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Reverse Bubbling and Nonuniqueness in the Harmonic Map Flow

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In this paper, we construct a new type of singularity which may occur in weak solutions of the harmonic map flow for two-dimensional domains. This "reverse bubbling" singularity may occur spontaneously, and enables us to construct solutions to the harmonic map heat equation which differ from the standard Struwe solution, despite agreeing for an arbitrarily long initial time interval.

1 Introduction

We consider the harmonic map heat flow for maps v from the 2-disc D to the 2-sphere S^2. This flow, introduced by Eells and Sampson [7] is the L^2-gradient flow for the harmonic map energy

$$E(v) = \frac{1}{2}\int_D |\nabla v|^2, \tag{1.1}$$

where we see v as a map into $S^2 \hookrightarrow \mathbb{R}^3$, and such a flow $u : D \times [0,\infty) \to S^2 \hookrightarrow \mathbb{R}^3$ obeys the equation

$$\frac{\partial u}{\partial t} = \Delta u + u|\nabla u|^2. \tag{1.2}$$

The basic theory for the heat flow in cases such as this where the domain is two-dimensional (the dimension for which the energy is conformally invariant) and the

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target need not be nonpositively curved, was developed by Struwe in [12] and we assume a familiarity with this work in the proofs (together with the geometric picture behind it, without which some of our deductions might seem a little mysterious). See also Chang [2] where the situation in which the domain has boundary is explicitly dealt with.

That theory established, for appropriate initial and boundary data, the existence of a weak solution $u \in W^{1,2}_{loc}(D \times [0,\infty), S^2)$ which is smooth in $D \times (0,\infty)$ away from at most a finite number of singular points (and is unique within a restricted class of solutions). We refer to this solution as the "Struwe solution" to distinguish it from the new examples we construct. Approaching a singular time, the now-familiar phenomenon of energy concentration occurs and by an appropriate rescaling we may extract a harmonic 2-sphere which we call a bubble (see [12] or [13] for details). Struwe's solution abandons the bubbles and continues the flow starting with the weak limit of the flow approaching the singular time.

In the new solutions we construct, we find the reverse occurring. A perfectly smooth flow may spontaneously develop a singularity by the addition of an "infinitely concentrated" bubble. This bubble, and its associated energy, then distributes itself as time increases; we refer to this procedure as "reverse bubbling."

We adopt the shorthand $u(t)$ to represent the map $D \to S^2$ obtained by taking the trace of u onto $D \times \{t\}$. One characteristic of the example of our main theorem is that although $E(u(t))$ is forced by the equation to be a nonincreasing function for nonsingular t, it jumps up as reverse bubbling occurs. Indeed, Freire has shown in [8, 9] (see also Rivière [11]) that the class of flows in $W^{1,2}_{loc}(D \times [0,\infty), S^2)$ for which $E(u(t))$ is a nonincreasing function of t, consists only of the Struwe solution.

In [13], we conjectured[1] (for general target $\mathcal{N}$) based on the possibility of reverse bubbling, that it should be possible to weaken the hypothesis that $E(u(t))$ is a nonincreasing function of t to the hypothesis that

$$\limsup_{t \downarrow T} E(u(t)) < E(u(T)) + \inf\left\{E(v) \mid v : S^2 \to \mathcal{N} \text{ is nonconstant and harmonic}\right\}, \tag{1.3}$$

for all $T > 0$, which in the case under consideration here where the target is S^2, is simply that

$$\limsup_{t \downarrow T} E(u(t)) < E(u(T)) + 4\pi. \tag{1.4}$$

Progress in this direction has recently been made by Harpes in [10]. Our main result

[1] Originally mistyped with weak inequality instead of strict inequality.

shows that this would be sharp by constructing an example of an alternative to the Struwe solution for which we have equality in (1.4) at some time $t = T$. Our technique may be used to construct a multitude of examples, but we state our main result as the existence of one of these.

Theorem 1.1. There exist a harmonic map $u_0 : D \to S^2$ and a weak solution $u \in W^{1,2}_{loc}(D \times [0, \infty), S^2)$ to the harmonic map heat equation (1.2) such that $u(t) = u_0$ for all $t \in [0, 1]$, but $u(t) \neq u_0$ for $t > 1$.

