# Bifurcation measure and postcritically finite rational maps

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Bassanelli and Berteloot [BB07] have defined a bifurcation measure $\mu_{\rm bif}$ on the moduli space $\mathcal{M}_d$ of rational maps of degree $d \geq 2$. They have proved that it is a positive measure of finite mass and that its support is contained in the closure of the set $\mathcal{Z}_d$ of conjugacy classes of rational maps of degree $d$ having $2d-2$ indifferent cycles.

Denote by $\mathcal{X}_d$ the set of conjugacy classes of strictly postcritically finite rational maps of degree $d$ which are not flexible Lattès examples. We prove that $\text{Supp}(\mu_{\rm bif}) = \overline{\mathcal{X}_d} = \overline{\mathcal{Z}_d}$. Our proof is based on a transversality result due to the second author.

A similar result was obtained with different techniques by Dujardin and Favre [DF] for the bifurcation measure on moduli spaces of polynomials of degree $d \geq 2$.

## Introduction

A result of Lyubich [Lyu83] asserts that for each rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$, there is a unique probability measure $\mu_f$ of maximal entropy $\log d$. It is ergodic, satisfies $f^*\mu_f = d \cdot \mu_f$ and is carried outside the exceptional set of $f$. This measure is called the *equilibrium measure* of $f$.

The *Lyapunov exponent* of $f$ with respect to the measure $\mu_f$ can be defined by

$$L(f) := \int_{\mathbb{P}^1} \log\|Df\| \, {\rm d}\mu_f,$$

where $\|\cdot\|$ is any smooth metric on $\mathbb{P}^1$. Since $\mu_f$ is ergodic, the quantity ${\rm e}^{L(f)}$ records the average rate of expansion of $f$ along a typical orbit with respect to $\mu_f$.

Denote by $\text{Rat}_d$ the set of rational maps of degree $d$. It is a smooth complex manifold of dimension $2d+1$. The function $L : \text{Rat}_d \to \mathbb{R}$ is continuous [Mañ88] and plurisubharmonic [DeM03] on $\text{Rat}_d$. The positive $(1,1)$-current

$$T_{\rm bif} := dd^c L$$

is called the *bifurcation current* in $\mathrm{Rat}_d$.

The *bifurcation locus* in $\mathrm{Rat}_d$ is the closure of the set of discontinuity of the map $f \in \mathrm{Rat}_d \mapsto J_f$, where $J_f$ stands for the Julia set of $f$ and the continuity is for the Hausdorff topology for compact subsets of $\mathbb{P}^1$. DeMarco [DeM03] proved that the support of the bifurcation current $T_{\mathrm{bif}}$ is equal to the bifurcation locus.

Given $f \in \mathrm{Rat}_d$, denote by $\mathcal{O}(f)$ the set of rational maps which are conjugate to $f$ by a Möbius transformation. It is a smooth submanifold of $\mathrm{Rat}_d$ of complex dimension 3. It is isomorphic to $\mathrm{Aut}(\mathbb{P}^1)/\Gamma$, where $\mathrm{Aut}(\mathbb{P}^1) \simeq \mathrm{PSL}(2,\mathbb{C})$ is the group of Möbius transformations, and where $\Gamma \subset \mathrm{Aut}(\mathbb{P}^1)$ is the finite subgroup of Möbius transformations which commute with $f$.

The *moduli space* $\mathcal{M}_d$ is the quotient $\mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$, where $\mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathrm{Rat}_d$ by conjugation. It is an orbifold of complex dimension $2d-2$. It is a normal, quasiprojective variety. We denote by $\Pi : \mathrm{Rat}_d \to \mathcal{M}_d$ the canonical projection.

The Lyapunov exponent is invariant under holomorphic conjugacy, thus is constant on the orbits $\mathcal{O}(f)$. The map $L : \mathrm{Rat}_d \to \mathbb{R}$ descends to a map $\hat{L} : \mathcal{M}_d \to \mathbb{R}$ which is continuous, bounded from below and plurisubharmonic on $\mathcal{M}_d$ (see [BB07] proposition 6.2). The measure

$$\mu_{\mathrm{bif}} := (dd^c \hat{L})^{\wedge(2d-2)}$$

is called the *bifurcation measure* on $\mathcal{M}_d$. By construction, we have

$$\Pi^* \mu_{\mathrm{bif}} = {T_{\mathrm{bif}}}^{\wedge(2d-2)}.$$

The *critical set* $\mathcal{C}(f)$ of a rational map $f$ is the set of critical points of $f$. The postcritical set is the set:

$$\mathcal{P}(f) = \bigcup_{\omega \in \mathcal{C}(f)} \bigcup_{n \geq 1} f^{\circ n}(\omega).$$

A rational map $f$ is *postcritically finite* if $\mathcal{P}(f)$ is finite. It is *strictly postcritically finite* if $\mathcal{P}(f)$ is finite and if $f$ does not have any superattracting cycle. It is a *Lattès map* if it is obtained as the quotient of an affine map $A : z \mapsto az+b$ on a complex torus $\mathbb{C}/\Lambda$: there is a finite-to-one holomorphic map $\Theta : \mathbb{C}/\Lambda \to \mathbb{P}^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{A} & \mathbb{C}/\Lambda \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Theta} \\ \mathbb{P}^1 & \xrightarrow[f]{} & \mathbb{P}^1. \end{array}$$

A Lattès example is strictly postcritically finite (see [Mil06]). It is a *flexible Lattès example* if we can choose $\Theta$ with degree 2 and $A(z) = az + b$ with $a > 1$ an integer.

Let us introduce the following notation:

- $\mathcal{X}_d \subset \mathcal{M}_d$ for the set of conjugacy classes of strictly postcritically finite rational maps of degree $d$ which are not flexible Lattès examples;
- $\mathcal{X}_d^* \subset \mathcal{X}_d$ for the subset of conjugacy classes of strictly postcritically finite rational maps of degree $d$ which have simple critical points and satisfy $\mathcal{C}(f) \cap \mathcal{P}(f) = \emptyset$;
- $\mathcal{Z}_d \subset \mathcal{M}_d$ for the set of conjugacy classes of maps having $2d - 2$ indifferent cycles (we do not count multiplicities).

Bassanelli and Berteloot [BB07] proved that

- the bifurcation measure does not vanish identically on $\mathcal{M}_d$ and has finite mass,
- the conjugacy class of any non-flexible Lattès map lies in the support of $\mu_{\mathrm{bif}}$,
- the support of $\mu_{\mathrm{bif}}$ is contained in the closure of $\mathcal{Z}_d$.

Our main result is the following.

**Main Theorem** *The support of the bifurcation measure in $\mathcal{M}_d$ is:*

$$\mathrm{Supp}(\mu_{\mathrm{bif}}) = \overline{\mathcal{X}_d^*} = \overline{\mathcal{X}_d} = \overline{\mathcal{Z}_d}.$$

**Remark** We shall prove in section 1 below that

$$\overline{\mathcal{X}_d^*} = \overline{\mathcal{X}_d} = \overline{\mathcal{Z}_d}. \tag{0.1}$$

Then, thanks to the results in [BB07], we will only have to prove the inclusion

$$\mathcal{X}_d^* \subset \mathrm{Supp}(\mu_{\mathrm{bif}}).$$

**Remark** We do not know how to prove that the conjugacy classes of flexible Lattès examples lie in the support of $\mu_{\mathrm{bif}}$. It would be enough to prove that every flexible Lattès example can be approximated by strictly postcritically finite rational maps which are not Lattès examples.

**Remark** The underlying idea of the proof is a potential-theoretic interpretation of a result of Tan Lei [Tan90] concerning the similarities between the Mandelbrot set and Julia sets.

Our proof relies on a transversality result that we will now present.

Assume $f_0 \in \mathrm{Rat}_d$ is a strictly postcritically finite rational map which has $2d-2$ distinct critical points $\omega_j(f_0)$, $j \in \{1, \ldots, 2d-2\}$, and satisfies $\mathcal{C}(f_0) \cap \mathcal{P}(f_0) = \emptyset$. There are integers $\ell_j \geq 1$ such that

$$\alpha_j(f_0) := f_0^{\circ \ell_j}\big(\omega_j(f_0)\big)$$

are periodic points of $f_0$. The critical points are simple and the periodic points are repelling. By the Implicit Function Theorem, there are

- analytic germs $\omega_j : (\mathrm{Rat}_d, f_0) \to \mathbb{P}^1$ following the critical points of $f$ as $f$ ranges in a neighborhood of $f_0$ in $\mathrm{Rat}_d$ and

- analytic germs $\alpha_j : (\mathrm{Rat}_d, f_0) \to \mathbb{P}^1$ following periodic points of $f$ as $f$ ranges in a neighborhood of $f_0$ in $\mathrm{Rat}_d$.

Let $F : (\mathrm{Rat}_d, f_0) \to (\mathbb{P}^1)^{2d-2}$ and $G : (\mathrm{Rat}_d, f_0) \to (\mathbb{P}^1)^{2d-2}$ be defined by :

$$F := \begin{pmatrix} F_1 \\ \vdots \\ F_{2d-2} \end{pmatrix} \quad \text{with} \quad F_j(f) := f^{\circ \ell_j}\big(\omega_j(f)\big) \quad \text{and} \quad G := \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{2d-2} \end{pmatrix}.$$

Denote by $D_{f_0}F$ and $D_{f_0}G$ the differentials of $F$ and $G$ at $f_0$.

