Dujardin and Favre [DF] for the bifurcation measure on moduli spaces of polynomials of degree $d \geq 2$.

Introduction

A result of Lyubich [Lyu83] asserts that for each rational map $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree $d \geq 2$, there is a unique probability measure μ_f of maximal entropy $\log d$. It is ergodic, satisfies $f^* \mu_f = d \cdot \mu_f$ and is carried outside the exceptional set of f . This measure is called the *equilibrium measure* of f .

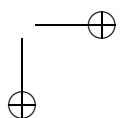
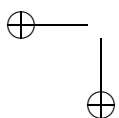
The *Lyapunov exponent* of f with respect to the measure μ_f can be defined by

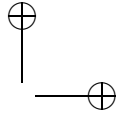
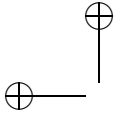
$$L(f) := \int_{\mathbb{P}^1} \log \|Df\| \, d\mu_f,$$

where $\|\cdot\|$ is any smooth metric on \mathbb{P}^1 . Since μ_f is ergodic, the quantity $e^{L(f)}$ records the average rate of expansion of f along a typical orbit with respect to μ_f .

Denote by Rat_d the set of rational maps of degree d . It is a smooth complex manifold of dimension $2d + 1$. The function $L : \text{Rat}_d \rightarrow \mathbb{R}$ is continuous [Mañ88] and plurisubharmonic [DeM03] on Rat_d . The positive $(1, 1)$ -current

$$T_{\text{bif}} := dd^c L$$





is called the *bifurcation current* in Rat_d .

The *bifurcation locus* in Rat_d is the closure of the set of discontinuity of the map $f \in \text{Rat}_d \mapsto J_f$, where J_f stands for the Julia set of f and the continuity is for the Hausdorff topology for compact subsets of \mathbb{P}^1 . DeMarco [DeM03] proved that the support of the bifurcation current T_{bif} is equal to the bifurcation locus.

Given $f \in \text{Rat}_d$, denote by $\mathcal{O}(f)$ the set of rational maps which are conjugate to f by a Möbius transformation. It is a smooth submanifold of Rat_d of complex dimension 3. It is isomorphic to $\text{Aut}(\mathbb{P}^1)/\Gamma$, where $\text{Aut}(\mathbb{P}^1) \simeq \text{PSL}(2, \mathbb{C})$ is the group of Möbius transformations, and where $\Gamma \subset \text{Aut}(\mathbb{P}^1)$ is the finite subgroup of Möbius transformations which commute with f .

The *moduli space* \mathcal{M}_d is the quotient $\text{Rat}_d/\text{Aut}(\mathbb{P}^1)$, where $\text{Aut}(\mathbb{P}^1)$ acts on Rat_d by conjugation. It is an orbifold of complex dimension $2d-2$. It is a normal, quasiprojective variety. We denote by $\Pi : \text{Rat}_d \rightarrow \mathcal{M}_d$ the canonical projection.

The Lyapunov exponent is invariant under holomorphic conjugacy, thus is constant on the orbits $\mathcal{O}(f)$. The map $L : \text{Rat}_d \rightarrow \mathbb{R}$ descends to a map $\hat{L} : \mathcal{M}_d \rightarrow \mathbb{R}$ which is continuous, bounded from below and plurisubharmonic on \mathcal{M}_d (see [BB07] proposition 6.2). The measure

$$\mu_{\text{bif}} := (dd^c \hat{L})^{\wedge (2d-2)}$$

is called the *bifurcation measure* on \mathcal{M}_d . By construction, we have

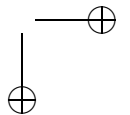
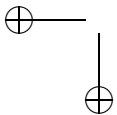
$$\Pi^* \mu_{\text{bif}} = T_{\text{bif}}^{\wedge (2d-2)}.$$

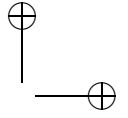
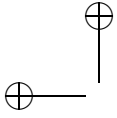
The *critical set* $\mathcal{C}(f)$ of a rational map f is the set of critical points of f . The postcritical set is the set:

$$\mathcal{P}(f) = \bigcup_{\omega \in \mathcal{C}(f)} \bigcup_{n \geq 1} f^{\circ n}(\omega).$$

A rational map f is *postcritically finite* if $\mathcal{P}(f)$ is finite. It is *strictly postcritically finite* if $\mathcal{P}(f)$ is finite and if f does not have any superattracting cycle. It is a *Lattès map* if it is obtained as the quotient of an affine map $A : z \mapsto az+b$ on a complex torus \mathbb{C}/Λ : there is a finite-to-one holomorphic map $\Theta : \mathbb{C}/\Lambda \rightarrow \mathbb{P}^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{A} & \mathbb{C}/\Lambda \\ \Theta \downarrow & & \downarrow \Theta \\ \mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1. \end{array}$$





A Lattès example is strictly postcritically finite (see [Mil06]). It is a *flexible Lattès example* if we can choose Θ with degree 2 and $A(z) = az + b$ with $a > 1$ an integer.

Let us introduce the following notation:

- $\mathcal{X}_d \subset \mathcal{M}_d$ for the set of conjugacy classes of strictly postcritically finite rational maps of degree d which are not flexible Lattès examples;
- $\mathcal{X}_d^* \subset \mathcal{X}_d$ for the subset of conjugacy classes of strictly postcritically finite rational maps of degree d which have simple critical points and satisfy $\mathcal{C}(f) \cap \mathcal{P}(f) = \emptyset$;
- $\mathcal{Z}_d \subset \mathcal{M}_d$ for the set of conjugacy classes of maps having $2d - 2$ indifferent cycles (we do not count multiplicities).

Bassanelli and Berteloot [BB07] proved that

- the bifurcation measure does not vanish identically on \mathcal{M}_d and has finite mass,
- the conjugacy class of any non-flexible Lattès map lies in the support of μ_{bif} ,
- the support of μ_{bif} is contained in the closure of \mathcal{Z}_d .

Our main result is the following.

Main Theorem *The support of the bifurcation measure in \mathcal{M}_d is:*

$$\text{Supp}(\mu_{\text{bif}}) = \overline{\mathcal{X}_d^*} = \overline{\mathcal{X}_d} = \overline{\mathcal{Z}_d}.$$

Remark We shall prove in section 1 below that

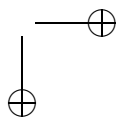
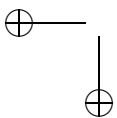
$$\overline{\mathcal{X}_d^*} = \overline{\mathcal{X}_d} = \overline{\mathcal{Z}_d}. \quad (0.1)$$

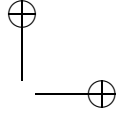
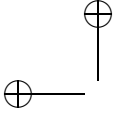
Then, thanks to the results in [BB07], we will only have to prove the inclusion

$$\mathcal{X}_d^* \subset \text{Supp}(\mu_{\text{bif}}).$$

Remark We do not know how to prove that the conjugacy classes of flexible Lattès examples lie in the support of μ_{bif} . It would be enough to prove that every flexible Lattès example can be approximated by strictly postcritically finite rational maps which are not Lattès examples.

Remark The underlying idea of the proof is a potential-theoretic interpretation of a result of Tan Lei [Tan90] concerning the similarities between the Mandelbrot set and Julia sets.





Our proof relies on a transversality result that we will now present.

Assume $f_0 \in \text{Rat}_d$ is a strictly postcritically finite rational map which has $2d - 2$ distinct critical points $\omega_j(f_0)$, $j \in \{1, \dots, 2d - 2\}$, and satisfies $\mathcal{C}(f_0) \cap \mathcal{P}(f_0) = \emptyset$. There are integers $\ell_j \geq 1$ such that

$$\alpha_j(f_0) := f_0^{\circ \ell_j}(\omega_j(f_0))$$

are periodic points of f_0 . The critical points are simple and the periodic points are repelling. By the Implicit Function Theorem, there are

- analytic germs $\omega_j : (\text{Rat}_d, f_0) \rightarrow \mathbb{P}^1$ following the critical points of f as f ranges in a neighborhood of f_0 in Rat_d and
- analytic germs $\alpha_j : (\text{Rat}_d, f_0) \rightarrow \mathbb{P}^1$ following periodic points of f as f ranges in a neighborhood of f_0 in Rat_d .

Let $F : (\text{Rat}_d, f_0) \rightarrow (\mathbb{P}^1)^{2d-2}$ and $G : (\text{Rat}_d, f_0) \rightarrow (\mathbb{P}^1)^{2d-2}$ be defined by :

$$F := \begin{pmatrix} F_1 \\ \vdots \\ F_{2d-2} \end{pmatrix} \quad \text{with} \quad F_j(f) := f^{\circ \ell_j}(\omega_j(f)) \quad \text{and} \quad G := \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{2d-2} \end{pmatrix}.$$

Denote by $D_{f_0}F$ and $D_{f_0}G$ the differentials of F and G at f_0 .

The transversality result we are interested in is the following.

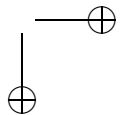
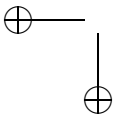
Proposition 1 *If f_0 is not a flexible Lattès example, then the linear map*

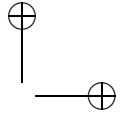
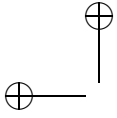
$$D_{f_0}F - D_{f_0}G : T_{f_0}\text{Rat}_d \rightarrow T_{\alpha_1(f_0)}\mathbb{P}^1 \times \cdots \times T_{\alpha_{2d-2}(f_0)}\mathbb{P}^1$$

has rank $2d - 2$. The kernel of $D_{f_0}F - D_{f_0}G$ is the tangent space to $\mathcal{O}(f_0)$ at f_0 .