The solution may be chosen so that $u(t) \in W^{1,2}(D, S^2)$ for all $t \geq 0$, and even to be smooth in $(D \times [0, \infty))\backslash(0, 1)$. Moreover,

$$\lim_{t \downarrow 1} E(u(t)) = E(u_0) + 4\pi. \tag{1.5}$$

□

The flow we construct is therefore smooth except at the origin at time $t = 1$. The map u_0 will be given explicitly later, as a symmetric conformal map from D onto a hemisphere of the target.

The construction of the flow from times $t = 1$ onwards consists of taking a limit of a sequence of flows with initial data which undergoes bubbling in the limit. The danger with this procedure is, intuitively, that the bubble may remain infinitely concentrated in the limit flow. Indeed we must show that the bubble distributes itself in "finite time." This is very reminiscent of the question of whether a bubble can concentrate itself in finite time, which was settled by Chang, Ding, and Ye in [4], and it turns out that we can borrow heavily from their construction.

In Section 5, we sketch a potentially useful extension of our main result in which a bubble develops in a flow, and is then immediately twisted and reattached, and the flow allowed to continue under reverse bubbling. Crucially (see Section 5) we achieve this without changing the homotopy class of the flow.

Aside from potential applications, this extension shows how it is crucial in Freire's hypothesis for uniqueness (that $E(u(t))$ should be nonincreasing in t) to take $u(t)$ to be the *trace* of u. Indeed, in the example of Section 5, we have $E(u(t)) \geq E(u(s))$ for all $t \leq s$, provided $s \neq 1$. (The value of $E(u(t))$ at $t = 1$ will be less than its limit from either side.)

We end this section by remarking that the "super-critical" case when the domain has dimension larger than two, is rather different from the critical two-dimensional case we consider here. In keeping with the super-critical cases of many other equations, the tendency for nonuniqueness and the formation of singularities is much greater (see [5, 6]) and occurs for different reasons.

2 Notation for corotationally symmetric maps

We will work with flows which have corotational symmetry, so that the PDE governing the heat flow may be written with one spatial variable. Many examples of heat flows with such symmetry have been considered prior to this work (cf. [4, 5, 13]).

As in Theorem 1.1, we consider maps between the 2-disc D and S^2. We use polar coordinates (r,ϕ) on the domain and spherical polar coordinates (α,φ) on the target. A point (α,φ) then corresponds to the point $(\sin\alpha\cos\varphi, \sin\alpha\sin\varphi, \cos\alpha) \in S^2 \hookrightarrow \mathbb{R}^3$. All the maps $D \to S^2$ we consider will enjoy the symmetric form

$$(r,\phi) \longrightarrow (\alpha(r),\phi), \tag{2.1}$$

for $\alpha : [0,1] \to \mathbb{R}$ with $\alpha(0)$ being an integer multiple of π. For our main result, α will take values only in $[0,\pi]$, although in principle we allow α to take any value. Given such a map, if we define $\mathcal{T}(\alpha) : [0,1] \to \mathbb{R}$ by

$$(\mathcal{T}(\alpha))(r) = \frac{\partial^2\alpha}{\partial r^2} + \frac{1}{r}\frac{\partial\alpha}{\partial r} - \frac{\sin\alpha\cos\alpha}{r^2}, \tag{2.2}$$

then $\mathcal{T}$ is essentially the tension of the map, and in particular, the map is harmonic if and only if $\mathcal{T}(\alpha) \equiv 0$. There are plenty of such harmonic maps, corresponding to maps $D \to S^2 \simeq \mathbb{C}\cup\{\infty\}$ of the form $z \to z/\lambda$, for $\lambda > 0$. In polar coordinates, these correspond to maps

$$\alpha(r) = 2\tan^{-1}\frac{r}{\lambda}. \tag{2.3}$$

All flows we consider will take the form $(r,\phi,t) \to (\alpha(r,t),\phi)$, in which case α evolves according to

$$\frac{\partial\alpha}{\partial t} = \mathcal{T}(\alpha) := \frac{\partial^2\alpha}{\partial r^2} + \frac{1}{r}\frac{\partial\alpha}{\partial r} - \frac{\sin\alpha\cos\alpha}{r^2}, \quad \alpha(0,t)=0,\ \alpha(1,t)=b, \tag{2.4}$$

for some fixed b, with given initial conditions $\alpha(\cdot,0)$.