The transversality result we are interested in is the following.

**Proposition 1** *If $f_0$ is not a flexible Lattès example, then the linear map*

$$D_{f_0}F - D_{f_0}G : T_{f_0}\mathrm{Rat}_d \to T_{\alpha_1(f_0)}\mathbb{P}^1 \times \cdots \times T_{\alpha_{2d-2}(f_0)}\mathbb{P}^1$$

*has rank $2d-2$. The kernel of $D_{f_0}F - D_{f_0}G$ is the tangent space to $\mathcal{O}(f_0)$ at $f_0$.*

There are several proofs of this result. Here, we include a proof due to the second author. A different proof was obtained by van Strien [vS00]. His proof covers a more general setting where the critical orbits are allowed to be preperiodic to a hyperbolic set. Our proof covers the case of maps commuting with a non trivial group of Möbius transformations (a slight modification of van Strien's proof probably also covers this case).

**Remark** A similar transversality theorem holds in the space of polynomials of degree $d$ and the arguments developed in this article lead to an alternative proof of the result of Dujardin and Favre.

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# 1 The sets $\mathcal{X}_d^*$, $\mathcal{X}_d$ and $\mathcal{Z}_d$

In this section, we prove equality (0.1): $\overline{\mathcal{X}_d^*} = \overline{\mathcal{X}_d} = \overline{\mathcal{Z}_d}$. Since $\mathcal{X}_d^* \subset \mathcal{X}_d$, it is enough to prove that

$$\mathcal{X}_d \subset \overline{\mathcal{Z}_d} \quad \text{and} \quad \mathcal{Z}_d \subset \overline{\mathcal{X}_d^*}.$$

## 1.1 Tools

Our proof relies on the following three results.

The first result is an immediate consequence of the Fatou-Shishikura inequality on the number of non repelling cycles of a rational map (see [Shi87] and/or [Eps99]). For $f \in \mathrm{Rat}_d$, denote by

- $N_{\mathrm{att}}(f)$ the number of attracting cycles of $f$,
- $N_{\mathrm{ind}}(f)$ the number of distinct indifferent cycles of $f$ and
- $N_{\mathrm{crit}}(f)$ the number of critical points of $f$, counting multiplicities, whose orbits are strictly preperiodic to repelling cycles.

**Theorem 1** *For any rational map $f \in \mathrm{Rat}_d$, we have:*

$$N_{\mathrm{crit}}(f) + N_{\mathrm{ind}}(f) + N_{\mathrm{att}}(f) \le 2d - 2.$$

The second result is a characterization of *stability* due to Mañe, Sad and Sullivan [MSS83] (compare with [McM94] section 4.1). Let $\Lambda$ be a complex manifold. A family of rational maps $\{f_\lambda\}_{\lambda\in\Lambda}$ is an *analytic family* if the map $(\lambda, z) \in \Lambda \times \mathbb{P}^1 \mapsto f_\lambda(z) \in \mathbb{P}^1$ is analytic. The family is *stable at* $\lambda_0 \in \Lambda$ if the number of attracting cycles of $f_\lambda$ is locally constant at $\lambda_0$.

**Theorem 2** *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be an analytic family of rational maps parametrized by a complex manifold $\Lambda$. The following assertions are equivalent.*

- *The family is stable at $\lambda_0$.*
- *For all $\mu \in S^1$ and $p \ge 1$, the number of cycles of $f_\lambda$ having multiplier $\mu$ and period $p$ is locally constant at $\lambda_0$.*

- *For all $\ell \geq 1$ and $p \geq 1$, the number of critical points $\omega$ of $f_\lambda$ such that $f^{\circ\ell}(\omega)$ is a repelling periodic point of period $p$ is locally constant at $\lambda_0$.*
- *The maximum period of an indifferent cycle of $f_\lambda$ is locally bounded at $\lambda_0$.*
- *The maximum period of a repelling cycle of $f_\lambda$ contained in the post-critical set $\mathcal{P}(f_\lambda)$ is locally bounded at $\lambda_0$.*

The third result is due to McMullen [McM87]. Assume $\Lambda$ is an irreducible quasiprojective complex variety. A family of rational maps $\{f_\lambda\}_{\lambda\in\Lambda}$ is an *algebraic family* if the map $(\lambda, z) \in \Lambda \times \mathbb{P}^1 \mapsto f_\lambda(z) \in \mathbb{P}^1$ is a rational mapping. The family is *trivial* if all its members are conjugate by Möbius transformations. The family is *stable* if it is stable at every $\lambda \in \Lambda$.

**Theorem 3** *A stable algebraic family of rational maps is either trivial or it is a family of flexible Lattès examples.*

## 1.2 Spaces of rational maps with marked critical points and marked periodic points

In the sequel, it will be convenient to consider the set $\mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_n}$ of rational maps of degree $d$ with marked critical points and marked periodic points of period dividing $n$. This can be defined as follows.

First, the unordered sets of $m$ points in $\mathbb{P}^1$ can be identified with $\mathbb{P}^m$. This yields a rational map $\sigma_m : (\mathbb{P}^1)^m \to \mathbb{P}^m$ which might be defined as follows:

$$\sigma_m\big([x_1 : y_1], \ldots [x_m : y_m]\big) = [a_0 : \cdots : a_m]$$

with

$$\sum_{j=0}^{m} a_j x^j y^{m-j} = \prod_{i=1}^{m} (xy_i - yx_i).$$

Second, to a rational map $f \in \mathrm{Rat}_d$, we associate the unordered set $\{\omega_1, \ldots, \omega_{2d-2}\}$ of its $2d-2$ critical points, listed with repetitions according to their multiplicities. This induces a rational map $\mathrm{Crit} : \mathrm{Rat}_d \to \mathbb{P}^{2d-2}$ which might be defined as follows: if $f\big([x : y]\big) = \big[P(x,y) : Q(x,y)\big]$ with $P$ and $Q$ homogeneous polynomials of degree $d$, then

$$\mathrm{Crit}(f) = [a_0 : \cdots : a_{2d-2}] \quad \text{with} \quad \sum_{j=0}^{2d-2} a_j x^j y^{2d-2-j} = \frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x}.$$

Third, given an integer $n \geq 1$, to a rational map $f \in \mathrm{Rat}_d$, we associate the unordered set $\{\alpha_1, \ldots, \alpha_{d^n+1}\}$ of the fixed points of $f^{\circ n}$, listed with

repetitions according to their multiplicities. This induces a rational map $\mathrm{Per}_n : \mathrm{Rat}_d \to \mathbb{P}^{2^n+1}$ which might be defined as follows:
if $f^{\circ n}\big([x:y]\big) = \big[P_n(x,y) : Q_n(x,y)\big]$ with $P_n$ and $Q_n$ homogeneous polynomials of degree $d^n$, then

$$\mathrm{Per}_n(f) = [a_0 : \cdots : a_{d^n+1}]$$

with

$$\sum_{j=0}^{d^n+1} a_j x^j y^{d^n+1-j} = yP_n(x,y) - xQ_n(x,y).$$

The space of rational maps of degree $d$ with marked critical points and marked periodic points of period dividing $n$ is the set:

$$\mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_n} := \left\{ \begin{array}{l} (f,\omega,\alpha) \in \mathrm{Rat}_d \times (\mathbb{P}^1)^{2d-2} \times (\mathbb{P}^1)^{d^n+1} \text{ such that} \\ \mathrm{Crit}(f) = \sigma_{2d-2}(\omega) \text{ and } \mathrm{Per}_n(f) = \sigma_{d^n+1}(\alpha) \end{array} \right\}.$$

Then, $\mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_n}$ is an algebraic subset[1] of $\mathrm{Rat}_d \times (\mathbb{P}^1)^{2d-2} \times (\mathbb{P}^1)^{d^n+1}$. Since there are $2d-2+d^n+1$ equations, the dimension of any irreducible component of $\mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_n}$ is at least that of $\mathrm{Rat}_d$, i.e. $2d+1$. Since the fibers of the projection $\mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_n} \to \mathrm{Rat}_d$ are finite (a rational map has finitely many critical points and finitely many periodic points of period dividing $n$), the dimension of any component is exactly $2d+1$.

## 1.3 The inclusion $\mathcal{X}_d \subset \overline{\mathcal{Z}_d}$

Assume $f_0 \in \mathrm{Rat}_d$ is a strictly postcritically finite rational map but not a flexible Lattès example. We must show that any neighborhood of $f_0$ in $\mathrm{Rat}_d$ contains a rational map with $2d-2$ indifferent cycles. This is an immediate consequence of the following lemma.