There are several proofs of this result. Here, we include a proof due to the second author. A different proof was obtained by van Strien [vS00]. His proof covers a more general setting where the critical orbits are allowed to be preperiodic to a hyperbolic set. Our proof covers the case of maps commuting with a non trivial group of Möbius transformations (a slight modification of van Strien's proof probably also covers this case).

Remark A similar transversality theorem holds in the space of polynomials of degree d and the arguments developed in this article lead to an alternative proof of the result of Dujardin and Favre.





Acknowledgements

We would like to thank our colleagues in Toulouse who brought this problem to our attention, in particular François Berteloot, Julien Duval and Vincent Guedj. We would like to thank John Hubbard for his constant support.

1 The sets \mathcal{X}_d^* , \mathcal{X}_d and \mathcal{Z}_d

In this section, we prove equality (0.1): $\overline{\mathcal{X}_d^*} = \overline{\mathcal{X}_d} = \overline{\mathcal{Z}_d}$. Since $\mathcal{X}_d^* \subset \mathcal{X}_d$, it is enough to prove that

$$\mathcal{X}_d \subset \overline{\mathcal{Z}_d} \quad \text{and} \quad \mathcal{Z}_d \subset \overline{\mathcal{X}_d^*}.$$

1.1 Tools

Our proof relies on the following three results.

The first result is an immediate consequence of the Fatou-Shishikura inequality on the number of non repelling cycles of a rational map (see [Shi87] and/or [Eps99]). For $f \in \text{Rat}_d$, denote by

- $N_{\text{att}}(f)$ the number of attracting cycles of f ,
- $N_{\text{ind}}(f)$ the number of distinct indifferent cycles of f and
- $N_{\text{crit}}(f)$ the number of critical points of f , counting multiplicities, whose orbits are strictly preperiodic to repelling cycles.

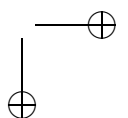
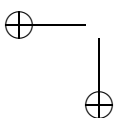
Theorem 1 *For any rational map $f \in \text{Rat}_d$, we have:*

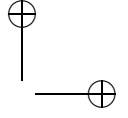
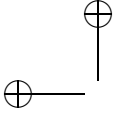
$$N_{\text{crit}}(f) + N_{\text{ind}}(f) + N_{\text{att}}(f) \leq 2d - 2.$$

The second result is a characterization of *stability* due to Mañé, Sad and Sullivan [MSS83] (compare with [McM94] section 4.1). Let Λ be a complex manifold. A family of rational maps $\{f_\lambda\}_{\lambda \in \Lambda}$ is an *analytic family* if the map $(\lambda, z) \in \Lambda \times \mathbb{P}^1 \mapsto f_\lambda(z) \in \mathbb{P}^1$ is analytic. The family is *stable* at $\lambda_0 \in \Lambda$ if the number of attracting cycles of f_λ is locally constant at λ_0 .

Theorem 2 *Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be an analytic family of rational maps parametrized by a complex manifold Λ . The following assertions are equivalent.*

- *The family is stable at λ_0 .*
- *For all $\mu \in S^1$ and $p \geq 1$, the number of cycles of f_λ having multiplier μ and period p is locally constant at λ_0 .*





- For all $\ell \geq 1$ and $p \geq 1$, the number of critical points ω of f_λ such that $f^{\circ \ell}(\omega)$ is a repelling periodic point of period p is locally constant at λ_0 .
- The maximum period of an indifferent cycle of f_λ is locally bounded at λ_0 .
- The maximum period of a repelling cycle of f_λ contained in the post-critical set $\mathcal{P}(f_\lambda)$ is locally bounded at λ_0 .

The third result is due to McMullen [McM87]. Assume Λ is an irreducible quasiprojective complex variety. A family of rational maps $\{f_\lambda\}_{\lambda \in \Lambda}$ is an *algebraic family* if the map $(\lambda, z) \in \Lambda \times \mathbb{P}^1 \mapsto f_\lambda(z) \in \mathbb{P}^1$ is a rational mapping. The family is *trivial* if all its members are conjugate by Möbius transformations. The family is *stable* if it is stable at every $\lambda \in \Lambda$.

Theorem 3 *A stable algebraic family of rational maps is either trivial or it is a family of flexible Lattès examples.*

1.2 Spaces of rational maps with marked critical points and marked periodic points

In the sequel, it will be convenient to consider the set $\text{Rat}_d^{\text{crit,per}_n}$ of rational maps of degree d with marked critical points and marked periodic points of period dividing n . This can be defined as follows.

First, the unordered sets of m points in \mathbb{P}^1 can be identified with \mathbb{P}^m . This yields a rational map $\sigma_m : (\mathbb{P}^1)^m \rightarrow \mathbb{P}^m$ which might be defined as follows:

$$\sigma_m([x_1 : y_1], \dots, [x_m : y_m]) = [a_0 : \dots : a_m]$$

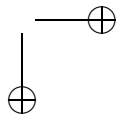
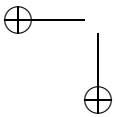
with

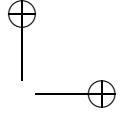
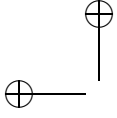
$$\sum_{j=0}^m a_j x^j y^{m-j} = \prod_{i=1}^m (xy_i - yx_i).$$

Second, to a rational map $f \in \text{Rat}_d$, we associate the unordered set $\{\omega_1, \dots, \omega_{2d-2}\}$ of its $2d-2$ critical points, listed with repetitions according to their multiplicities. This induces a rational map $\text{Crit} : \text{Rat}_d \rightarrow \mathbb{P}^{2d-2}$ which might be defined as follows: if $f([x : y]) = [P(x, y) : Q(x, y)]$ with P and Q homogeneous polynomials of degree d , then

$$\text{Crit}(f) = [a_0 : \dots : a_{2d-2}] \quad \text{with} \quad \sum_{j=0}^{2d-2} a_j x^j y^{2d-2-j} = \frac{\partial P}{\partial x} \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y} \frac{\partial Q}{\partial x}.$$

Third, given an integer $n \geq 1$, to a rational map $f \in \text{Rat}_d$, we associate the unordered set $\{\alpha_1, \dots, \alpha_{d^n+1}\}$ of the fixed points of $f^{\circ n}$, listed with





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repetitions according to their multiplicities. This induces a rational map $\text{Per}_n : \text{Rat}_d \rightarrow \mathbb{P}^{2^n+1}$ which might be defined as follows: if $f^{\circ n}([x : y]) = [P_n(x, y) : Q_n(x, y)]$ with P_n and Q_n homogeneous polynomials of degree d^n , then

$$\text{Per}_n(f) = [a_0 : \cdots : a_{d^n+1}]$$

with

$$\sum_{j=0}^{d^n+1} a_j x^j y^{d^n+1-j} = yP_n(x, y) - xQ_n(x, y).$$

The space of rational maps of degree d with marked critical points and marked periodic points of period dividing n is the set:

$$\text{Rat}_d^{\text{crit,per}_n} := \left\{ (f, \omega, \alpha) \in \text{Rat}_d \times (\mathbb{P}^1)^{2d-2} \times (\mathbb{P}^1)^{d^n+1} \text{ such that } \left. \begin{array}{l} \text{Crit}(f) = \sigma_{2d-2}(\omega) \text{ and } \text{Per}_n(f) = \sigma_{d^n+1}(\alpha) \end{array} \right\}.$$

Then, $\text{Rat}_d^{\text{crit,per}_n}$ is an algebraic subset¹ of $\text{Rat}_d \times (\mathbb{P}^1)^{2d-2} \times (\mathbb{P}^1)^{d^n+1}$. Since there are $2d-2+d^n+1$ equations, the dimension of any irreducible component of $\text{Rat}_d^{\text{crit,per}_n}$ is at least that of Rat_d , i.e. $2d+1$. Since the fibers of the projection $\text{Rat}_d^{\text{crit,per}_n} \rightarrow \text{Rat}_d$ are finite (a rational map has finitely many critical points and finitely many periodic points of period dividing n), the dimension of any component is exactly $2d+1$.

1.3 The inclusion $\mathcal{X}_d \subset \overline{\mathcal{Z}_d}$

Assume $f_0 \in \text{Rat}_d$ is a strictly postcritically finite rational map but not a flexible Lattès example. We must show that any neighborhood of f_0 in Rat_d contains a rational map with $2d-2$ indifferent cycles. This is an immediate consequence of the following lemma.

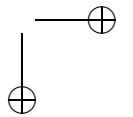
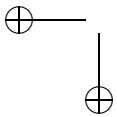
Lemma 1 *Assume $r \geq 1$ and $s \geq 0$ are integers such that $r + s = 2d - 2$. Assume $f_0 \in \text{Rat}_d$ is not a flexible Lattès example, $N_{\text{crit}}(f_0) = r$ and $N_{\text{ind}}(f_0) = s$. Then, arbitrarily close to f_0 , we can find a rational map f_1 such that $N_{\text{crit}}(f_1) = r - 1$ and $N_{\text{ind}}(f_1) = s + 1$.*

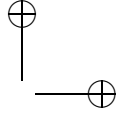
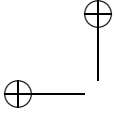
Proof: Let p_1, \dots, p_s be the periods of the indifferent cycles of f_0 . Denote by M their least common multiple. A point $\lambda \in \text{Rat}_d^{\text{crit,per}_M}$ is of the form

$$(g_\lambda, \omega_1(\lambda), \dots, \omega_{2d-2}(\lambda), \alpha_1(\lambda), \dots, \alpha_{d^M+1}(\lambda)).$$

We choose $\lambda_0 \in \text{Rat}_d^{\text{crit,per}_M}$ so that

¹By algebraic we mean a quasiprojective variety in some projective space. It is classical that any product of projective spaces embeds in some \mathbb{P}^N . Thus, it makes sense to speak of algebraic subsets of products of projective spaces





- $g_{\lambda_0} = f_0$,
- $\omega_1(\lambda_0), \dots, \omega_r(\lambda_0)$ are preperiodic to repelling cycles of f_0 and
- $\alpha_1(\lambda_0), \dots, \alpha_s(\lambda_0)$ are indifferent periodic points of f_0 belonging to distinct cycles.