3 Construction of the supersolution

The flow of Theorem 1.1 (at least from $t=1$ onwards) will be constructed as a limit of flows whose initial data converges weakly to u_0 in $W^{1,2}$, but which form a bubble at the origin in the limit. The crux of the argument is to show that we get more than the trivial flow (for which $u(t)=u_0$ for all t) in the limit, and that the energy concentrating in the initial data must distribute itself at a uniform rate, so that the limit flow distributes energy in "finite time."

As mentioned in the introduction, this is similar to the problem of the existence of finite time blowup, first demonstrated by Chang, Ding, and Ye [4] and we will in fact be able to modify their subsolution construction in order to obtain an appropriate supersolution. We adopt, where possible, a corresponding notation.

We have mentioned in Section 2 that if we define

$$\phi(r,\lambda) := 2\tan^{-1}\frac{r}{\lambda} = \cos^{-1}\left(\frac{\lambda^2 - r^2}{\lambda^2 + r^2}\right) \tag{3.1}$$

for $r \in [0,1]$ and $\lambda > 0$, then

$$\mathcal{T}(\phi(\cdot,\lambda)) \equiv 0 \tag{3.2}$$

for any $\lambda > 0$.

We choose $\varepsilon > 0$ once and for all, and define the distorted function

$$\theta(r,\lambda) = \phi\left(r^{1+\varepsilon},\lambda\right) \tag{3.3}$$

for $r \in [0,1]$ and $\lambda > 0$ (cf. [4]). A simple calculation confirms that for each $\lambda > 0$, θ satisfies

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} - (1+\varepsilon)^2\frac{\sin\theta\cos\theta}{r^2} = 0. \tag{3.4}$$

We may then choose $\mu = \mu(\varepsilon) > 0$ sufficiently large so that both

$$\cos\theta(r,\mu) \geq \frac{1}{1+\varepsilon} \tag{3.5}$$

for $r \in [0,1]$ and

$$\theta(1,\mu) = \phi(1,\mu) < \frac{\pi}{4}. \tag{3.6}$$

Next, for $\delta > 0$ to be chosen later (depending on the precise value of ε we took) we define $\lambda : [0,\infty) \to \mathbb{R}$ by

$$\lambda(t) = [(1-\varepsilon)\delta t]^{1/(1-\varepsilon)} \tag{3.7}$$

so that λ solves the ODE $\dot{\lambda} = \delta\lambda^{\varepsilon}$ for $t > 0$ with the initial condition $\lambda(0) = 0$.

Modulo the choice of δ still to be made, we propose as candidate for the supersolution, the function

$$f(r,t) = \phi\big(r,\lambda(t)\big) - \theta(r,\mu) \tag{3.8}$$

for $r \in [0,1]$ and $t > 0$.

Proposition 3.1. With f as above (for suitably chosen $\delta > 0$) the following hold:

(i) For $t > 0$ and $r \in [0,1]$,

$$\frac{\partial f}{\partial t} - \mathcal{T}(f(\cdot,t)) \geq 0. \tag{3.9}$$

(ii) There exists $T > 0$ such that for $t \in (0,T]$,

$$f(1,t) \in \left(\frac{\pi}{2},\pi\right). \tag{3.10}$$

(iii) For any $r \in (0,1]$,

$$\lim_{t\downarrow 0} f(r,t) = \pi - \theta(r,\mu), \tag{3.11}$$

with uniform convergence on any closed subinterval of $(0,1]$.

(iv) There holds

$$\lim_{t\downarrow 0} \frac{\partial f}{\partial r}(0,t) = \infty. \tag{3.12}$$

(v) For fixed $r \in [0,1]$, $f(r,t)$ is a nonincreasing function of t. □

Proof. For fixed $t > 0$, we have

$$\mathcal{T}(f(\cdot,t)) = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \left(\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r}\right) - \frac{\sin(\phi-\theta)\cos(\phi-\theta)}{r^2}, \tag{3.13}$$

where we abbreviate $\phi(r,\lambda(t))$ and $\theta(r,\mu)$ by ϕ and θ, respectively.