**Lemma 1** *Assume $r \geq 1$ and $s \geq 0$ are integers such that $r+s = 2d-2$. Assume $f_0 \in \mathrm{Rat}_d$ is not a flexible Lattès example, $N_{\mathrm{crit}}(f_0) = r$ and $N_{\mathrm{ind}}(f_0) = s$. Then, arbitrarily close to $f_0$, we can find a rational map $f_1$ such that $N_{\mathrm{crit}}(f_1) = r-1$ and $N_{\mathrm{ind}}(f_1) = s+1$.*

*Proof:* Let $p_1, \ldots, p_s$ be the periods of the indifferent cycles of $f_0$. Denote by $M$ their least common multiple. A point $\lambda \in \mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_M}$ is of the form

$$\big(g_\lambda, \omega_1(\lambda), \ldots, \omega_{2d-2}(\lambda), \alpha_1(\lambda), \ldots \alpha_{d^M+1}(\lambda)\big).$$

We choose $\lambda_0 \in \mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_M}$ so that

[1]By algebraic we mean a quasiprojective variety in some projective space. It is classical that any product of projective spaces embeds in some $\mathbb{P}^N$. Thus, it makes sense to speak of algebraic subsets of products of projective spaces

- $g_{\lambda_0} = f_0$,
- $\omega_1(\lambda_0), \ldots \omega_r(\lambda_0)$ are preperiodic to repelling cycles of $f_0$ and
- $\alpha_1(\lambda_0),\ldots, \alpha_s(\lambda_0)$ are indifferent periodic points of $f_0$ belonging to distinct cycles.

For $i \in [1, r]$, let $k_i \geq 1$ and $p_i \geq 1$ be the least integers such that

$$g_{\lambda_0}^{\circ k_i}\big(\omega_i(\lambda_0)\big) = g_{\lambda_0}^{\circ k_i+p_i}\big(\omega_i(\lambda_0)\big).$$

For $j \in [1, s]$, let $\mu_j$ be the multiplier of $g_{\lambda_0}^{\circ M}$ at $\alpha_j(\lambda_0)$. In particular, $\mu_j \in S^1$.

Consider the algebraic subset

$$\left\{\lambda \in \mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_M}\ ;\ \begin{array}{l} \forall i \in [1, r-1],\ g_\lambda^{\circ k_i}\big(\omega_i(\lambda)\big) = g_\lambda^{\circ k_i+p_i}\big(\omega_i(\lambda)\big) \text{ and} \\ \forall j \in [1, s],\ \text{the multiplier of } g_\lambda^{\circ M} \text{ at } \alpha_j(\lambda) \text{ is } \mu_j \end{array}\right\}$$

and let $\Lambda$ be an irreducible component containing $\lambda_0$. Note that there are $r - 1 + s = 2d - 3$ equations. It follows that the dimension of $\Lambda$ is at least $(2d + 1) - (2d - 3) = 4$. In particular, the image of $\Lambda$ by the projection $\mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_M} \to \mathrm{Rat}_d$ cannot be contained in $\mathcal{O}(f_0)$.

Embedding $\Lambda$ in some $\mathbb{P}^N$ (with $N$ sufficiently large) and slicing with an appropriate projective subspace, we deduce that there is an algebraic curve $\Gamma \subset \Lambda$ containing $\lambda_0$, whose image by the projection $\mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_M} \to \mathrm{Rat}_d$ is not contained in $\mathcal{O}(f_0)$. Desingularizing the algebraic curve, we obtain a smooth quasiprojective curve $\Sigma$ (i.e. a Riemann surface of finite type) and an algebraic map $\Sigma \to \Gamma$ which is surjective and generically one-to-one. We let $\sigma_0 \in \Sigma$ be a point which is mapped to $\lambda_0 \in \Gamma$. With an abuse of notation, we write $g_\sigma$, $\omega_i(\sigma)$ and $\alpha_j(\sigma)$ in place of $g_{\lambda(\sigma)}$, $\omega_i\big(\lambda(\sigma)\big)$ and $\alpha_j\big(\lambda(\sigma)\big)$. Then, we have an algebraic family of rational maps $\{g_\sigma\}_{\sigma\in\Sigma}$ parametrized by a smooth quasiprojective curve $\Sigma$, coming with marked critical points $\omega_i(\sigma)$ and marked periodic points $\alpha_j(\sigma)$.

The family is not trivial and $g_{\sigma_0}$ is not a flexible Lattès example. Assume that the family were stable at $\sigma_0$. Then, according to theorem 2, the number of critical points which are preperiodic to repelling cycles would be locally constant at $\sigma_0$. Thus, the critical orbit relation

$$g_\sigma^{\circ k_r}\big(\omega_r(\sigma)\big) = g_\sigma^{\circ k_r+p_r}\big(\omega_r(\sigma)\big)$$

would hold in a neighborhood of $\sigma_0$ in $\Sigma$, thus for all $\sigma \in \Sigma$ by analytic continuation. As a consequence, for all $\sigma \in \Sigma$, we would have $N_{\mathrm{crit}}(g_\sigma) + N_{\mathrm{ind}}(g_\sigma) = 2d - 2$. According to theorem 1, this would imply that $N_{\mathrm{att}}(g_\sigma) = 0$ for all $\sigma \in \Sigma$. The family would be stable which would contradict theorem 3.

Thus, the family $\{g_\sigma\}_{\sigma\in\Sigma}$ is not stable at $\sigma_0$. According to theorem 2, the maximum period of an indifferent cycle of $g_\sigma$ is not locally bounded at $\sigma_0$. Thus, we can find a parameter $\sigma_1 \in \Sigma$ arbitrarily close to $\sigma_0$ such that $g_{\sigma_1}$ has at least $s+1$ indifferent cycles. □

## 1.4 The inclusion $\mathcal{Z}_d \subset \overline{\mathcal{X}_d^*}$

The proof follows essentially the same lines as the previous one. We begin with a rational map $f_0 \in \mathrm{Rat}_d$ having $2d-2$ indifferent cycles. We must show that arbitrarily close to $f_0$, we can find a postcritically finite rational map $f_1 \in \mathrm{Rat}_d$ which has simple critical points, satisfies $\mathcal{C}(f_1)\cap\mathcal{P}(f_1)=\emptyset$ and is not a flexible Lattès example.

The following lemma implies (by induction) that arbitrarily close to $f_0$, we can find a rational map $f_1 \in \mathrm{Rat}_d$ whose postcritical set contains $2d-2$ distinct repelling cycles. Automatically, the $2d-2$ critical points of $f_1$ have to be simple, strictly preperiodic, with disjoint orbits. In particular, $\mathcal{C}(f_1)\cap\mathcal{P}(f_1)=\emptyset$. In addition, Lattès examples lie in a Zariski closed subset of $\mathrm{Rat}_d$. Since $f_0$ has indifferent cycles, it cannot be a Lattès examples. So, if $f_1$ is sufficiently close to $f_0$, then $f_1$ is not a Lattès example.

**Lemma 2** *Assume $r\geq 0$ and $s\geq 1$ are integers such that $r+s=2d-2$. Assume $f_0\in\mathrm{Rat}_d$ satisfies $N_{\mathrm{crit}}(f_0)=r$, $N_{\mathrm{ind}}(f_0)=s$, the postcritical set of $f_0$ containing $r$ distinct repelling cycles. Then, arbitrarily close to $f_0$, we can find a rational map $f_1$ which satisfies $N_{\mathrm{crit}}(f_1)=r+1$ and $N_{\mathrm{ind}}(f_1)=s-1$, the postcritical set of $f_1$ containing $r+1$ distinct repelling cycles.*

*Proof:* The proof is similar to the proof of lemma 1; we do not give all the details. First , note that since $s\geq 1$, $f_0$ has an indifferent cycle, and thus, is not a Lattès example. Second, we can find a non trivial algebraic family of rational maps $\{g_\sigma\}_{\sigma\in\Sigma}$ parametrized by a smooth quasiprojective curve $\Sigma$ with marked critical points $\omega_1(\sigma), \ldots, \omega_{2d-2}(\sigma)$ and marked periodic points $\alpha_1(\sigma), \ldots, \alpha_{d^M+1}(\sigma)$ such that

- $g_{\sigma_0}=f_0$,
- for $j\in[1,s]$, the periodic points $\alpha_j(\sigma_0)$ belong to distinct indifferent cycles of $f_0$,
- for all $\sigma\in\Sigma$ and all $i\in[1,r]$, the critical point $\omega_i(\sigma)$ is preperiodic to a periodic cycle of $g_\sigma$ and
- for all $\sigma\in\Sigma$ and all $j\in[1,s-1]$, the periodic points $\alpha_j(\sigma)$ are indifferent periodic points of $g_\sigma$.

If the family were stable at $\sigma_0$, the indifferent periodic point $\alpha_s(\sigma_0)$ would be persistently indifferent in a neighborhood of $\sigma_0$ in $\Sigma$, thus in all $\Sigma$ by analytic continuation. The relation $N_{\rm crit}(g_\sigma) + N_{\rm ind}(g_\sigma) = 2d - 2$ would hold throughout $\Sigma$. Thus, for all $\sigma \in \Sigma$, we would have $N_{\rm att}(g_\sigma) = 0$ and the family would be stable. This is not possible since $g_{\sigma_0}$ is not a Lattès example and since the family $\{g_\sigma\}_{\sigma\in\Sigma}$ is not trivial.