For $i \in [1, r]$, let $k_i \geq 1$ and $p_i \geq 1$ be the least integers such that

$$g_{\lambda_0}^{\circ k_i}(\omega_i(\lambda_0)) = g_{\lambda_0}^{\circ k_i + p_i}(\omega_i(\lambda_0)).$$

For $j \in [1, s]$, let μ_j be the multiplier of $g_{\lambda_0}^{\circ M}$ at $\alpha_j(\lambda_0)$. In particular, $\mu_j \in S^1$.

Consider the algebraic subset

$$\left\{ \lambda \in \text{Rat}_d^{\text{crit, per}_M} ; \quad \forall i \in [1, r-1], g_{\lambda}^{\circ k_i}(\omega_i(\lambda)) = g_{\lambda}^{\circ k_i + p_i}(\omega_i(\lambda)) \text{ and } \forall j \in [1, s], \text{ the multiplier of } g_{\lambda}^{\circ M} \text{ at } \alpha_j(\lambda) \text{ is } \mu_j \right\}$$

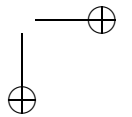
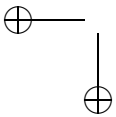
and let Λ be an irreducible component containing λ_0 . Note that there are $r-1+s=2d-3$ equations. It follows that the dimension of Λ is at least $(2d+1) - (2d-3) = 4$. In particular, the image of Λ by the projection $\text{Rat}_d^{\text{crit, per}_M} \rightarrow \text{Rat}_d$ cannot be contained in $\mathcal{O}(f_0)$.

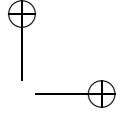
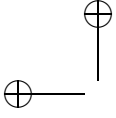
Embedding Λ in some \mathbb{P}^N (with N sufficiently large) and slicing with an appropriate projective subspace, we deduce that there is an algebraic curve $\Gamma \subset \Lambda$ containing λ_0 , whose image by the projection $\text{Rat}_d^{\text{crit, per}_M} \rightarrow \text{Rat}_d$ is not contained in $\mathcal{O}(f_0)$. Desingularizing the algebraic curve, we obtain a smooth quasiprojective curve Σ (i.e. a Riemann surface of finite type) and an algebraic map $\Sigma \rightarrow \Gamma$ which is surjective and generically one-to-one. We let $\sigma_0 \in \Sigma$ be a point which is mapped to $\lambda_0 \in \Gamma$. With an abuse of notation, we write g_{σ} , $\omega_i(\sigma)$ and $\alpha_j(\sigma)$ in place of $g_{\lambda(\sigma)}$, $\omega_i(\lambda(\sigma))$ and $\alpha_j(\lambda(\sigma))$. Then, we have an algebraic family of rational maps $\{g_{\sigma}\}_{\sigma \in \Sigma}$ parametrized by a smooth quasiprojective curve Σ , coming with marked critical points $\omega_i(\sigma)$ and marked periodic points $\alpha_j(\sigma)$.

The family is not trivial and g_{σ_0} is not a flexible Lattès example. Assume that the family were stable at σ_0 . Then, according to theorem 2, the number of critical points which are preperiodic to repelling cycles would be locally constant at σ_0 . Thus, the critical orbit relation

$$g_{\sigma}^{\circ k_r}(\omega_r(\sigma)) = g_{\sigma}^{\circ k_r + p_r}(\omega_r(\sigma))$$

would hold in a neighborhood of σ_0 in Σ , thus for all $\sigma \in \Sigma$ by analytic continuation. As a consequence, for all $\sigma \in \Sigma$, we would have $N_{\text{crit}}(g_{\sigma}) + N_{\text{ind}}(g_{\sigma}) = 2d-2$. According to theorem 1, this would imply that $N_{\text{att}}(g_{\sigma}) = 0$ for all $\sigma \in \Sigma$. The family would be stable which would contradict theorem 3.





1. The sets \mathcal{X}_d^* , \mathcal{X}_d and \mathcal{Z}_d

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Thus, the family $\{g_\sigma\}_{\sigma \in \Sigma}$ is not stable at σ_0 . According to theorem 2, the maximum period of an indifferent cycle of g_σ is not locally bounded at σ_0 . Thus, we can find a parameter $\sigma_1 \in \Sigma$ arbitrarily close to σ_0 such that g_{σ_1} has at least $s + 1$ indifferent cycles. \square

1.4 The inclusion $\mathcal{Z}_d \subset \overline{\mathcal{X}_d^*}$

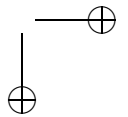
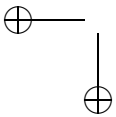
The proof follows essentially the same lines as the previous one. We begin with a rational map $f_0 \in \text{Rat}_d$ having $2d - 2$ indifferent cycles. We must show that arbitrarily close to f_0 , we can find a postcritically finite rational map $f_1 \in \text{Rat}_d$ which has simple critical points, satisfies $\mathcal{C}(f_1) \cap \mathcal{P}(f_1) = \emptyset$ and is not a flexible Lattès example.

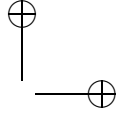
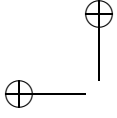
The following lemma implies (by induction) that arbitrarily close to f_0 , we can find a rational map $f_1 \in \text{Rat}_d$ whose postcritical set contains $2d - 2$ distinct repelling cycles. Automatically, the $2d - 2$ critical points of f_1 have to be simple, strictly preperiodic, with disjoint orbits. In particular, $\mathcal{C}(f_1) \cap \mathcal{P}(f_1) = \emptyset$. In addition, Lattès examples lie in a Zariski closed subset of Rat_d . Since f_0 has indifferent cycles, it cannot be a Lattès examples. So, if f_1 is sufficiently close to f_0 , then f_1 is not a Lattès example.

Lemma 2 *Assume $r \geq 0$ and $s \geq 1$ are integers such that $r + s = 2d - 2$. Assume $f_0 \in \text{Rat}_d$ satisfies $N_{\text{crit}}(f_0) = r$, $N_{\text{ind}}(f_0) = s$, the postcritical set of f_0 containing r distinct repelling cycles. Then, arbitrarily close to f_0 , we can find a rational map f_1 which satisfies $N_{\text{crit}}(f_1) = r + 1$ and $N_{\text{ind}}(f_1) = s - 1$, the postcritical set of f_1 containing $r + 1$ distinct repelling cycles.*

Proof: The proof is similar to the proof of lemma 1; we do not give all the details. First, note that since $s \geq 1$, f_0 has an indifferent cycle, and thus, is not a Lattès example. Second, we can find a non trivial algebraic family of rational maps $\{g_\sigma\}_{\sigma \in \Sigma}$ parametrized by a smooth quasiprojective curve Σ with marked critical points $\omega_1(\sigma), \dots, \omega_{2d-2}(\sigma)$ and marked periodic points $\alpha_1(\sigma), \dots, \alpha_{dM+1}(\sigma)$ such that

- $g_{\sigma_0} = f_0$,
- for $j \in [1, s]$, the periodic points $\alpha_j(\sigma_0)$ belong to distinct indifferent cycles of f_0 ,
- for all $\sigma \in \Sigma$ and all $i \in [1, r]$, the critical point $\omega_i(\sigma)$ is preperiodic to a periodic cycle of g_σ and
- for all $\sigma \in \Sigma$ and all $j \in [1, s - 1]$, the periodic points $\alpha_j(\sigma)$ are indifferent periodic points of g_σ .





If the family were stable at σ_0 , the indifferent periodic point $\alpha_s(\sigma_0)$ would be persistently indifferent in a neighborhood of σ_0 in Σ , thus in all Σ by analytic continuation. The relation $N_{\text{crit}}(g_\sigma) + N_{\text{ind}}(g_\sigma) = 2d - 2$ would hold throughout Σ . Thus, for all $\sigma \in \Sigma$, we would have $N_{\text{att}}(g_\sigma) = 0$ and the family would be stable. This is not possible since g_{σ_0} is not a Lattès example and since the family $\{g_\sigma\}_{\sigma \in \Sigma}$ is not trivial.

It follows that the period of a repelling cycle contained in the postcritical set of g_σ is not locally bounded at σ_0 . In addition, for $i \in [1, r]$, the critical points $\omega_i(\sigma_0)$ of g_{σ_0} are preperiodic to distinct repelling cycles. Thus, we can find a rational map g_{σ_1} , with σ_1 arbitrarily close to σ_0 , such that g_{σ_1} has $r + 1$ critical points preperiodic to distinct repelling cycles. \square

2 Transversality

In this section, we prove proposition 1. Assume $f_0 \in \text{Rat}_d$ is a strictly postcritically finite rational map which has $2d - 2$ distinct critical points and satisfies $\mathcal{C}(f_0) \cap \mathcal{P}(f_0) = \emptyset$. Assume the analytic germs

$$F : (\text{Rat}_d, f_0) \rightarrow (\mathbb{P}^1)^{2d-2} \quad \text{and} \quad G : (\text{Rat}_d, f_0) \rightarrow (\mathbb{P}^1)^{2d-2}$$

are defined as in the introduction. We want to show that the rank of $D_{f_0}F - D_{f_0}G$ is $2d - 2$.