By virtue of (3.4) and the fact that $\mathcal{T}(\phi(\cdot,\lambda(t))) \equiv 0$, we then have

$$\mathcal{T}(f(\cdot,t)) = r^{-2}\big[\sin\phi\cos\phi - (1+\varepsilon)^2\sin\theta\cos\theta - \sin(\phi-\theta)\cos(\phi-\theta)\big]. \tag{3.14}$$

Following [4] we may rewrite this as

$$\mathcal{T}(f(\cdot,t)) = r^{-2}\big[-(1+\varepsilon)^2\sin\theta\cos\theta + \cos(2\phi-\theta)\sin\theta\big], \tag{3.15}$$

which we may estimate using (3.5) to give

$$\mathcal{T}(f(\cdot,t)) \leq r^{-2}\big[-(1+\varepsilon)\sin\theta + \cos(2\phi-\theta)\sin\theta\big] \leq -r^2\varepsilon\sin\theta, \tag{3.16}$$

since $\sin\theta \geq 0$. Expanding out θ in terms of r and μ then gives us

$$\sin\theta = \frac{2\mu r^{1+\varepsilon}}{\mu^2 + r^{2(1+\varepsilon)}} \geq \frac{2\mu r^{1+\varepsilon}}{\mu^2+1}, \tag{3.17}$$

and hence

$$\mathcal{T}(f(\cdot,t)) \leq -\frac{2\mu\varepsilon r^{\varepsilon-1}}{\mu^2+1}. \tag{3.18}$$

Next we turn to the time derivative of f. Since

$$\frac{\partial\phi}{\partial\lambda}(r,\lambda) = -\frac{2r}{\lambda^2+r^2}, \tag{3.19}$$

we have

$$\frac{\partial f}{\partial t}(r,t) = \frac{\partial\phi}{\partial\lambda}(r,\lambda(t))\lambda'(t) = -\frac{2r}{\lambda^2(t)+r^2}\delta\lambda^{\varepsilon}(t). \tag{3.20}$$

Therefore, by (3.18),

$$\frac{\partial f}{\partial t} - \mathcal{T}(f) \geq -\frac{2r\delta\lambda^{\varepsilon}(t)}{\lambda^2(t)+r^2} + \frac{2\mu\varepsilon r^{\varepsilon-1}}{\mu^2+1} = 2r^{\varepsilon-1}\left(\frac{\mu\varepsilon}{\mu^2+1} - \frac{\delta\left(\frac{r}{\lambda}\right)^{2-\varepsilon}}{1+\left(\frac{r}{\lambda}\right)^2}\right) \geq 0 \tag{3.21}$$

provided that we choose $\delta > 0$ sufficiently small (depending on ε and μ) since the function $s \to s^{2-\varepsilon}/(1+s^2)$ is bounded above for $s \geq 0$.

This completes the proof of part (i).

For part (ii), we choose $T > 0$ sufficiently small so that $\lambda(T) < \tan(\pi/8)$. Then, using (3.6), we may calculate for $t \in (0,T]$

$$\begin{aligned} f(1,t) &= \phi\big(1,\lambda(t)\big) - \theta(1,\mu) = 2\tan^{-1}\frac{1}{\lambda(t)} - \theta(1,\mu) \\ &= 2\left(\frac{\pi}{2} - \tan^{-1}\lambda(t)\right) - \theta(1,\mu) \geq \pi - 2\tan^{-1}\lambda(T) - \theta(1,\mu) \\ &> \pi - \frac{\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}. \end{aligned} \tag{3.22}$$

The final three parts are straightforward calculations. ■

4 Construction of the example of Theorem 1.1

We define explicitly the map u_0 of Theorem 1.1 in terms of polar coordinates, as $(r, \varphi) \to (g(r), \varphi)$ where

$$g(r) := \pi - \phi(r, 1). \tag{4.1}$$

The map u_0 is then clearly harmonic, and since it is conformal, its energy must equal the area of its image. Since its image is precisely a hemisphere, we have $E(u_0) = 2\pi$. (Indeed u_0 is energy minimising amongst all maps sharing its boundary values.)