It follows that the period of a repelling cycle contained in the postcritical set of $g_\sigma$ is not locally bounded at $\sigma_0$. In addition, for $i \in [1, r]$, the critical points $\omega_i(\sigma_0)$ of $g_{\sigma_0}$ are preperiodic to distinct repelling cycles. Thus, we can find a rational map $g_{\sigma_1}$, with $\sigma_1$ arbitrarily close to $\sigma_0$, such that $g_{\sigma_1}$ has $r + 1$ critical points preperiodic to distinct repelling cycles. □

# 2 Transversality

In this section, we prove proposition 1. Assume $f_0 \in {\rm Rat}_d$ is a strictly postcritically finite rational map which has $2d - 2$ distinct critical points and satisfies $\mathcal{C}(f_0) \cap \mathcal{P}(f_0) = \emptyset$. Assume the analytic germs

$$F : ({\rm Rat}_d, f_0) \to (\mathbb{P}^1)^{2d-2} \quad \text{and} \quad G : ({\rm Rat}_d, f_0) \to (\mathbb{P}^1)^{2d-2}$$

are defined as in the introduction. We want to show that the rank of $D_{f_0}F - D_{f_0}G$ is $2d - 2$.

Since $T_{f_0}{\rm Rat}_d$ has complex dimension $2d+1$, the rank of $D_{f_0}F - D_{f_0}G$ is $2d - 2$ if and only if the kernel has dimension 3.

Note that on $\mathcal{O}(f_0)$, we have $F \equiv G$. Thus, every tangent vector to $\mathcal{O}(f_0)$ at $f_0$ is in the kernel of $D_{f_0}F - D_{f_0}G$. Since $\mathcal{O}(f_0)$ has complex dimension 3, we must therefore show that every vector in the kernel of $D_{f_0}F - D_{f_0}G$ is tangent to $\mathcal{O}(f_0)$.

Thus, it is enough to show that if $\{f_t\}$ is a holomorphic family of rational maps of degree $d \geq 2$, parametrized by a neighborhood of $t = 0$ in $\mathbb{C}$ and if

$$\xi := \frac{df_t}{dt}\Big|_{t=0}$$

belongs to ${\rm Ker}(D_{f_0}F - D_{f_0}G)$, then $\xi$ is a tangent vector to $\mathcal{O}(f_0)$.

## 2.1 The tangent vectors to ${\rm Rat}_d$.

Assume $\{f_t\}$ is a holomorphic family of rational maps of degree $d \geq 2$, parametrized by a neighborhood of $t = 0$ in $\mathbb{C}$, and assume $\xi := \frac{df_t}{dt}\Big|_{t=0}$. Then, for all $z \in \mathbb{P}^1$, the vector $\xi(z)$ is tangent to the curve $t \mapsto f_t(z)$ at $t = 0$. This vector belongs to $T_{f_0(z)}\mathbb{P}^1$.

In other words, if $f \in \mathrm{Rat}_d$ and if $\xi \in T_f\mathrm{Rat}_d$, then for all $z \in \mathbb{P}^1$, $\xi(z) \in T_{f(z)}\mathbb{P}^1$. If $z$ is not a critical point of $f$, then

$$D_z f : T_z\mathbb{P}^1 \to T_{f(z)}\mathbb{P}^1$$

is an isomorphism and we set:

$$\eta_\xi(z) := \big(D_z f\big)^{-1}\big(\xi(z)\big).$$

We obtain a vector field $\eta_\xi$ defined and holomorphic on $\mathbb{P}^1 \setminus \mathcal{C}(f)$.

**Remark** Assume $\{f_t\}$ is a holomorphic family of rational maps, parametrized by a neighborhood of $t = 0$ in $\mathbb{C}$, with $\xi = \frac{df_t}{dt}\Big|_{t=0}$. If $z_0$ is not a critical point of $f_0$, the equation $f_t(z) = f_0(z_0)$ defines implicitly a map $z(t)$, holomorphic in a neighborhood of $t = 0$ with $z(0) = z_0$. The vector $\eta_\xi(z_0) \in T_{z_0}\mathbb{P}^1$ is tangent to the curve $t \mapsto z(t)$ at $t = 0$.

**Remark** If $f$ has simple critical points, then the vector field $\eta_\xi$ has simple poles or removable singularities along $\mathcal{C}(f)$. This can be seen by working in coordinates, using the fact that $f'$ has simple zeroes at points of $\mathcal{C}(f)$.

**Proposition 2** *Assume $f \in \mathrm{Rat}_d$ and assume $\{\phi_t\}$ is an analytic family of Möbius transformations parametrized by a neighborhood of $t = 0$ in $\mathbb{C}$, with $\phi_0 = \mathrm{Id}$. Let $\theta$ be the holomorphic vector field on $\mathbb{P}^1$ defined by $\theta := \frac{d\phi_t}{dt}\Big|_{t=0}$. Then,*

$$\xi = \frac{d(\phi_t \circ f \circ \phi_t^{-1})}{dt}\Big|_{t=0} \quad \Longleftrightarrow \quad \eta_\xi = f^*\theta - \theta.$$

**Remark** The vector field $f^*\theta$ is meromorphic on $\mathbb{P}^1$ and holomorphic on $\mathbb{P}^1 \setminus \mathcal{C}(f)$. If $z$ is not a critical point of $f$, then

$$(f^*\theta)(z) := \big(D_z f\big)^{-1}\big(\theta(f(z))\big).$$

*Proof:* Set $f_t := \phi_t \circ f \circ \phi_t^{-1}$. Note that $f_0 = f$. Differentiating the equality $\phi_t \circ f(z) = f_t \circ \phi_t(z)$ with respect to $t$ and evaluating at $t = 0$, we see that $\xi = \frac{df_t}{dt}\Big|_{t=0}$ if and only if for all $z \in \mathbb{P}^1$, we have

$$\theta\big(f(z)\big) = \xi(z) + D_z f\big(\theta(z)\big). \tag{2.2}$$

If equality (2.2) holds, then applying $(D_z f)^{-1}$ to both sides, we obtain the following equality of vector fields:

$$f^*\theta = \eta_\xi + \theta.$$

Conversely, if $f^*\theta = \eta_\xi + \theta$, then for all $z \in \mathbb{P}^1 \setminus \mathcal{C}(f)$, we have the equality of vectors $f^*\theta(z) = \eta_\xi(z) + \theta(z)$. Applying $D_z f$ to both sides shows that equality (2.2) hold on $\mathbb{P}^1 \setminus \mathcal{C}(f)$, thus on $\mathbb{P}^1$ by analytic continuation. □

**Corollary 1** *The vector $\xi \in T_f \mathrm{Rat}_d$ is tangent to $\mathcal{O}(f)$ if and only if there is a vector field $\theta$, holomorphic on $\mathbb{P}^1$, such that:*

$$\eta_\xi = f^*\theta - \theta.$$

*Proof:* If $\xi \in T_f\mathcal{O}(f)$, there is a holomorphic family of Möbius transformations $\phi_t$ such that $\phi_0 = \mathrm{Id}$ and $\xi = \frac{d(\phi_t \circ f \circ \phi_t^{-1})}{dt}\Big|_{t=0}$. The vector field $\theta := \frac{d\phi_t}{dt}\Big|_{t=0}$ is holomorphic on $\mathbb{P}^1$, and according to the previous proposition, $\eta_\xi = f^*\theta - \theta$.

Conversely, $\eta_\xi = f^*\theta - \theta$ with $\theta$ holomorphic on $\mathbb{P}^1$, we can find a family of Möbius transformations $\phi_t$ such that $\phi_0 = \mathrm{Id}$ and $\frac{d\phi_t}{dt}\Big|_{t=0} = \theta$. According to the previous proposition, the family $\{f_t := \phi_t \circ f \circ \phi_t^{-1}\}$ satisfies $\xi = \frac{df_t}{dt}\Big|_{t=0}$, and since $f_t \in \mathcal{O}(f)$, we have $\xi \in T_f\mathcal{O}(f)$. □

## 2.2 Following critical points and critical values

In the following proposition, $\{f_t\}$ is a holomorphic family of rational maps, parametrized by a neighborhood of $t = 0$ in $\mathbb{C}$. We set $\xi := \frac{df_t}{dt}\Big|_{t=0}$ and let $\eta_\xi$ be the corresponding meromorphic vector field on $\mathbb{P}^1$. We assume that $\omega_0$ is a simple critical point of $f_0$ and that $v_0$ is the corresponding critical value. Finally, we let

$$\omega : (\mathbb{C}, 0) \to (\mathbb{P}^1, \omega_0) \quad \text{and} \quad v : (\mathbb{C}, 0) \to (\mathbb{P}^1, v_0)$$

be holomorphic germs following respectively a critical point of $f_t$ and the corresponding critical value of $f_t$ (in view of the Implicit Function Theorem, since $\omega_0$ is a simple critical point of $f_0$, those germs exist and are unique).

**Proposition 3** *Assume $\theta$ is a vector field, defined and holomorphic in a neighborhood of $v_0$. Then, $f_0^*\theta - \eta_\xi$ is holomorphic in a neighborhood of $\omega_0$ if and only if*

$$\theta(v_0) = \frac{dv(t)}{dt}\Big|_{t=0}.$$

*In that case,*

$$(f_0^*\theta - \eta_\xi)(\omega_0) = \frac{d\omega(t)}{dt}\Big|_{t=0}.$$

*Proof:* Let us first show that there exists a vector field $\theta_0$, defined and holomorphic in a neighborhood of $v_0$, such that $f_0^*\theta_0 - \eta_\xi$ is holomorphic in a neighborhood of $\omega_0$.