Since $T_{f_0}\text{Rat}_d$ has complex dimension $2d + 1$, the rank of $D_{f_0}F - D_{f_0}G$ is $2d - 2$ if and only if the kernel has dimension 3.

Note that on $\mathcal{O}(f_0)$, we have $F \equiv G$. Thus, every tangent vector to $\mathcal{O}(f_0)$ at f_0 is in the kernel of $D_{f_0}F - D_{f_0}G$. Since $\mathcal{O}(f_0)$ has complex dimension 3, we must therefore show that every vector in the kernel of $D_{f_0}F - D_{f_0}G$ is tangent to $\mathcal{O}(f_0)$.

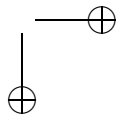
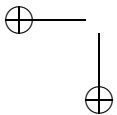
Thus, it is enough to show that if $\{f_t\}$ is a holomorphic family of rational maps of degree $d \geq 2$, parametrized by a neighborhood of $t = 0$ in \mathbb{C} and if

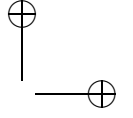
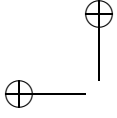
$$\xi := \left. \frac{df_t}{dt} \right|_{t=0}$$

belongs to $\text{Ker}(D_{f_0}F - D_{f_0}G)$, then ξ is a tangent vector to $\mathcal{O}(f_0)$.

2.1 The tangent vectors to Rat_d .

Assume $\{f_t\}$ is a holomorphic family of rational maps of degree $d \geq 2$, parametrized by a neighborhood of $t = 0$ in \mathbb{C} , and assume $\xi := \left. \frac{df_t}{dt} \right|_{t=0}$. Then, for all $z \in \mathbb{P}^1$, the vector $\xi(z)$ is tangent to the curve $t \mapsto f_t(z)$ at $t = 0$. This vector belongs to $T_{f_0(z)}\mathbb{P}^1$.





In other words, if $f \in \text{Rat}_d$ and if $\xi \in T_f \text{Rat}_d$, then for all $z \in \mathbb{P}^1$, $\xi(z) \in T_{f(z)} \mathbb{P}^1$. If z is not a critical point of f , then

$$D_z f : T_z \mathbb{P}^1 \rightarrow T_{f(z)} \mathbb{P}^1$$

is an isomorphism and we set:

$$\eta_\xi(z) := (D_z f)^{-1}(\xi(z)).$$

We obtain a vector field η_ξ defined and holomorphic on $\mathbb{P}^1 \setminus \mathcal{C}(f)$.

Remark Assume $\{f_t\}$ is a holomorphic family of rational maps, parametrized by a neighborhood of $t = 0$ in \mathbb{C} , with $\xi = \left. \frac{df_t}{dt} \right|_{t=0}$. If z_0 is not a critical point of f_0 , the equation $f_t(z) = f_0(z_0)$ defines implicitly a map $z(t)$, holomorphic in a neighborhood of $t = 0$ with $z(0) = z_0$. The vector $\eta_\xi(z_0) \in T_{z_0} \mathbb{P}^1$ is tangent to the curve $t \mapsto z(t)$ at $t = 0$.

Remark If f has simple critical points, then the vector field η_ξ has simple poles or removable singularities along $\mathcal{C}(f)$. This can be seen by working in coordinates, using the fact that f' has simple zeroes at points of $\mathcal{C}(f)$.

Proposition 2 Assume $f \in \text{Rat}_d$ and assume $\{\phi_t\}$ is an analytic family of Möbius transformations parametrized by a neighborhood of $t = 0$ in \mathbb{C} , with $\phi_0 = \text{Id}$. Let θ be the holomorphic vector field on \mathbb{P}^1 defined by $\theta := \left. \frac{d\phi_t}{dt} \right|_{t=0}$. Then,

$$\xi = \left. \frac{d(\phi_t \circ f \circ \phi_t^{-1})}{dt} \right|_{t=0} \iff \eta_\xi = f^* \theta - \theta.$$

Remark The vector field $f^* \theta$ is meromorphic on \mathbb{P}^1 and holomorphic on $\mathbb{P}^1 \setminus \mathcal{C}(f)$. If z is not a critical point of f , then

$$(f^* \theta)(z) := (D_z f)^{-1}(\theta(f(z))).$$

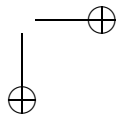
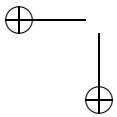
Proof: Set $f_t := \phi_t \circ f \circ \phi_t^{-1}$. Note that $f_0 = f$. Differentiating the equality $\phi_t \circ f(z) = f_t \circ \phi_t(z)$ with respect to t and evaluating at $t = 0$, we see that $\xi = \left. \frac{df_t}{dt} \right|_{t=0}$ if and only if for all $z \in \mathbb{P}^1$, we have

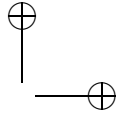
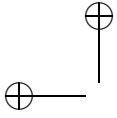
$$\theta(f(z)) = \xi(z) + D_z f(\theta(z)). \quad (2.2)$$

If equality (2.2) holds, then applying $(D_z f)^{-1}$ to both sides, we obtain the following equality of vector fields:

$$f^* \theta = \eta_\xi + \theta.$$

Conversely, if $f^* \theta = \eta_\xi + \theta$, then for all $z \in \mathbb{P}^1 \setminus \mathcal{C}(f)$, we have the equality of vectors $f^* \theta(z) = \eta_\xi(z) + \theta(z)$. Applying $D_z f$ to both sides shows that equality (2.2) hold on $\mathbb{P}^1 \setminus \mathcal{C}(f)$, thus on \mathbb{P}^1 by analytic continuation. \square





Corollary 1 *The vector $\xi \in T_f \text{Rat}_d$ is tangent to $\mathcal{O}(f)$ if and only if there is a vector field θ , holomorphic on \mathbb{P}^1 , such that:*

$$\eta_\xi = f^* \theta - \theta.$$

Proof: If $\xi \in T_f \mathcal{O}(f)$, there is a holomorphic family of Möbius transformations ϕ_t such that $\phi_0 = \text{Id}$ and $\xi = \left. \frac{d(\phi_t \circ f \circ \phi_t^{-1})}{dt} \right|_{t=0}$. The vector field $\theta := \left. \frac{d\phi_t}{dt} \right|_{t=0}$ is holomorphic on \mathbb{P}^1 , and according to the previous proposition, $\eta_\xi = f^* \theta - \theta$.

Conversely, $\eta_\xi = f^* \theta - \theta$ with θ holomorphic on \mathbb{P}^1 , we can find a family of Möbius transformations ϕ_t such that $\phi_0 = \text{Id}$ and $\left. \frac{d\phi_t}{dt} \right|_{t=0} = \theta$. According to the previous proposition, the family $\{f_t := \phi_t \circ f \circ \phi_t^{-1}\}$ satisfies $\xi = \left. \frac{df_t}{dt} \right|_{t=0}$, and since $f_t \in \mathcal{O}(f)$, we have $\xi \in T_f \mathcal{O}(f)$. \square

2.2 Following critical points and critical values

In the following proposition, $\{f_t\}$ is a holomorphic family of rational maps, parametrized by a neighborhood of $t = 0$ in \mathbb{C} . We set $\xi := \left. \frac{df_t}{dt} \right|_{t=0}$ and let η_ξ be the corresponding meromorphic vector field on \mathbb{P}^1 . We assume that ω_0 is a simple critical point of f_0 and that v_0 is the corresponding critical value. Finally, we let

$$w : (\mathbb{C}, 0) \rightarrow (\mathbb{P}^1, \omega_0) \quad \text{and} \quad v : (\mathbb{C}, 0) \rightarrow (\mathbb{P}^1, v_0)$$

be holomorphic germs following respectively a critical point of f_t and the corresponding critical value of f_t (in view of the Implicit Function Theorem, since ω_0 is a simple critical point of f_0 , those germs exist and are unique).

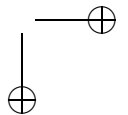
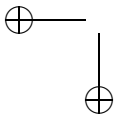
Proposition 3 *Assume θ is a vector field, defined and holomorphic in a neighborhood of v_0 . Then, $f_0^* \theta - \eta_\xi$ is holomorphic in a neighborhood of ω_0 if and only if*

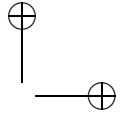
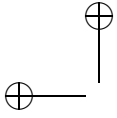
$$\theta(v_0) = \left. \frac{dv(t)}{dt} \right|_{t=0}.$$

In that case,

$$(f_0^* \theta - \eta_\xi)(\omega_0) = \left. \frac{d\omega(t)}{dt} \right|_{t=0}.$$

Proof: Let us first show that there exists a vector field θ_0 , defined and holomorphic in a neighborhood of v_0 , such that $f_0^* \theta_0 - \eta_\xi$ is holomorphic in a neighborhood of ω_0 .