We construct the flow promised in Theorem 1.1 by passing to the limit of a sequence of flows with increasingly concentrated initial data. For simplicity, we construct with the singularity to occur at time $t = 0$, and then translate it to $t = 1$ at the end. In order to pass to the limit of a sequence of flows, we require compactness which we obtain from the following uniform regularity result of Struwe, based on [12, Lemma 3.10].

In order to state his result we define, for $R > 0$, $x \in D$ and $v \in W^{1,2}(D, S^2)$, the local energy

$$E(v; x, R) := \frac{1}{2}\int_{D\cap B(x,R)} |\nabla v|^2, \tag{4.2}$$

where $B(x, R)$ is a disc centred at x, of radius R, in $\mathbb{R}^2$.

Lemma 4.1 (see [12]). There exists $\varepsilon_0 > 0$ such that if $a < b$ and $v : D \times [a, b] \to S^2$ is a smooth solution to (1.2), and $R > 0$ is chosen so that

$$E\big(v(t); x, R\big) \leq \varepsilon_0 \tag{4.3}$$

for all $x \in D$ and $t \in [a, b]$, then the C^k norm of v (for any $k \in \mathbb{N}$) over any $\Omega \subset\subset D \times (a, b)$ is bounded uniformly in terms of Ω, k, and R. □

Remark 4.2. To avoid later complications, we assume, without loss of generality, that $\sqrt{\varepsilon_0/4\pi} < \pi/2$.

We now construct a sequence $u_n : D \times [0, \infty] \to S^2$ of smooth flows of the corotationally symmetric form $(r, \varphi) \to (\Psi_n(r, t), \varphi)$. To do this, we write down initial maps of the form

$$(r, \varphi) \longrightarrow \big(\psi_n(r), \varphi\big) \tag{4.4}$$

and take the corresponding Struwe solution, which we will show to be free from singularities for each $n \in \mathbb{N}$. Note that by appealing to the uniqueness of smooth solutions (see [12]) the flow will retain the corotational symmetry, and Ψ_n will be well defined.

We define the functions $\psi_n : [0, 1] \to [0, \pi]$ by

$$\psi_n(r) = \min\left\{2\tan^{-1}(nr), g(r)\right\}. \tag{4.5}$$

Note that there is a unique point $r_n \in (0, 1)$ such that $2\tan^{-1}(nr_n) = g(r_n)$. We then have that $\psi_n(r) = 2\tan^{-1}(nr)$ for $r \in [0, r_n]$ and $\psi_n(r) = g(r)$ for $r \in [r_n, 1]$. Moreover, the following properties hold, and are preserved under the passage to subsequences (simultaneously for ψ_n and r_n):

(i) ψ_n is continuous on $[0, 1]$, and smooth except at r_n.
(ii) $\psi_n(0) = 0$, and $\psi_n'(0) \in (0, \infty)$.
(iii) $\psi_n(r) \le g(r)$ for all $r \in [0, 1]$, with equality for $r \in [r_n, 1]$.
(iv) $r_n \to 0$ as $n \to \infty$.
(v) $\psi_n(1) = \pi/2$.
(vi) $\psi_n(r) \ge \psi_m(r)$ if $n \ge m$, for all $r \in [0, 1]$.
(vii) $E(\psi_n) < E(u_0) + 4\pi = 6\pi$ for all $n \in \mathbb{N}$, and $E(\psi_n) \uparrow 6\pi$ as $n \to \infty$.

In (vii) above, we have used the suggestive notation $E(\psi_n)$ to denote the energy of the map $D \to S^2$ given by (4.4).

By virtue of property (vii), since the energy of the flow is nonincreasing for the Struwe solution, we see that

$$E\big(u_n(t)\big) < 6\pi, \tag{4.6}$$

for all $n \in \mathbb{N}$ and $t \ge 0$, and hence that the flows u_n must be free from singularities. Indeed, any singularity would absorb at least 4π of the energy (that being the minimum energy of a bubble in this situation) which would leave less than 2π of energy which is less than the minimum energy of a map $D \to S^2$ with the given boundary conditions (which constrain the area, and therefore the energy to be at least 2π). Note that we could also rule out singularities using a maximum principle argument—see below—by arguing that we must have $\Psi_n(r, t) \le 2\tan^{-1}(nr) \le 2\tan^{-1} n$ for $r \in [0, 1]$ and $t \ge 0$ and therefore that the image of u_n is barred from a geodesic disc in the target of radius $2\tan^{-1}(1/n)$. Since any bubble must cover the target S^2, no bubbling—and therefore no singularities—can occur. We will use this method during the extension of Section 5.