Choose a holomorphic family of local biholomorphisms $\phi_t$ sending $v_0$ to $v(t)$, with $\phi_0 = \mathrm{Id}$. Note that there is a holomorphic family of local biholomorphisms $\psi_t$ sending $\omega_0$ to $\omega(t)$, with $\psi_0 = \mathrm{Id}$ and

$$\phi_t \circ f_0 = f_t \circ \psi_t.$$

Note that

$$\theta_0 := \frac{d\phi_t}{dt}\Big|_{t=0}$$

is defined and holomorphic in a neighborhood of $v_0$ and

$$\zeta_0 := \frac{d\psi_t}{dt}\Big|_{t=0}$$

is defined and holomorphic in a neighborhood of $\omega_0$. Differentiating the equality $\phi_t \circ f_0(z) = f_t \circ \psi_t(z)$ with respect to $t$ and evaluating at $t = 0$ yields the following equality of vectors:

$$\theta_0\big(f_0(z)\big) = \xi(z) + D_z f_0\big(\zeta_0(z)\big).$$

Applying $(D_z f_0)^{-1}$ to both sides of the equation yields the equality of vector fields:

$$f_0^*\theta_0 = \eta_\xi + \zeta_0.$$

As required, $f_0^*\theta_0 - \eta_\xi = \zeta_0$ is holomorphic in a neighborhood of $\omega_0$. In addition,

$$\theta_0(v_0) = \frac{dv(t)}{dt}\Big|_{t=0}$$

and

$$(f_0^*\theta_0 - \eta_\xi)(\omega_0) = \zeta_0(\omega_0) = \frac{d\omega(t)}{dt}\Big|_{t=0}.$$

Let us now assume that $\theta$ is a vector field, defined and holomorphic in a neighborhood of $v_0$. If $\theta$ agrees with $\theta_0$ at $v_0$, then $\theta - \theta_0$. Consequently, the pullback $f_0^*(\theta - \theta_0)$ is holomorphic vanishes at $\omega_0$. It follows that $f_0^*\theta - \eta_\xi$ is holomorphic and agrees with $f_0^*\theta_0 - \eta_\xi$ at $\omega_0$.

Finally, if $\theta$ does not agree with $\theta_0$ at $v_0$, then $\theta - \theta_0$ is not vanishing at $v_0$. Thus, the vector field $f_0^*(\theta - \theta_0)$ has a pole at $\omega_0$. As a consequence, $f_0^*\theta - \eta_\xi$ has a pole at $\omega_0$. □

## 2.3 Proof of proposition 1

Let us recall the hypotheses. The rational map $f_0 \in \mathrm{Rat}_d$ is postcritically finite. It has $2d-2$ distinct critical points $\omega_j(f_0)$ and $\mathcal{C}(f_0) \cap \mathcal{P}(f_0) = \emptyset$. In particular, there are integers $\ell_j \geq 1$ such that

$$\alpha_j(f_0) := f_0^{\circ \ell_j}\big(\omega_j(f_0)\big)$$

are periodic points of $f_0$. The critical points are simple and the periodic points are repelling. By the Implicit Function Theorem, there are

- analytic germs $\omega_j : (\mathrm{Rat}_d, f_0) \to \mathbb{P}^1$ following the critical points of $f$ and
- analytic germs $\alpha_j : (\mathrm{Rat}_d, f_0) \to \mathbb{P}^1$ following periodic points of $f$.

The analytic germs

$$F : (\mathrm{Rat}_d, f_0) \to (\mathbb{P}^1)^{2d-2} \quad \text{and} \quad G : (\mathrm{Rat}_d, f_0) \to (\mathbb{P}^1)^{2d-2}$$

are defined by

$$F{:=}\begin{pmatrix} F_1 \\ \vdots \\ F_{2d-2} \end{pmatrix} \quad \text{with} \quad F_j(f){:=}f^{\circ \ell_j}\big(\omega_j(f)\big) \quad \text{and} \quad G{:=}\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{2d-2} \end{pmatrix}.$$

Assume $\{f_t\}$ is a holomorphic family of rational maps of degree $d \geq 2$, parametrized by a neighborhood of $t = 0$ in $\mathbb{C}$. Suppose

$$\xi := \frac{df_t}{dt}\Big|_{t=0} \in \mathrm{Ker}(D_{f_0}F - D_{f_0}G).$$

Let $\eta_\xi$ be the associated holomorphic vector field on $\mathbb{P}^1 \setminus \mathcal{C}(f_0)$. We want to show that $\xi \in T_f\mathcal{O}(f)$, and for this purpose, we will show that there is a globally holomorphic vector field $\theta$ such that

$$\eta_\xi = f_0^*\theta - \theta.$$

First, we define $\theta$ on $\mathcal{P}_{f_0}$ as follows. If $z_0 \in \mathcal{P}_{f_0}$, then

- $z_0$ is not a precritical point of $f_0$ and
- $z_0$ is preperiodic to a repelling periodic point of $f_0$.

It follows from the Implicit Function Theorem that there is a holomorphic germ $z : (\mathbb{C}, 0) \to (\mathbb{P}^1, z_0)$ such that $z(t)$ is a preperiodic point of $f_t$ and this germ is unique. Indeed, there are integers $\ell \geq 0$ and $p \geq 1$ such that

$$f_0^{\circ \ell}(z_0) = \alpha_0 \quad \text{with} \quad f_0^{\circ p}(\alpha_0) = \alpha_0.$$

Since $\alpha_0$ is a repelling periodic point of $f_0$, $(f_0^{\circ p})'(\alpha_0) \neq 1$ and since $z_0$ is not precritical, $(f_0^{\circ \ell})'(z_0) \neq 0$ whence

$$(f_0^{\circ \ell + p})'(z_0) = (f_0^{\circ p})'(\alpha_0) \cdot (f_0^{\circ \ell})'(z_0) \neq (f_0^{\circ \ell})'(z_0).$$

We set

$$\theta(z_0):=\frac{dz(t)}{dt}\Big|_{t=0}.$$

Second, we have the following equality, valid on the postcritical set $\mathcal{P}(f_0)$:

$$f_0^*\theta = \theta + \eta_\xi.$$

Indeed, assume that $z_0 \in \mathcal{P}(f_0)$ and $w_0 = f_0(z_0) \in \mathcal{P}(f_0)$. Then,

$$\theta(z_0) = \frac{dz(t)}{dt}\Big|_{t=0} \quad \text{and} \quad \theta(w_0) = \frac{dw(t)}{dt}\Big|_{t=0} \quad \text{with} \quad w(t) = f_t\big(z(t)\big).$$

Evaluating the derivative of the equation $w(t) = f_t\big(z(t)\big)$ at $t = 0$ yields:

$$\theta(w_0) = D_{z_0}f_0\big(\theta(z_0)\big) + \xi(z_0).$$

Applying $(D_{z_0}f_0)^{-1}$ to both sides yields the required equation.

Third, we may reformulate the condition that $\xi$ belongs to the kernel of $D_{f_0}F - D_{f_0}G$ as follows. Let $v_j : (\mathrm{Rat}_d, f_0) \to \mathbb{P}^1$ be the analytic germs following the critical values of $f$:

$$v_j(f):=f\big(\omega_j(f)\big).$$

**Lemma 3** *If $\xi$ belongs to the kernel of $D_{f_0}F - D_{f_0}G$, then for all integer $j \in \{1, \ldots, 2d-2\}$, we have*

$$\theta\big(v_j(f_0)\big) = \frac{dv_j(f_t)}{dt}\Big|_{t=0}.$$

*Proof:* To lighten notation, let us leave aside the index $j$. Choose a critical point $\omega(f_0)$ and let $v(f_0)$ be the corresponding critical value. We have $f_0^{\circ\ell}\big(\omega(f_0)\big) = \alpha(f_0)$ for some integer $\ell \geq 1$ and if $\xi$ belongs to the kernel of $D_{f_0}F - D_{f_0}G$, then

$$\frac{df_t^{\circ\ell}\big(\omega(f_t)\big)}{dt}\Big|_{t=0} = \frac{d\alpha(f_t)}{dt}\Big|_{t=0}.$$

In addition, we have a point $z_t$ depending holomorphically on $t$, such that

$$z_0 = v(f_0) \quad \text{and} \quad f_t^{\circ\ell-1}(z_t) = \alpha(f_t).$$

Using

$$\frac{df_t^{\circ\ell-1}\big(v(f_t)\big)}{dt}\Big|_{t=0} = \frac{df_t^{\circ\ell-1}(z_t)}{dt}\Big|_{t=0},$$

we easily derive

$$D_{v(f_0)}f_0^{\circ\ell-1}\left(\frac{dv(f_t)}{dt}\Big|_{t=0}\right) = D_{z_0}f_0^{\circ\ell-1}\left(\frac{dz_t}{dt}\Big|_{t=0}\right).$$

Since $v(f_0) = z_0$ and since there are no critical points in the forward orbit of $v(f_0)$, the map $D_{v(f_0)} f_0^{\circ \ell - 1} = D_{z_0} f_0^{\circ \ell - 1}$ is invertible. So:

$$\frac{dv(f_t)}{dt}\Big|_{t=0} = \frac{dz_t}{dt}\Big|_{t=0} = \theta\big(v(f_0)\big). \qquad \square$$

We now come to the heart of the argument.