Choose a holomorphic family of local biholomorphisms ϕ_t sending v_0 to $v(t)$, with $\phi_0 = \text{Id}$. Note that there is a holomorphic family of local biholomorphisms ψ_t sending ω_0 to $\omega(t)$, with $\psi_0 = \text{Id}$ and

$$\phi_t \circ f_0 = f_t \circ \psi_t.$$

Note that

$$\theta_0 := \left. \frac{d\phi_t}{dt} \right|_{t=0}$$

is defined and holomorphic in a neighborhood of v_0 and

$$\zeta_0 := \left. \frac{d\psi_t}{dt} \right|_{t=0}$$

is defined and holomorphic in a neighborhood of ω_0 . Differentiating the equality $\phi_t \circ f_0(z) = f_t \circ \psi_t(z)$ with respect to t and evaluating at $t = 0$ yields the following equality of vectors:

$$\theta_0(f_0(z)) = \xi(z) + D_z f_0(\zeta_0(z)).$$

Applying $(D_z f_0)^{-1}$ to both sides of the equation yields the equality of vector fields:

$$f_0^* \theta_0 = \eta_\xi + \zeta_0.$$

As required, $f_0^* \theta_0 - \eta_\xi = \zeta_0$ is holomorphic in a neighborhood of ω_0 . In addition,

$$\theta_0(v_0) = \left. \frac{dv(t)}{dt} \right|_{t=0}$$

and

$$(f_0^* \theta_0 - \eta_\xi)(\omega_0) = \zeta_0(\omega_0) = \left. \frac{d\omega(t)}{dt} \right|_{t=0}.$$

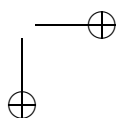
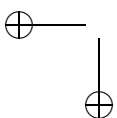
Let us now assume that θ is a vector field, defined and holomorphic in a neighborhood of v_0 . If θ agrees with θ_0 at v_0 , then $\theta - \theta_0$. Consequently, the pullback $f_0^*(\theta - \theta_0)$ is holomorphic vanishes at ω_0 . It follows that $f_0^* \theta - \eta_\xi$ is holomorphic and agrees with $f_0^* \theta_0 - \eta_\xi$ at ω_0 .

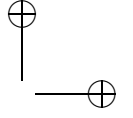
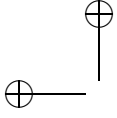
Finally, if θ does not agree with θ_0 at v_0 , then $\theta - \theta_0$ is not vanishing at v_0 . Thus, the vector field $f_0^*(\theta - \theta_0)$ has a pole at ω_0 . As a consequence, $f_0^* \theta - \eta_\xi$ has a pole at ω_0 . \square

2.3 Proof of proposition 1

Let us recall the hypotheses. The rational map $f_0 \in \text{Rat}_d$ is postcritically finite. It has $2d - 2$ distinct critical points $\omega_j(f_0)$ and $\mathcal{C}(f_0) \cap \mathcal{P}(f_0) = \emptyset$. In particular, there are integers $\ell_j \geq 1$ such that

$$\alpha_j(f_0) := f_0^{\circ \ell_j}(\omega_j(f_0))$$





are periodic points of f_0 . The critical points are simple and the periodic points are repelling. By the Implicit Function Theorem, there are

- analytic germs $\omega_j : (\text{Rat}_d, f_0) \rightarrow \mathbb{P}^1$ following the critical points of f and
- analytic germs $\alpha_j : (\text{Rat}_d, f_0) \rightarrow \mathbb{P}^1$ following periodic points of f .

The analytic germs

$$F : (\text{Rat}_d, f_0) \rightarrow (\mathbb{P}^1)^{2d-2} \quad \text{and} \quad G : (\text{Rat}_d, f_0) \rightarrow (\mathbb{P}^1)^{2d-2}$$

are defined by

$$F := \begin{pmatrix} F_1 \\ \vdots \\ F_{2d-2} \end{pmatrix} \quad \text{with} \quad F_j(f) := f^{\circ \ell_j}(\omega_j(f)) \quad \text{and} \quad G := \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{2d-2} \end{pmatrix}.$$

Assume $\{f_t\}$ is a holomorphic family of rational maps of degree $d \geq 2$, parametrized by a neighborhood of $t = 0$ in \mathbb{C} . Suppose

$$\xi := \left. \frac{df_t}{dt} \right|_{t=0} \in \text{Ker}(D_{f_0}F - D_{f_0}G).$$

Let η_ξ be the associated holomorphic vector field on $\mathbb{P}^1 \setminus \mathcal{C}(f_0)$. We want to show that $\xi \in T_f\mathcal{O}(f)$, and for this purpose, we will show that there is a globally holomorphic vector field θ such that

$$\eta_\xi = f_0^*\theta - \theta.$$

First, we define θ on \mathcal{P}_{f_0} as follows. If $z_0 \in \mathcal{P}_{f_0}$, then

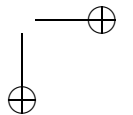
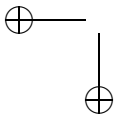
- z_0 is not a precritical point of f_0 and
- z_0 is preperiodic to a repelling periodic point of f_0 .

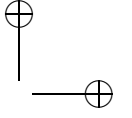
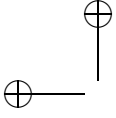
It follows from the Implicit Function Theorem that there is a holomorphic germ $z : (\mathbb{C}, 0) \rightarrow (\mathbb{P}^1, z_0)$ such that $z(t)$ is a preperiodic point of f_t and this germ is unique. Indeed, there are integers $\ell \geq 0$ and $p \geq 1$ such that

$$f_0^{\circ \ell}(z_0) = \alpha_0 \quad \text{with} \quad f_0^{\circ p}(\alpha_0) = \alpha_0.$$

Since α_0 is a repelling periodic point of f_0 , $(f_0^{\circ p})'(\alpha_0) \neq 1$ and since z_0 is not precritical, $(f_0^{\circ \ell})'(z_0) \neq 0$ whence

$$(f_0^{\circ \ell+p})'(z_0) = (f_0^{\circ p})'(\alpha_0) \cdot (f_0^{\circ \ell})'(z_0) \neq (f_0^{\circ \ell})'(z_0).$$





We set

$$\theta(z_0) := \frac{dz(t)}{dt} \Big|_{t=0}.$$

Second, we have the following equality, valid on the postcritical set $\mathcal{P}(f_0)$:

$$f_0^* \theta = \theta + \eta_\xi.$$

Indeed, assume that $z_0 \in \mathcal{P}(f_0)$ and $w_0 = f_0(z_0) \in \mathcal{P}(f_0)$. Then,

$$\theta(z_0) = \frac{dz(t)}{dt} \Big|_{t=0} \quad \text{and} \quad \theta(w_0) = \frac{dw(t)}{dt} \Big|_{t=0} \quad \text{with} \quad w(t) = f_t(z(t)).$$

Evaluating the derivative of the equation $w(t) = f_t(z(t))$ at $t = 0$ yields:

$$\theta(w_0) = D_{z_0} f_0(\theta(z_0)) + \xi(z_0).$$

Applying $(D_{z_0} f_0)^{-1}$ to both sides yields the required equation.

Third, we may reformulate the condition that ξ belongs to the kernel of $D_{f_0} F - D_{f_0} G$ as follows. Let $v_j : (\text{Rat}_d, f_0) \rightarrow \mathbb{P}^1$ be the analytic germs following the critical values of f :

$$v_j(f) := f(\omega_j(f)).$$

Lemma 3 *If ξ belongs to the kernel of $D_{f_0} F - D_{f_0} G$, then for all integer $j \in \{1, \dots, 2d - 2\}$, we have*

$$\theta(v_j(f_0)) = \frac{dv_j(f_t)}{dt} \Big|_{t=0}.$$

Proof: To lighten notation, let us leave aside the index j . Choose a critical point $\omega(f_0)$ and let $v(f_0)$ be the corresponding critical value. We have $f_0^{\circ \ell}(\omega(f_0)) = \alpha(f_0)$ for some integer $\ell \geq 1$ and if ξ belongs to the kernel of $D_{f_0} F - D_{f_0} G$, then

$$\frac{df_t^{\circ \ell}(\omega(f_t))}{dt} \Big|_{t=0} = \frac{d\alpha(f_t)}{dt} \Big|_{t=0}.$$

In addition, we have a point z_t depending holomorphically on t , such that

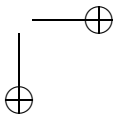
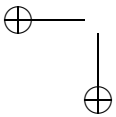
$$z_0 = v(f_0) \quad \text{and} \quad f_t^{\circ \ell - 1}(z_t) = \alpha(f_t).$$

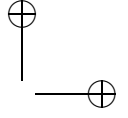
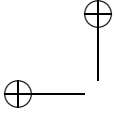
Using

$$\frac{df_t^{\circ \ell - 1}(v(f_t))}{dt} \Big|_{t=0} = \frac{df_t^{\circ \ell - 1}(z_t)}{dt} \Big|_{t=0},$$

we easily derive

$$D_{v(f_0)} f_0^{\circ \ell - 1} \left(\frac{dv(f_t)}{dt} \Big|_{t=0} \right) = D_{z_0} f_0^{\circ \ell - 1} \left(\frac{dz_t}{dt} \Big|_{t=0} \right).$$





Since $v(f_0) = z_0$ and since there are no critical points in the forward orbit of $v(f_0)$, the map $D_{v(f_0)}f_0^{\circ\ell-1} = D_{z_0}f_0^{\circ\ell-1}$ is invertible. So:

$$\left. \frac{dv(f_t)}{dt} \right|_{t=0} = \left. \frac{dz_t}{dt} \right|_{t=0} = \theta(v(f_0)). \quad \square$$

We now come to the heart of the argument.

Lemma 4 *The vector field θ is the restriction to $\mathcal{P}(f_0)$ of a vector field which is holomorphic on \mathbb{P}^1 .*

Proof: A rational map whose postcritical set contains only two points is conjugate to $z \mapsto z^{\pm d}$. It follows that $\mathcal{P}(f_0)$ contains at three points.