We now exploit the maximum principle to constrain the evolution of the u_n with respect to each other, and with respect to the "supersolution" f constructed in Section 3.

Lemma 4.3. If $n, m \in \mathbb{N}$ with $n \geq m$, then for all $r \in [0, 1]$

(i) $\Psi_n(r, t) \geq \Psi_m(r, t)$ for all $t \geq 0$,

(ii) $\Psi_n(r, t) \leq g(r)$ for all $t \geq 0$,

(iii) $\Psi_n(r, t) \leq f(r, t)$ for all $t > 0$. □

Proof. Since $\psi_n(r) \geq \psi_m(r)$ for all $r \in [0, 1]$, we may apply the maximum principle to deduce that $\Psi_n(r, t) \geq \Psi_m(r, t)$ for all $t \geq 0$. Details of how to make such an application of the maximum principle for this equation may be found in [3]. Similarly, part (ii) follows because $\psi_n(r) \leq g(r)$ for all $r \in [0, 1]$.

For part (iii), we observe that by choosing $\tau > 0$ sufficiently small (depending on n) we can achieve $f(r, \tau) \geq \psi_n(r)$ for all $r \in [0, 1]$. (To check this, observe that $\psi_n(r) \leq g(r) < \pi - \theta(r, \mu)$ for all $r \in (0, 1]$, and then use parts (iii), (iv), and (v) of Proposition 3.1.) Since f is a supersolution (parts (i) and (ii) of Proposition 3.1), we may again apply the maximum principle to deduce that $\Psi_n(r, t) \leq f(r, t + \tau)$ for all $t > 0$. However, by part (v) of Proposition 3.1, we then have

$$\Psi_n(r, t) \leq f(r, t + \tau) \leq f(r, t) \tag{4.7}$$

for all $t > 0$. ■

Lemma 4.3 will be applied to control the energy concentration in the flows u_n.

Lemma 4.4. There exist $T > 0$ and $N \in \mathbb{N}$ such that for any $t_0 \in (0, T)$, there exists $R > 0$ (dependent on t_0) such that $E(u_n(t); x, R) < \varepsilon_0$ for all $t \in [t_0, T]$, $x \in D$ and $n \geq N$. □

Proof. Using the fact that $\sup_{r \in [0,1]} \psi_n(r) \to \pi$ as $n \to \infty$, we can choose $N \in \mathbb{N}$ sufficiently large so that

$$\sup_{r \in [0,1]} \psi_N(r) \geq \pi - \frac{1}{2}\sqrt{\frac{\varepsilon_0}{4\pi}}, \tag{4.8}$$

and thus find $\hat{r} \in (0, 1)$ such that

$$\psi_N(\hat{r}) \geq \pi - \frac{1}{2}\sqrt{\frac{\varepsilon_0}{4\pi}}. \tag{4.9}$$

We can then choose $T > 0$ sufficiently small so that for any $t \in [0, T]$,

$$\Psi_N(\hat{r}, t) \geq \pi - \sqrt{\frac{\varepsilon_0}{4\pi}}. \tag{4.10}$$

We also reduce T, if necessary, to be less than the T of part (ii) of Proposition 3.1. By appealing to Lemma 4.3 (part (i)) we then have

$$\Psi_n(\hat{r}, t) \geq \pi - \sqrt{\frac{\varepsilon_0}{4\pi}}, \tag{4.11}$$

for any $n \geq N$, and $t \in [0, T]$.