**Lemma 4** *The vector field $\theta$ is the restriction to $\mathcal{P}(f_0)$ of a vector field which is holomorphic on $\mathbb{P}^1$.*

*Proof:* A rational map whose postcritical set contains only two points is conjugate to $z \mapsto z^{\pm d}$. It follows that $\mathcal{P}(f_0)$ which contains at three points.

Choose three distinct points $z_1$, $z_2$, $z_3$ in $\mathcal{P}(f_0)$ and let $\tilde{\theta}$ be the unique holomorphic vector field on $\mathbb{P}^1$ which coincides with $\theta$ at $z_1$, $z_2$ and $z_3$. We must show that $\tilde{\theta}$ coincides with $\theta$ at any other point of $\mathcal{P}(f_0)$. If there are only three points in $\mathcal{P}(f_0)$, we are done. Otherwise, let $z_4$ be a point in $\mathcal{P}(f_0) \setminus \{z_1, z_2, z_3\}$. Consider a quadratic differential $q$ which is holomorphic on $\mathbb{P}^1 \setminus \{z_1, z_2, z_3, z_4\}$ and has simple poles at $z_1$, $z_2$, $z_3$ and $z_4$. Then, the 1-form $q \otimes \tilde{\theta}$ is holomorphic on $\mathbb{P}^1$ and has a most simple poles on $\{z_1, z_2, z_3, z_4\}$. The sum of residues of a meromorphic 1-form on $\mathbb{P}^1$ is 0. It follows that $\theta$ and $\tilde{\theta}$ coincide at $z_4$ if and only if

$$0 = \sum_{z \in \{z_1, z_2, z_3, z_4\}} \mathrm{Res}(q \otimes \tilde{\theta}, z) = \sum_{z \in \{z_1, z_2, z_3, z_4\}} \mathrm{Res}(q \otimes \theta, z) = 0.^2$$

Denote by $\mathcal{Q}(f_0)$ the set of quadratic differentials which are holomorphic on $\mathbb{P}^1 \setminus \mathcal{P}(f_0)$ and have at most simple poles on $\mathcal{P}(f_0)$. By the above discussion, in order to prove the lemma, it is enough to show that for all $q \in \mathcal{Q}(f_0)$, we have:

$$\langle q, \theta \rangle := 2i\pi \sum_{z \in \mathcal{P}(f_0)} \mathrm{Res}(q \otimes \theta, z) = 0.$$

For this purpose, choose a vector field which is $\mathcal{C}^\infty$ on $\mathbb{P}^1$, holomorphic in a neighborhood of $\mathcal{P}(f_0)$ and which coincides with $\theta$ on $\mathcal{P}(f_0)$. To lighten notation, let us still denote it by $\theta$. Without loss of generality, we may assume that $U$ is a finite union of disjoint disks. Then, according to the residue theorem and to Stokes' theorem, for all $q \in \mathcal{Q}(f_0)$, we have:

$$\langle q, \theta \rangle = \int_{\partial U} q \otimes \theta = -\int_{\mathbb{P}^1 \setminus \overline{U}} q \otimes \bar{\partial}\theta = -\int_{\mathbb{P}^1} q \otimes \bar{\partial}\theta.$$

[2] To define the residue, we choose any holomorphic extension of $\theta$ to a neighborhood of $\mathcal{P}(f_0)$ (compare with [HS94] for example).

For the last equality, we used $\partial\theta = 0$ in $\overline{U}$. If $q \in \mathcal{Q}(f_0)$, we therefore have:

$$\big\langle (f_0)_* q, \theta \big\rangle = -\int_{\mathbb{P}^1} (f_0)_* q \otimes \bar\partial\theta = -\int_{\mathbb{P}^1} q \otimes f_0^*\bar\partial\theta = -\int_{\mathbb{P}^1} q \otimes \bar\partial\big(f_0^*\theta - \eta_\xi\big)$$

since $\bar\partial\eta_\xi \equiv 0$. However according to proposition 3, $f_0^*\theta - \eta_\xi$ is a $\mathcal{C}^\infty$ vector field on $\mathbb{P}^1$. Since $f_0^*\theta - \eta_\xi$ coincides with $\theta$ on $\mathcal{P}(f_0)$, we see that:

$$\forall q \in \mathcal{Q}(f_0), \quad \big\langle (f_0)_* q, \theta \big\rangle = \langle q, \theta \rangle .$$

Consider the linear map $\nabla : \mathcal{Q}(f_0) \to \mathcal{Q}(f_0)$ defined by $\nabla q = (f_0)_* q - q$. We have just seen that for all $q \in \mathcal{Q}(f_0)$, we have $\langle \nabla q, \theta \rangle = 0$.

We will now use the fact that $f_0$ is not a flexible Lattès example. Under this hypothesis $\nabla : \mathcal{Q}(f_0) \to \mathcal{Q}(f_0)$ is injective (see [DH93]), thus surjective. It follows that for all $q \in \nabla\big(\mathcal{Q}(f_0)\big) = \mathcal{Q}(f_0)$, we have $\langle q, \theta \rangle = 0$. □

We still denote by $\theta$ the holomorphic extension to $\mathbb{P}^1$. Note that the vector field $f_0^*\theta$ is defined and holomorphic on $\mathbb{P}^1 \setminus \mathcal{C}(f_0)$. According to lemma 4, for $j \in \{1, \ldots, 2d-2\}$, we have:

$$\theta\big(v_j(f_0)\big) = \frac{df_t\big(\omega_j(t)\big)}{dt}\Big|_{t=0}.$$

According to proposition 3, the vector field $f_0^*\theta - \eta_\xi$ is holomorphic in a neighborhood of $\mathcal{C}(f_0)$, thus on $\mathbb{P}^1$. It coincides with $\theta$ on $\mathcal{P}(f_0)$ which contains at least three points. Two holomorphic vector fields on $\mathbb{P}^1$ which coincide at three points are equal: indeed, in any affine coordinate, their expressions are polynomials of degree $\le 2$. As a consequence, we have the required equality:

$$f_0^*\theta - \eta_\xi = \theta.$$

# 3 The bifurcation measure

We will now prove that $\mathcal{X}_d^* \subset \mathrm{Supp}(\mu_{\mathrm{bif}})$. This will complete the proof of our main theorem.

The idea is the following. Assume $f_0 \in \mathrm{Rat}_d$ is a strictly postcritically finite rational map which is not a flexible Lattès example, has simple critical points and satisfies $\mathcal{C}(f_0) \cap \mathcal{P}(f_0) = \emptyset$. It is enough to show that there is a $2d-2$-dimensional complex manifold $\Sigma$ transverse to $\mathcal{O}(f_0)$ at $f_0$, and a basis of neighborhoods $\Sigma_n$ of $f_0$ in $\Sigma$, such that

$$\int_{\Sigma_n} (T_{\mathrm{bif}})^{\wedge(2d-2)} > 0.$$

Let $\omega_j : (\mathrm{Rat}_d, f_0) \to \mathbb{P}^1$ be analytic germs following the critical points of $f$. There are integers $\ell_j \geq 1$ such that $\alpha_j(f_0){:=}f_0^{\circ \ell_j}\big(\omega_j(f_0)\big)$ are repelling periodic points of $f_0$ of periods $p_j$. Let $V_j$ be simply connected neighborhoods of $\alpha_j(f_0)$ such that $\overline{V}_j \subset W_j{:=}f_0^{\circ p_j}(V_j)$ and such that $f_0^{\circ p_j} : V_j \to W_j$ are isomorphisms.

Denote by $M$ the least common multiple of the periods $p_j$ and define $\Sigma_n \subset \Sigma$ as the connected component containing $f_0$ of the set

$$\big\{f \in \Sigma\ ;\ \forall j \in \{1, \ldots, 2d-2\},\ f^{\circ \ell_j + nM}\big(\omega_j(f)\big) \in V_j\big\}.$$

Our proof essentially consists in proving that

$$\int_{\Sigma_n} (T_{\mathrm{bif}})^{\wedge(2d-2)} \underset{n\to+\infty}{\sim} \frac{(2d-2)!}{d^{nM(2d-2)+\sum \ell_j}} \cdot \prod_{j=1}^{2d-2} \mu_{f_0}(V_j)$$

where $\mu_{f_0}$ is the equilibrium measure of $f_0$. Since the disks $V_j$ intersect the Julia set of $f_0$, we have $\mu_{f_0}(V_j) > 0$ for all $j$, and so, for $n$ large enough,

$$\int_{\Sigma_n} (T_{\mathrm{bif}})^{\wedge(2d-2)} > 0.$$

## 3.1 Another definition of the bifurcation current

We will use a second definition of the bifurcation current $T_{\mathrm{bif}}$ due to DeMarco [DeM01] (see [DeM03] or [BB07] for the equivalence of the two definitions). The current $T_{\mathrm{bif}}$ can be defined locally by considering the behavior of the critical orbits as follows.