Choose three distinct points z_1, z_2, z_3 in $\mathcal{P}(f_0)$ and let $\tilde{\theta}$ be the unique holomorphic vector field on \mathbb{P}^1 which coincides with θ at z_1, z_2 and z_3 . We must show that $\tilde{\theta}$ coincides with θ at any other point of $\mathcal{P}(f_0)$. If there are only three points in $\mathcal{P}(f_0)$, we are done. Otherwise, let z_4 be a point in $\mathcal{P}(f_0) \setminus \{z_1, z_2, z_3\}$. Consider a quadratic differential q which is holomorphic on $\mathbb{P}^1 \setminus \{z_1, z_2, z_3, z_4\}$ and has simple poles at z_1, z_2, z_3 and z_4 . Then, the 1-form $q \otimes \tilde{\theta}$ is holomorphic on \mathbb{P}^1 and has at most simple poles on $\{z_1, z_2, z_3, z_4\}$. The sum of residues of a meromorphic 1-form on \mathbb{P}^1 is 0. It follows that θ and $\tilde{\theta}$ coincide at z_4 if and only if

$$0 = \sum_{z \in \{z_1, z_2, z_3, z_4\}} \text{Res}(q \otimes \tilde{\theta}, z) = \sum_{z \in \{z_1, z_2, z_3, z_4\}} \text{Res}(q \otimes \theta, z) = 0.^2$$

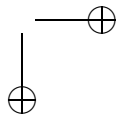
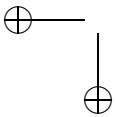
Denote by $\mathcal{Q}(f_0)$ the set of quadratic differentials which are holomorphic on $\mathbb{P}^1 \setminus \mathcal{P}(f_0)$ and have at most simple poles on $\mathcal{P}(f_0)$. By the above discussion, in order to prove the lemma, it is enough to show that for all $q \in \mathcal{Q}(f_0)$, we have:

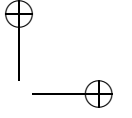
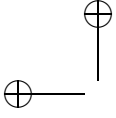
$$\langle q, \theta \rangle := 2i\pi \sum_{z \in \mathcal{P}(f_0)} \text{Res}(q \otimes \theta, z) = 0.$$

For this purpose, choose a vector field which is \mathcal{C}^∞ on \mathbb{P}^1 , holomorphic in a neighborhood of $\mathcal{P}(f_0)$ and which coincides with θ on $\mathcal{P}(f_0)$. To lighten notation, let us still denote it by θ . Without loss of generality, we may assume that U is a finite union of disjoint disks. Then, according to the residue theorem and to Stokes' theorem, for all $q \in \mathcal{Q}(f_0)$, we have:

$$\langle q, \theta \rangle = \int_{\partial U} q \otimes \theta = - \int_{\mathbb{P}^1 \setminus \bar{U}} q \otimes \bar{\partial}\theta = - \int_{\mathbb{P}^1} q \otimes \bar{\partial}\theta.$$

²To define the residue, we choose any holomorphic extension of θ to a neighborhood of $\mathcal{P}(f_0)$ (compare with [HS94] for example).





For the last equality, we used $\partial\theta = 0$ in \bar{U} . If $q \in \mathcal{Q}(f_0)$, we therefore have:

$$\langle (f_0)_*q, \theta \rangle = - \int_{\mathbb{P}^1} (f_0)_*q \otimes \bar{\partial}\theta = - \int_{\mathbb{P}^1} q \otimes f_0^*\bar{\partial}\theta = - \int_{\mathbb{P}^1} q \otimes \bar{\partial}(f_0^*\theta - \eta_\xi)$$

since $\bar{\partial}\eta_\xi \equiv 0$. However according to proposition 3, $f_0^*\theta - \eta_\xi$ is a \mathcal{C}^∞ vector field on \mathbb{P}^1 . Since $f_0^*\theta - \eta_\xi$ coincides with θ on $\mathcal{P}(f_0)$, we see that:

$$\forall q \in \mathcal{Q}(f_0), \quad \langle (f_0)_*q, \theta \rangle = \langle q, \theta \rangle.$$

Consider the linear map $\nabla : \mathcal{Q}(f_0) \rightarrow \mathcal{Q}(f_0)$ defined by $\nabla q = (f_0)_*q - q$. We have just seen that for all $q \in \mathcal{Q}(f_0)$, we have $\langle \nabla q, \theta \rangle = 0$.

We will now use the fact that f_0 is not a flexible Lattès example. Under this hypothesis $\nabla : \mathcal{Q}(f_0) \rightarrow \mathcal{Q}(f_0)$ is injective (see [DH93]), thus surjective. It follows that for all $q \in \nabla(\mathcal{Q}(f_0)) = \mathcal{Q}(f_0)$, we have $\langle q, \theta \rangle = 0$. \square

We still denote by θ the holomorphic extension to \mathbb{P}^1 . Note that the vector field $f_0^*\theta$ is defined and holomorphic on $\mathbb{P}^1 \setminus \mathcal{C}(f_0)$. According to lemma 4, for $j \in \{1, \dots, 2d - 2\}$, we have:

$$\theta(v_j(f_0)) = \left. \frac{df_t(\omega_j(t))}{dt} \right|_{t=0}.$$

According to proposition 3, the vector field $f_0^*\theta - \eta_\xi$ is holomorphic in a neighborhood of $\mathcal{C}(f_0)$, thus on \mathbb{P}^1 . It coincides with θ on $\mathcal{P}(f_0)$ which contains at least three points. Two holomorphic vector fields on \mathbb{P}^1 which coincide at three points are equal: indeed, in any affine coordinate, their expressions are polynomials of degree ≤ 2 . As a consequence, we have the required equality:

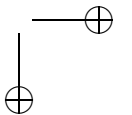
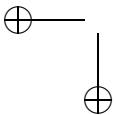
$$f_0^*\theta - \eta_\xi = \theta.$$

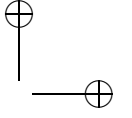
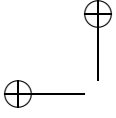
3 The bifurcation measure

We will now prove that $\mathcal{X}_d^* \subset \text{Supp}(\mu_{\text{bif}})$. This will complete the proof of our main theorem.

The idea is the following. Assume $f_0 \in \text{Rat}_d$ is a strictly postcritically finite rational map which is not a flexible Lattès example, has simple critical points and satisfies $\mathcal{C}(f_0) \cap \mathcal{P}(f_0) = \emptyset$. It is enough to show that there is a $2d - 2$ -dimensional complex manifold Σ transverse to $\mathcal{O}(f_0)$ at f_0 , and a basis of neighborhoods Σ_n of f_0 in Σ , such that

$$\int_{\Sigma_n} (T_{\text{bif}})^{\wedge(2d-2)} > 0.$$





Let $\omega_j : (\text{Rat}_d, f_0) \rightarrow \mathbb{P}^1$ be analytic germs following the critical points of f . There are integers $\ell_j \geq 1$ such that $\alpha_j(f_0) := f_0^{\circ \ell_j}(\omega_j(f_0))$ are repelling periodic points of f_0 of periods p_j . Let V_j be simply connected neighborhoods of $\alpha_j(f_0)$ such that $\overline{V_j} \subset W_j := f_0^{\circ p_j}(V_j)$ and such that $f_0^{\circ p_j} : V_j \rightarrow W_j$ are isomorphisms.

Denote by M the least common multiple of the periods p_j and define $\Sigma_n \subset \Sigma$ as the connected component containing f_0 of the set

$$\{f \in \Sigma ; \forall j \in \{1, \dots, 2d-2\}, f^{\circ \ell_j + nM}(\omega_j(f)) \in V_j\}.$$

Our proof essentially consists in proving that

$$\int_{\Sigma_n} (T_{\text{bif}})^{\wedge(2d-2)} \underset{n \rightarrow +\infty}{\sim} \frac{(2d-2)!}{d^{nM(2d-2) + \sum \ell_j}} \cdot \prod_{j=1}^{2d-2} \mu_{f_0}(V_j)$$

where μ_{f_0} is the equilibrium measure of f_0 . Since the disks V_j intersect the Julia set of f_0 , we have $\mu_{f_0}(V_j) > 0$ for all j , and so, for n large enough,

$$\int_{\Sigma_n} (T_{\text{bif}})^{\wedge(2d-2)} > 0.$$

3.1 Another definition of the bifurcation current

We will use a second definition of the bifurcation current T_{bif} due to De-Marco [DeM01] (see [DeM03] or [BB07] for the equivalence of the two definitions). The current T_{bif} can be defined locally by considering the behavior of the critical orbits as follows.

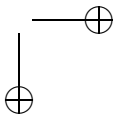
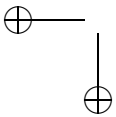
We still denote by $\pi : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{P}^1$ the canonical projection. Assume $f_0 \in \text{Rat}_d$ has only simple critical points and let U be a sufficiently small neighborhood of f_0 so that there are:

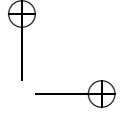
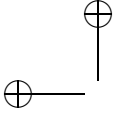
- holomorphic functions $\{\omega_j : U \rightarrow \mathbb{P}^1\}_{j \in \{1, \dots, 2d-2\}}$ following the critical points of f ,
- holomorphic functions $\{\tilde{\omega}_j : U \rightarrow \mathbb{C}^2 \setminus \{0\}\}_{j \in \{1, \dots, 2d-2\}}$ such that $\pi \circ \tilde{\omega}_j = \omega_j$ and
- a family $\{P_f\}_{f \in U}$ of non-degenerate homogeneous polynomials of degree d depending analytically on $f \in U$ such that $\pi \circ P_f = f \circ \pi$.