Now suppose that $t_0 \in (0, T)$. We choose $\delta \in (0, 1)$ sufficiently small so that $f(r, t_0) < \sqrt{\varepsilon_0/4\pi}$ for $r \in [0, \delta]$. By applying part (iii) of Lemma 4.3 and part (v) of Proposition 3.1 we then have for $t \in [t_0, T]$, $r \in [0, \delta]$, and $n \in \mathbb{N}$, that

$$\Psi_n(r, t) \leq f(r, t) \leq f\big(r, t_0\big) < \sqrt{\frac{\varepsilon_0}{4\pi}}. \tag{4.12}$$

We now consider what the constraints (4.11) and (4.12) tell us about the image of u_n. Keeping in mind Remark 4.2, we must have $0 < \delta < \hat{r} < 1$. For each $t \in [t_0, T]$ and $n \geq N$, as r varies within $[\delta, \hat{r}]$, $\Psi_n(r, t)$ must cover the interval $[\sqrt{\varepsilon_0/4\pi}, \pi - \sqrt{\varepsilon_0/4\pi}\,]$ which corresponds to the image of u_n containing the sphere with discs of radius $\sqrt{\varepsilon_0/4\pi}$ removed from each pole. In particular, its area over this region of the domain must be at least $4\pi - 2\pi(\sqrt{\varepsilon_0/4\pi})^2 = 4\pi - \varepsilon_0/2$. (Here we are using the fact that the area of a geodesic disc of radius r on the sphere is less than πr^2.)

Meanwhile, since $\Psi_n(1, t) = \pi/2$ (for all $n \in \mathbb{N}$ and $t \geq 0$) for each $t \in [t_0, T]$ and $n \geq N$, as r varies within $[\hat{r}, 1]$, $\Psi_n(r, t)$ must cover the interval $[\pi/2, \pi - \sqrt{\varepsilon_0/4\pi}]$ which corresponds to the image of u_n containing a hemisphere with a disc of radius $\sqrt{\varepsilon_0/4\pi}$ removed from the centre. In particular, its area over this region of the domain must be at least $2\pi - \pi(\sqrt{\varepsilon_0/4\pi})^2 = 2\pi - \varepsilon_0/4$.

Combining these two regions of the domain, we see that the total area of the image (counted with multiplicity) over the part of the domain with $r \geq \delta$ must be at least $6\pi - 3\varepsilon_0/4$. Since energy must exceed area, we find that for each $t \in [t_0, T]$ and $n \geq N$,

$$E\big(u_n(t)\big) - E\big(u_n(t); 0, \delta\big) \geq 6\pi - \frac{3\varepsilon_0}{4}. \tag{4.13}$$

We are now in a position to prove Lemma 4.4 with T and N as above, by contradiction. Suppose that the lemma is false. Then for some $t_0 \in (0, T)$, after passing to a subsequence in n, we may take sequences $\{x_n\} \subset D$, $\{t_n\} \subset [t_0, T]$ and $R_n \downarrow 0$ such that $E(u_n(t_n); x_n, R_n) \geq \varepsilon_0$. Without loss of generality, we may assume that $\{x_n\}$ is convergent in $\overline{D}$, and by exploiting the corotational symmetry and the universal bound $E(u_n(t)) < 6\pi$ (see (4.6)) we must have $x_n \to 0$. By adjusting the numbers R_n, we can then reduce to assuming that $x_n = 0$ for all n.

But then we have (provided that we take n sufficiently large so that $R_n < \delta$) that

$$\begin{aligned} E\big(u_n(t_n)\big) &\geq E\big(u_n(t_n); 0, R_n\big) + \Big(E\big(u_n(t_n)\big) - E\big(u_n(t_n); 0, \delta\big)\Big) \\ &\geq \varepsilon_0 + 6\pi - \frac{3\varepsilon_0}{4} = 6\pi + \frac{\varepsilon_0}{4}, \end{aligned} \tag{4.14}$$

by (4.13), which contradicts the energy bound (4.6). ■

We may now combine Lemma 4.4, restricting the concentration of energy in u_n, with the regularity result, Lemma 4.1, of Struwe.

This tells us that for any $k \in \mathbb{N}$, there exists a bound on the C^k norm of u_n (for $n \geq N$) over any $\Omega \subset\subset D \times (t_0, T]$ in terms of Ω, R, and k. Moreover, since we can select R dependent only on Ω, we may take the upper bound on the C^k norm to depend only on Ω and k.

In particular, we may pass to a subsequence to obtain convergence (in any C^k norm) of u_n to a new solution u on Ω. By a diagonal argument in which we allow t_0 to decrease to zero, and Ω to fill out $D \times [t_0, T]$, we may then extend to a smooth solution u on the whole region $D \times (0, T)$.