We still denote by $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ the canonical projection. Assume $f_0 \in \mathrm{Rat}_d$ has only simple critical points and let $U$ be a sufficiently small neighborhood of $f_0$ so that there are:

- holomorphic functions $\big\{\omega_j : U \to \mathbb{P}^1\big\}_{j\in\{1,\ldots,2d-2\}}$ following the critical points of $f$,
- holomorphic functions $\big\{\tilde{\omega}_j : U \to \mathbb{C}^2 \setminus \{0\}\big\}_{j\in\{1,\ldots,2d-2\}}$ such that $\pi \circ \tilde{\omega}_j = \omega_j$ and
- a family $\{P_f\}_{f\in U}$ of non-degenerate homogeneous polynomials of degree $d$ depending analytically on $f \in U$ such that $\pi \circ P_f = f \circ \pi$.

The map $G : U \times \mathbb{C}^2 \to \mathbb{R}$ defined by

$$G(f, z){:=} \lim_{n\to+\infty} \frac{1}{d^n} \log\big\|P_f^{\circ n}(z)\big\|$$

is plurisubharmonic on $U \times \mathbb{C}^2$.

For $j \in \{1, \ldots, 2d-2\}$, define $G_j : U \to \mathbb{R}$ by:

$$G_j(f) := G\big(f, \tilde{\omega}_j(f)\big).$$

DeMarco [DeM03] proved that

- if $P_f$ is chosen so that $\mathrm{Res}(P_f) = 1$,[3] and
- if $\tilde{\omega}_j$ are chosen so that $\det(D_z P_f) = \prod_{j=1}^{2d-2} \big(z \wedge \tilde{\omega}_j(f)\big)$,

then

$$\sum_{j=1}^{2d-2} G_j(f) = L(f) + \log d.$$

As a consequence,

$$T_{\mathrm{bif}}\big|_U = \sum_{j=1}^{2d-2} dd^c G_j.$$

## 3.2 A parametrized family of rational maps

From now on, we assume that $f_0 \in \mathrm{Rat}_d$ is a strictly postcritically finite rational map which is not a flexible Lattès example, has simple critical points and satisfies $\mathcal{C}(f_0) \cap \mathcal{P}(f_0) = \emptyset$.

Let $\omega_j : (\mathrm{Rat}_d, f_0) \to \mathbb{P}^1$ be analytic germs following the critical points of $f$ and for $n \geq 1$, let $v_j^n : (\mathrm{Rat}_d, f_0) \to \mathbb{P}^1$ be the analytic germs defined by

$$v_j^n(f) = f^{\circ n}\big(\omega_j(f)\big).$$

There are integers $\ell_j \geq 1$ and $p_j \geq 1$ such that

$$\alpha_j(f_0) := v_j^{\ell_j}(f_0)$$

are repelling periodic points of $f_0$ of periods $p_j$. Let $\alpha_j : (\mathrm{Rat}_d, f_0) \to \mathbb{P}^1$ be analytic germs following those periodic points and let

$$\phi_{j,f} : \big(\mathbb{P}^1, \alpha_j(f)\big) \to (\mathbb{C}, 0)$$

be linearizing maps depending analytically on $f$, i.e.

$$\phi_{j,f} \circ f^{\circ p_j} = \mu_j(f) \cdot \phi_{j,f}$$

with

[3] If $z = (z_1, z_2)$ and $w = (w_1, w_2)$, then $z \wedge w = z_1 w_2 - z_2 w_1$ and if $F(z) = \Big(\prod_i z \wedge a_i, \prod_j z \wedge b_j\Big)$, then $\mathrm{Res}(F) = \prod_{i,j} a_i \wedge b_j$.

- $\phi_{j,f}\big(\alpha_j(f)\big) = 0$,
- $\phi_{j,f}$ is locally injective at $\alpha_j(f)$ and
- $\mu_j(f)$ is the multiplier of $f^{\circ p_j}$ at $\alpha_j(f)$.

Consider the analytic map $\tau : (\mathrm{Rat}_d, f_0) \to (\mathbb{C}^{2d-2}, 0)$ defined by

$$\tau{:=}(\tau_1, \dots, \tau_{2d-2}) \quad \text{with} \quad \tau_j(f){:=}\phi_{j,f} \circ v_j^{\ell_j}(f).$$

**Lemma 5** *The rank of $D_{f_0}\tau : T_{f_0}\mathrm{Rat}_d \to \mathbb{C}^{2d-2}$ is $2d-2$. The kernel is $T_{f_0}\mathcal{O}(f_0)$.*

*Proof:* Differentiating the equality $\phi_{j,f}\big(\alpha_j(f)\big) = 0$ with respect to $f$, we have

$$\frac{d\phi_{j,f}}{df}\Big|_{\left(f_0,\alpha_j(f_0)\right)} + D_{\alpha_j(f_0)}\phi_{j,f_0} \circ D_{f_0}\alpha_j = 0.$$

As a consequence, we have

$$\begin{aligned} D_{f_0}\tau_j &= \frac{d\phi_{j,f}}{df}\Big|_{\left(f_0,\alpha_j(f_0)\right)} + D_{\alpha_j(f_0)}\phi_{j,f_0} \circ D_{f_0}v_j^{\ell_j} \\ &= D_{\alpha_j(f_0)}\phi_{j,f_0} \circ \big(D_{f_0}v_j^{\ell_j} - D_{f_0}\alpha_j\big). \end{aligned}$$

Thus, $\xi \in T_{f_0}\mathrm{Rat}_d$ belongs to the kernel of $D_{f_0}\tau$ if and only if it belongs to the kernel of $D_{f_0}v_j^{\ell_j} - D_{f_0}\alpha_j$ for all $j \in \{1, \dots, 2d-2\}$, i.e. if and only if it belongs to the kernel of $D_{f_0}F - D_{f_0}G$. According to proposition 1, such a $\xi$ belongs to $T_{f_0}\mathcal{O}(f_0)$. Thus, the kernel $D_{f_0}\tau$ has dimension 3 and the image has dimension $2d-2$ (the dimension of $T_{f_0}\mathrm{Rat}_d$ is $2d+1$). □

**Corollary 2** *If $r > 0$ is sufficiently small and if $\Delta{:=}D(0,r)$, then there is a section*

$$\chi : \Delta^{2d-2} \to \mathrm{Rat}_d \quad \textit{with} \quad \chi(0) = f_0 \quad \textit{and} \quad \tau \circ \chi = \mathrm{Id}.$$

From now on, we assume that such a section $\chi : \Delta^{2d-2} \to \mathrm{Rat}_d$ has been chosen and we let $\Sigma \subset \mathrm{Rat}_d$ be the complex manifold of dimension $2d-2$ defined by

$$\Sigma{:=}\chi(\Delta^{2d-2}).$$

Then, $\Sigma$ is an analytic family of rational maps parametrized by the positions of the points $f^{\circ \ell_j}\big(\omega_j(f)\big)$ in the linearizing coordinates $\phi_{j,f}$.

## 3.3 The proof

Let us now assume that $V_j \subset \mathbb{P}^1$ is a sufficiently small neighborhood of $\alpha_j(f_0)$ so that

- $\phi_{j,f_0}$ is defined and univalent on a neighborhood of $\overline{V}_j$ and
- $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ has an analytic section $\sigma_j : V_j \to \mathbb{C}^2 \setminus \{0\}$ defined on $V_j$.

Decreasing $r$ if necessary, we may assume that $\Delta$ is relatively compact in $\phi_{j,f_0}(V_j)$ for all $j \in \{1, \dots, 2d-2\}$.

Let us now denote by $M$ the least common multiple of the periods $p_j$ and set $m_j = M/p_j$. For $n \geq 0$, we define $\chi_n : \Delta^{2d-2} \to \Sigma$ by

$$\chi_n(x_1, \dots, x_{2d-2}) := \chi\left(\frac{x_1}{\big(\mu_1(f_0)\big)^{nm_1}}, \dots, \frac{x_{2d-2}}{\big(\mu_{2d-2}(f_0)\big)^{nm_{2d-2}}}\right)$$

and set

$$\Sigma_n := \chi_n(\Delta^{2d-2}) \subset \Sigma.$$

**Lemma 6** *As $n \to +\infty$, the sequences of functions*

$$\left(\frac{\mu_j \circ \chi_n}{\mu_j(f_0)}\right)^n : \Delta^{2d-2} \to \mathbb{C}$$

*converge uniformly to* 1.