The map $G : U \times \mathbb{C}^2 \rightarrow \mathbb{R}$ defined by

$$G(f, z) := \lim_{n \rightarrow +\infty} \frac{1}{d^n} \log \|P_f^{\circ n}(z)\|$$

is plurisubharmonic on $U \times \mathbb{C}^2$.





3. The bifurcation measure

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For $j \in \{1, \dots, 2d-2\}$, define $G_j : U \rightarrow \mathbb{R}$ by:

$$G_j(f) := G(f, \tilde{\omega}_j(f)).$$

DeMarco [DeM03] proved that

- if P_f is chosen so that $\text{Res}(P_f) = 1$,³ and
- if $\tilde{\omega}_j$ are chosen so that $\det(D_z P_f) = \prod_{j=1}^{2d-2} (z \wedge \tilde{\omega}_j(f))$,

then

$$\sum_{j=1}^{2d-2} G_j(f) = L(f) + \log d.$$

As a consequence,

$$T_{\text{bif}}|_U = \sum_{j=1}^{2d-2} dd^c G_j.$$

3.2 A parametrized family of rational maps

From now on, we assume that $f_0 \in \text{Rat}_d$ is a strictly postcritically finite rational map which is not a flexible Lattès example, has simple critical points and satisfies $\mathcal{C}(f_0) \cap \mathcal{P}(f_0) = \emptyset$.

Let $\omega_j : (\text{Rat}_d, f_0) \rightarrow \mathbb{P}^1$ be analytic germs following the critical points of f and for $n \geq 1$, let $v_j^n : (\text{Rat}_d, f_0) \rightarrow \mathbb{P}^1$ be the analytic germs defined by

$$v_j^n(f) = f^{\circ n}(\omega_j(f)).$$

There are integers $\ell_j \geq 1$ and $p_j \geq 1$ such that

$$\alpha_j(f_0) := v_j^{\ell_j}(f_0)$$

are repelling periodic points of f_0 of periods p_j . Let $\alpha_j : (\text{Rat}_d, f_0) \rightarrow \mathbb{P}^1$ be analytic germs following those periodic points and let

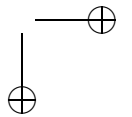
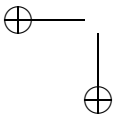
$$\phi_{j,f} : (\mathbb{P}^1, \alpha_j(f)) \rightarrow (\mathbb{C}, 0)$$

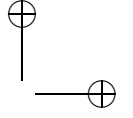
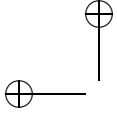
be linearizing maps depending analytically on f , i.e.

$$\phi_{j,f} \circ f^{\circ p_j} = \mu_j(f) \cdot \phi_{j,f}$$

with

³If $z = (z_1, z_2)$ and $w = (w_1, w_2)$, then $z \wedge w = z_1 w_2 - z_2 w_1$ and if $F(z) = (\prod_i z \wedge a_i, \prod_j z \wedge b_j)$, then $\text{Res}(F) = \prod_{i,j} a_i \wedge b_j$.





- $\phi_{j,f}(\alpha_j(f)) = 0$,
- $\phi_{j,f}$ is locally injective at $\alpha_j(f)$ and
- $\mu_j(f)$ is the multiplier of $f^{\circ p_j}$ at $\alpha_j(f)$.

Consider the analytic map $\tau : (\text{Rat}_d, f_0) \rightarrow (\mathbb{C}^{2d-2}, 0)$ defined by

$$\tau := (\tau_1, \dots, \tau_{2d-2}) \quad \text{with} \quad \tau_j(f) := \phi_{j,f} \circ v_j^{\ell_j}(f).$$

Lemma 5 *The rank of $D_{f_0}\tau : T_{f_0}\text{Rat}_d \rightarrow \mathbb{C}^{2d-2}$ is $2d - 2$. The kernel is $T_{f_0}\mathcal{O}(f_0)$.*

Proof: Differentiating the equality $\phi_{j,f}(\alpha_j(f)) = 0$ with respect to f , we have

$$\left. \frac{d\phi_{j,f}}{df} \right|_{(f_0, \alpha_j(f_0))} + D_{\alpha_j(f_0)}\phi_{j,f_0} \circ D_{f_0}\alpha_j = 0.$$

As a consequence, we have

$$\begin{aligned} D_{f_0}\tau_j &= \left. \frac{d\phi_{j,f}}{df} \right|_{(f_0, \alpha_j(f_0))} + D_{\alpha_j(f_0)}\phi_{j,f_0} \circ D_{f_0}v_j^{\ell_j} \\ &= D_{\alpha_j(f_0)}\phi_{j,f_0} \circ (D_{f_0}v_j^{\ell_j} - D_{f_0}\alpha_j). \end{aligned}$$

Thus, $\xi \in T_{f_0}\text{Rat}_d$ belongs to the kernel of $D_{f_0}\tau$ if and only if it belongs to the kernel of $D_{f_0}v_j^{\ell_j} - D_{f_0}\alpha_j$ for all $j \in \{1, \dots, 2d - 2\}$, i.e. if and only if it belongs to the kernel of $D_{f_0}F - D_{f_0}G$. According to proposition 1, such a ξ belongs to $T_{f_0}\mathcal{O}(f_0)$. Thus, the kernel $D_{f_0}\tau$ has dimension 3 and the image has dimension $2d - 2$ (the dimension of $T_{f_0}\text{Rat}_d$ is $2d + 1$). \square

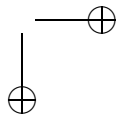
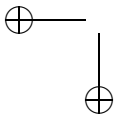
Corollary 2 *If $r > 0$ is sufficiently small and if $\Delta := D(0, r)$, then there is a section*

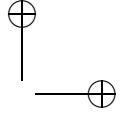
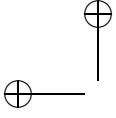
$$\chi : \Delta^{2d-2} \rightarrow \text{Rat}_d \quad \text{with} \quad \chi(0) = f_0 \quad \text{and} \quad \tau \circ \chi = \text{Id}.$$

From now on, we assume that such a section $\chi : \Delta^{2d-2} \rightarrow \text{Rat}_d$ has been chosen and we let $\Sigma \subset \text{Rat}_d$ be the complex manifold of dimension $2d - 2$ defined by

$$\Sigma := \chi(\Delta^{2d-2}).$$

Then, Σ is an analytic family of rational maps parametrized by the positions of the points $f^{\circ \ell_j}(\omega_j(f))$ in the linearizing coordinates $\phi_{j,f}$.





3.3 The proof

Let us now assume that $V_j \subset \mathbb{P}^1$ is a sufficiently small neighborhood of $\alpha_j(f_0)$ so that

- ϕ_{j,f_0} is defined and univalent on a neighborhood of \bar{V}_j and
- $\pi : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{P}^1$ has an analytic section $\sigma_j : V_j \rightarrow \mathbb{C}^2 \setminus \{0\}$ defined on V_j .

Decreasing r if necessary, we may assume that Δ is relatively compact in $\phi_{j,f_0}(V_j)$ for all $j \in \{1, \dots, 2d-2\}$.

Let us now denote by M the least common multiple of the periods p_j and set $m_j = M/p_j$. For $n \geq 0$, we define $\chi_n : \Delta^{2d-2} \rightarrow \Sigma$ by

$$\chi_n(x_1, \dots, x_{2d-2}) := \chi \left(\frac{x_1}{(\mu_1(f_0))^{nm_1}}, \dots, \frac{x_{2d-2}}{(\mu_{2d-2}(f_0))^{nm_{2d-2}}} \right)$$

and set

$$\Sigma_n := \chi_n(\Delta^{2d-2}) \subset \Sigma.$$

Lemma 6 *As $n \rightarrow +\infty$, the sequences of functions*

$$\left(\frac{\mu_j \circ \chi_n}{\mu_j(f_0)} \right)^n : \Delta^{2d-2} \rightarrow \mathbb{C}$$

converge uniformly to 1.

Proof: Since the map $\chi : \Delta^{2d-2} \rightarrow \text{Rat}_d$ and the germs $\mu_j : (\text{Rat}_d, f_0) \rightarrow \mathbb{C}$ are analytic, there is a constant $C > 0$ such that

$$\left\| \frac{\mu_j \circ \chi_n}{\mu_j(f_0)} - 1 \right\|_{\infty} \leq C\lambda^n$$

with

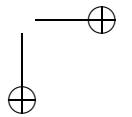
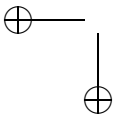
$$\lambda := \min_{j \in \{1, \dots, 2d-2\}} \left\{ \frac{1}{|\mu_j(f_0)|^{m_j}} \right\}.$$

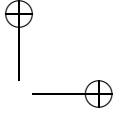
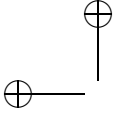
It follows easily that

$$\left\| \left(\frac{\mu_j \circ \chi_n}{\mu_j(f_0)} \right)^n - 1 \right\|_{\infty} \xrightarrow{n \rightarrow +\infty} 0. \quad \square$$

Corollary 3 *For n sufficiently large and all $j \in \{1, \dots, 2d-2\}$, we have*

$$v_j^{\ell_j + nM}(\Sigma_n) \subset V_j.$$





Proof: Since $\Delta = D(0, r)$ is relatively compact in $\phi_{j, f_0}(V_j)$, there is a neighborhood U of f_0 in Rat_d and a $\kappa > 1$ such that for all $f \in U$ and all $j \in \{1, \dots, 2d-2\}$, $D(0, \kappa r) \subset \phi_{j, f}(V_j)$. Let us choose n sufficiently large so that $\chi_n(\Delta^{2d-2}) \subset U$ and so that

$$\forall f \in \chi_n(\Delta^{2d-2}), \forall j \in \{1, \dots, 2d-2\}, \quad \left| \frac{\mu_j(f)}{\mu_j(f_0)} \right|^{nm_j} < \kappa.$$