*Proof:* Since the map $\chi : \Delta^{2d-2} \to \mathrm{Rat}_d$ and the germs $\mu_j : (\mathrm{Rat}_d, f_0) \to \mathbb{C}$ are analytic, there is a constant $C > 0$ such that

$$\left\|\frac{\mu_j \circ \chi_n}{\mu_j(f_0)} - 1\right\|_\infty \leq C\lambda^n$$

with

$$\lambda := \min_{j \in \{1,\dots,2d-2\}} \left\{\frac{1}{\big|\mu_j(f_0)\big|^{m_j}}\right\}.$$

It follows easily that

$$\left\|\left(\frac{\mu_j \circ \chi_n}{\mu_j(f_0)}\right)^n - 1\right\|_\infty \underset{n \to +\infty}{\longrightarrow} 0.$$

□

**Corollary 3** *For $n$ sufficiently large and all $j \in \{1, \dots, 2d-2\}$, we have*

$$v_j^{\ell_j + nM}(\Sigma_n) \subset V_j.$$

*Proof:* Since $\Delta = D(0,r)$ is relatively compact in $\phi_{j,f_0}(V_j)$, there is a neighborhood $U$ of $f_0$ in $\mathrm{Rat}_d$ and a $\kappa > 1$ such that for all $f \in U$ and all $j \in \{1,\ldots,2d-2\}$, $D(0,\kappa r) \subset \phi_{j,f}(V_j)$. Let us choose $n$ sufficiently large so that $\chi_n(\Delta^{2d-2}) \subset U$ and so that

$$\forall f \in \chi_n(\Delta^{2d-2}),\ \forall j \in \{1,\ldots,2d-2\}, \quad \left|\frac{\mu_j(f)}{\mu_j(f_0)}\right|^{nm_j} < \kappa.$$

If $f \in \Sigma_n$, then $f = \chi_n(x_1,\ldots,x_{2d-2})$ with $(x_1,\ldots,x_{2d-2}) \in \Delta^{2d-2}$,

$$\phi_{j,f}\big(v_j^{\ell_j}(f)\big) = \frac{x_j}{\big(\mu_j(f_0)\big)^{nm_j}} \quad \text{and} \quad \big(\mu_j(f)\big)^{nm_j} \cdot \frac{x_j}{\big(\mu_j(f_0)\big)^{nm_j}} \in \phi_{j,f}(V_j).$$

In addition, $\phi_{j,f} \circ f^{\circ p_j} = \mu_j(f) \cdot \phi_{j,f}$. Thus,

$$v_j^{\ell_j+nM}(f) = v_j^{\ell_j+nm_jp_j}(f) = \phi_{j,f}^{-1}\left(\big(\mu_j(f)\big)^{nm_j} \cdot \frac{x_j}{\big(\mu_j(f_0)\big)^{nm_j}}\right) \in V_j.$$

□

From now on, we assume that $n$ is sufficiently large so that for all integer $j \in \{1,\ldots,2d-2\}$, we have

$$v_j^{\ell_j+nM}(\Sigma_n) \subset V_j.$$

Define plurisubharmonic functions $G_j^n : \Delta^{2d-2} \to \mathbb{R}$ by

$$G_j^n(\mathbf{x}) := G\big(\chi_n(\mathbf{x}), \sigma_j \circ v_j^{\ell_j+nM} \circ \chi_n(\mathbf{x})\big).$$

**Lemma 7** *For $j \in \{1,\ldots,2d-2\}$, we have:*

$$dd^c G_j^n = d^{\ell_j+nM} \cdot \chi_n^*(dd^c G_j).$$

*Proof:* Assume $f \in \Sigma_n$ and $j \in \{1,\ldots,2d-2\}$. Note that

$$\pi \circ P_f^{\circ \ell_j+nM}\big(\tilde\omega_j(f)\big) = f^{\circ \ell_j+nM}\big(\omega_j(f)\big) = v_j^{\ell_j+nM}(f) = \pi \circ \sigma_j \circ v_j^{\ell_j+nM}(f).$$

Thus,

$$P_f^{\circ n}\big(\tilde\omega_j(f)\big) = \lambda_j^n(f) \cdot \sigma \circ v_j^{\ell_j+nM}(f)$$

for some holomorphic functions $\lambda_j^n : \Sigma_n \to \mathbb{C}^*$. Note that

$$G\big(f, P_f(z)\big) = d \cdot G(f,z) \quad \text{and} \quad \forall \lambda \in \mathbb{C}^*, \quad G(f,\lambda z) = G(f,z) + \log|\lambda|.$$

Thus, if $f = \chi_n(\mathbf{x})$ with $\mathbf{x} \in \Delta^{2d-2}$, then

$$\begin{aligned} G_j^n(\mathbf{x}) &= G\big(f, P_f^{\circ \ell_j + nM}\big(\tilde{\omega}_j(f)\big)\big) - \log\big|\lambda_j^n(f)\big| \\ &= d^{\ell_j + nM} G\big(f, \tilde{\omega}_j(f)\big) - \log\big|\lambda_j^n(f)\big| \\ &= d^{\ell_j + nM} G_j(f) - \log\big|\lambda_j^n(f)\big| \\ &= d^{\ell_j + nM} G_j \circ \chi_n(\mathbf{x}) - \log\big|\lambda_j^n \circ \chi_n(\mathbf{x})\big|. \end{aligned}$$

Since $\log|\lambda_j^n \circ \chi_n|$ is pluriharmonic on $\Delta^{2d-2}$, we have

$$dd^c G_j^n = d^{\ell_j + nM} \cdot \chi_n^*(dd^c G_j). \qquad \square$$

It follows that

$$\chi_n^*(T_{\text{bif}}) = \sum_{j=1}^{2d-2} \chi_n^*(dd^c G_j) = \frac{1}{d^{nM}} \sum_{j=1}^{2d-2} \frac{1}{d^{\ell_j}} dd^c G_j^n.$$

**Lemma 8** *We have*

$$\lim_{n \to +\infty} \int_{\Delta^{2d-2}} \left( \sum_{j=1}^{2d-2} \frac{1}{d^{\ell_j}} dd^c G_j^n \right)^{\wedge(2d-2)} = \frac{(2d-2)!}{d^{\sum \ell_j}} \cdot \prod_{j=1}^{2d-2} \mu_{f_0}\left(\phi_{j,f_0}^{-1}(\Delta)\right).$$

*Proof:* If $f = \chi_n(x_1, \ldots, x_{2d-2})$, then

$$v_j^{\ell_j + nM}(f) = \phi_{j,f}^{-1}\left( \left( \frac{\mu_j(f)}{\mu_j(f_0)} \right)^{nm_j} \cdot x_j \right).$$

Thus, as $n \to +\infty$, the sequence of functions $v_j^{\ell_j + nM} \circ \chi_n : \Delta^{2d-2} \to V_j$ converge uniformly to

$$(x_1, \ldots, x_{2d-2}) \mapsto \phi_{j,f_0}^{-1}(x_j).$$

As a consequence, as $n \to +\infty$, the sequences of functions $G_j^n : \Delta^{2d-2} \to \mathbb{R}$ converge uniformly to

$$G_j^\infty : (x_1, \ldots, x_{2d-2}) \mapsto G\big(f_0, \sigma_j \circ \phi_{j,f_0}^{-1}(x_j)\big).$$

Thanks to the uniform convergence of the potentials, we can write:

$$\lim_{n \to +\infty} \int_{\Delta^{2d-2}} \left( \sum_{j=1}^{2d-2} \frac{1}{d^{\ell_j}} dd^c G_j^n \right)^{\wedge(2d-2)} = \int_{\Delta^{2d-2}} \left( \sum_{j=1}^{2d-2} \frac{1}{d^{\ell_j}} dd^c G_j^\infty \right)^{\wedge(2d-2)}.$$

Note that $G_j^\infty$ only depends on the $j$-th coordinate. It follows that

$$\left(\sum_{j=1}^{2d-2} \frac{1}{d^{\ell_j}} dd^c G_j^\infty\right)^{\wedge(2d-2)} = \frac{(2d-2)!}{d^{\sum \ell_j}} \cdot \bigwedge_{j=1}^{2d-2} dd^c G_j^\infty.$$

In addition, $G_j^\infty(x_1, \ldots, x_{2d-2}) = g_j \circ \phi_{j,f_0}^{-1}(x_j)$ with $g_j : V_j \to \mathbb{R}$ the subharmonic function defined by $g_j(z) = G\big(f_0, \sigma_j(z)\big)$. We have

$$dd^c g_j = \mu_{f_0}\big|_{V_j}.$$

Therefore, according to Fubini's theorem,

$$\begin{aligned}\int_{\Delta^{2d-2}} \left(\bigwedge_{j=1}^{2d-2} dd^c G_j^\infty\right) &= \prod_{j=1}^{2d-2} \left(\int_\Delta dd^c \left(g_j \circ \phi_{j,f_0}^{-1}\right)\right) \\ &= \prod_{j=1}^{2d-2} \left(\int_{\phi_{j,f_0}^{-1}(\Delta)} dd^c g_j\right) \\ &= \prod_{j=1}^{2d-2} \mu_{f_0}\left(\phi_{j,f_0}^{-1}(\Delta)\right). \qquad \square\end{aligned}$$

We can now complete the proof. Since the points $\alpha_j(f_0)$ are repelling periodic points of $f_0$, they are in the support of the equilibrium measure $\mu_{f_0}$. Thus,

$$\mu_{f_0}\left(\phi_{j,f_0}^{-1}(\Delta)\right) > 0.$$

As a consequence,

$$\lim_{n\to+\infty} d^{nM(2d-2)} \cdot \left(\int_{\Sigma_n} T_{\mathrm{bif}}{}^{\wedge(2d-2)}\right) = \frac{(2d-2)!}{d^{\sum \ell_j}} \cdot \prod_{j=1}^{2d-2} \mu_{f_0}\left(\phi_{j,f_0}^{-1}(\Delta)\right) > 0.$$

Thus, $f_0$ is in the support of $T_{\mathrm{bif}}{}^{\wedge(2d-2)}$ and the conjugacy class of $f_0$ is in the support of $\mu_{\mathrm{bif}}$.

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