If $f \in \Sigma_n$, then $f = \chi_n(x_1, \dots, x_{2d-2})$ with $(x_1, \dots, x_{2d-2}) \in \Delta^{2d-2}$,

$$\phi_{j, f}(v_j^{\ell_j}(f)) = \frac{x_j}{(\mu_j(f_0))^{nm_j}} \quad \text{and} \quad (\mu_j(f))^{nm_j} \cdot \frac{x_j}{(\mu_j(f_0))^{nm_j}} \in \phi_{j, f}(V_j).$$

In addition, $\phi_{j, f} \circ f^{\circ p_j} = \mu_j(f) \cdot \phi_{j, f}$. Thus,

$$v_j^{\ell_j + nM}(f) = v_j^{\ell_j + nm_j p_j}(f) = \phi_{j, f}^{-1} \left((\mu_j(f))^{nm_j} \cdot \frac{x_j}{(\mu_j(f_0))^{nm_j}} \right) \in V_j. \quad \square$$

From now on, we assume that n is sufficiently large so that for all integer $j \in \{1, \dots, 2d-2\}$, we have

$$v_j^{\ell_j + nM}(\Sigma_n) \subset V_j.$$

Define plurisubharmonic functions $G_j^n : \Delta^{2d-2} \rightarrow \mathbb{R}$ by

$$G_j^n(\mathbf{x}) := G(\chi_n(\mathbf{x}), \sigma_j \circ v_j^{\ell_j + nM} \circ \chi_n(\mathbf{x})).$$

Lemma 7 For $j \in \{1, \dots, 2d-2\}$, we have:

$$dd^c G_j^n = d^{\ell_j + nM} \cdot \chi_n^*(dd^c G_j).$$

Proof: Assume $f \in \Sigma_n$ and $j \in \{1, \dots, 2d-2\}$. Note that

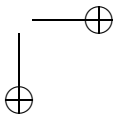
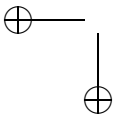
$$\pi \circ P_f^{\circ \ell_j + nM}(\tilde{\omega}_j(f)) = f^{\circ \ell_j + nM}(\omega_j(f)) = v_j^{\ell_j + nM}(f) = \pi \circ \sigma_j \circ v_j^{\ell_j + nM}(f).$$

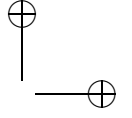
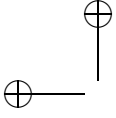
Thus,

$$P_f^{\circ n}(\tilde{\omega}_j(f)) = \lambda_j^n(f) \cdot \sigma \circ v_j^{\ell_j + nM}(f)$$

for some holomorphic functions $\lambda_j^n : \Sigma_n \rightarrow \mathbb{C}^*$. Note that

$$G(f, P_f(z)) = d \cdot G(f, z) \quad \text{and} \quad \forall \lambda \in \mathbb{C}^*, \quad G(f, \lambda z) = G(f, z) + \log |\lambda|.$$





3. The bifurcation measure

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Thus, if $f = \chi_n(\mathbf{x})$ with $\mathbf{x} \in \Delta^{2d-2}$, then

$$\begin{aligned} G_j^n(\mathbf{x}) &= G(f, P_f^{\circ \ell_j + nM}(\tilde{\omega}_j(f))) - \log|\lambda_j^n(f)| \\ &= d^{\ell_j + nM} G(f, \tilde{\omega}_j(f)) - \log|\lambda_j^n(f)| \\ &= d^{\ell_j + nM} G_j(f) - \log|\lambda_j^n(f)| \\ &= d^{\ell_j + nM} G_j \circ \chi_n(\mathbf{x}) - \log|\lambda_j^n \circ \chi_n(\mathbf{x})|. \end{aligned}$$

Since $\log|\lambda_j^n \circ \chi_n|$ is pluriharmonic on Δ^{2d-2} , we have

$$dd^c G_j^n = d^{\ell_j + nM} \cdot \chi_n^*(dd^c G_j). \quad \square$$

It follows that

$$\chi_n^*(T_{\text{bif}}) = \sum_{j=1}^{2d-2} \chi_n^*(dd^c G_j) = \frac{1}{d^{nM}} \sum_{j=1}^{2d-2} \frac{1}{d^{\ell_j}} dd^c G_j^n.$$

Lemma 8 *We have*

$$\lim_{n \rightarrow +\infty} \int_{\Delta^{2d-2}} \left(\sum_{j=1}^{2d-2} \frac{1}{d^{\ell_j}} dd^c G_j^n \right)^{\wedge(2d-2)} = \frac{(2d-2)!}{d^{\sum \ell_j}} \cdot \prod_{j=1}^{2d-2} \mu_{f_0}(\phi_{j,f_0}^{-1}(\Delta)).$$

Proof: If $f = \chi_n(x_1, \dots, x_{2d-2})$, then

$$v_j^{\ell_j + nM}(f) = \phi_{j,f}^{-1} \left(\left(\frac{\mu_j(f)}{\mu_j(f_0)} \right)^{nm_j} \cdot x_j \right).$$

Thus, as $n \rightarrow +\infty$, the sequence of functions $v_j^{\ell_j + nM} \circ \chi_n : \Delta^{2d-2} \rightarrow V_j$ converge uniformly to

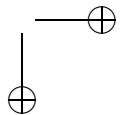
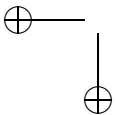
$$(x_1, \dots, x_{2d-2}) \mapsto \phi_{j,f_0}^{-1}(x_j).$$

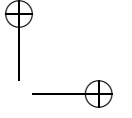
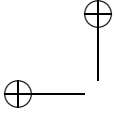
As a consequence, as $n \rightarrow +\infty$, the sequences of functions $G_j^n : \Delta^{2d-2} \rightarrow \mathbb{R}$ converge uniformly to

$$G_j^\infty : (x_1, \dots, x_{2d-2}) \mapsto G(f_0, \sigma_j \circ \phi_{j,f_0}^{-1}(x_j)).$$

Thanks to the uniform convergence of the potentials, we can write:

$$\lim_{n \rightarrow +\infty} \int_{\Delta^{2d-2}} \left(\sum_{j=1}^{2d-2} \frac{1}{d^{\ell_j}} dd^c G_j^n \right)^{\wedge(2d-2)} = \int_{\Delta^{2d-2}} \left(\sum_{j=1}^{2d-2} \frac{1}{d^{\ell_j}} dd^c G_j^\infty \right)^{\wedge(2d-2)}.$$





Note that G_j^∞ only depends on the j -th coordinate. It follows that

$$\left(\sum_{j=1}^{2d-2} \frac{1}{d^{\ell_j}} dd^c G_j^\infty \right)^{\wedge(2d-2)} = \frac{(2d-2)!}{d^{\sum \ell_j}} \cdot \bigwedge_{j=1}^{2d-2} dd^c G_j^\infty.$$

In addition, $G_j^\infty(x_1, \dots, x_{2d-2}) = g_j \circ \phi_{j, f_0}^{-1}(x_j)$ with $g_j : V_j \rightarrow \mathbb{R}$ the subharmonic function defined by $g_j(z) = G(f_0, \sigma_j(z))$. We have

$$dd^c g_j = \mu_{f_0}|_{V_j}.$$

Therefore, according to Fubini's theorem,

$$\begin{aligned} \int_{\Delta^{2d-2}} \left(\bigwedge_{j=1}^{2d-2} dd^c G_j^\infty \right) &= \prod_{j=1}^{2d-2} \left(\int_{\Delta} dd^c (g_j \circ \phi_{j, f_0}^{-1}) \right) \\ &= \prod_{j=1}^{2d-2} \left(\int_{\phi_{j, f_0}^{-1}(\Delta)} dd^c g_j \right) \\ &= \prod_{j=1}^{2d-2} \mu_{f_0}(\phi_{j, f_0}^{-1}(\Delta)). \quad \square \end{aligned}$$

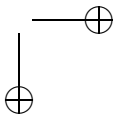
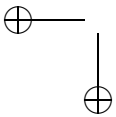
We can now complete the proof. Since the points $\alpha_j(f_0)$ are repelling periodic points of f_0 , they are in the support of the equilibrium measure μ_{f_0} . Thus,

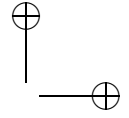
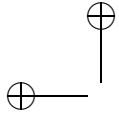
$$\mu_{f_0}(\phi_{j, f_0}^{-1}(\Delta)) > 0.$$

As a consequence,

$$\lim_{n \rightarrow +\infty} d^{nM(2d-2)} \cdot \left(\int_{\Sigma_n} T_{\text{bif}}^{\wedge(2d-2)} \right) = \frac{(2d-2)!}{d^{\sum \ell_j}} \cdot \prod_{j=1}^{2d-2} \mu_{f_0}(\phi_{j, f_0}^{-1}(\Delta)) > 0.$$

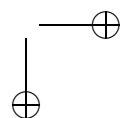
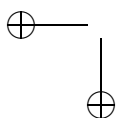
Thus, f_0 is in the support of $T_{\text{bif}}^{\wedge(2d-2)}$ and the conjugacy class of f_0 is in the support of μ_{bif} .

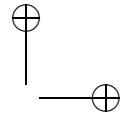
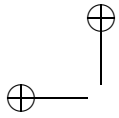




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