By virtue of the construction, u will preserve some of the features of the u_n. For example, it will necessarily be corotationally symmetric, it will be extendable continuously to the boundary of D for each $t \geq 0$ (with the same boundary conditions as the u_n) and also extendable continuously to $t = 0$ over any $\Omega \subset D$ whose closure does not contain the origin (with values at $t = 0$ agreeing with u_0). Moreover, such a smooth solution must have energy decreasing in time.

Having constructed the solution on this potentially short time interval, it is now easy to extend for all time. For example, we may take the Struwe solution $\tilde{u}$ starting at time $t = T/2$, with initial data $u(T/2)$. By the uniqueness assertion of Struwe [12] (or the more general one of Freire [8, 9] which is also written for domains with boundary as well as holding for weaker flows) $\tilde{u}$ is obliged to agree with u over $D \times (T/2, T)$. Moreover, we cannot have singularities in the flow because of energy considerations, as before. We can therefore use $\tilde{u}$ to extend u for all time.

We point out that in the limit $t \downarrow 0$, the map $u(t) : D \to S^2$ converges weakly in $W^{1,2}$ to u_0, and develops a bubble with energy 4π at the origin. There is "no loss of energy" in that

$$\lim_{t \downarrow 0} E(u(t)) = 6\pi = 4\pi + E\big(u_0\big). \tag{4.15}$$

To complete the proof of Theorem 1.1, we simply translate in time the solution we have just constructed so that it begins at $t = 1$, and set $u(t) = u_0$ for $t \in [0, 1]$. This flow is then a solution to the harmonic map heat equation (1.2) for $t \in [0, 1]$ and $t > 1$ separately, and is continuous on $D \times [0, \infty)$ except at the point $(0, 1)$ (i.e., the origin at time $t = 1$). Moreover, we have $u \in W^{1,2}(D \times [0, s))$ for any $s > 0$, so standard arguments guarantee that u is a weak solution to the harmonic map heat equation. With the hindsight of the theory developed by Struwe in [12], it is then straightforward to prove that u is smooth away from the one singular point. That completes the construction, and the proof of Theorem 1.1.

5 Reattachment of bubbles

In this section, we sketch how the construction made to prove Theorem 1.1 may be used to construct an example of a flow which initially agrees with the Struwe solution, developing a singularity in finite-time, but for which we may rotate and reattach the bubble (at the same point and time, and with the same orientation) and continue with a flow which redistributes the energy. In other words, we may have bubbling *and* reverse bubbling at the same instant of time.

As touched upon in the introduction, this is potentially extremely useful for applications of the harmonic map flow to understand the topology of spaces of maps, since under this procedure the homotopy class is preserved (as opposed to when we remove the bubble altogether) and we might be able to consider multi-dimensional families of heat flows within a fixed homotopy class, keeping track of changes in topology in terms of "twistings" of bubbles as they develop and are reattached. Note that the flow immediately before and after the singularity will not be homotopic via corotationally symmetric maps—we must pass via nonsymmetric maps.

To construct such an example (still working under the corotational ansatz) we may start with an initial map given by $(r, \varphi) \to (h(r), \varphi)$ with boundary conditions $h(0) = 2\pi$ and $h(1) = \pi/2$, and such that $h(r) \geq g(r)$ for all $r \in [0, 1]$, with equality except for r in a small region near 0. We also ask that $h'(r) < 0$ for all $r \in [0, 1]$, and that $E(h)$ is just a little over the 6π above which it must be under these conditions (where again we use the suggestive notation $E(h)$ for the energy of the corresponding map $D \to S^2$). In particular, we ask that $E(h) - 6\pi \ll \varepsilon_0$.

If we write the subsequent heat flow as $(r, \varphi, t) \to (H(r, t), \varphi)$ for some H, then by the theory of Chang, Ding, and Ye [4] blowup must occur, at time $t = T$, say. Although we cannot expect the condition $h'(r) < 0$ to imply $H'(r, t) < 0$, we claim that for each $t \geq 0$, the function $H(\cdot, t) - \pi$ will have a single zero. (The only way this could fail would be via a pitchfork bifurcation, which can be ruled out by a direct argument.) Moreover, if we write $r(t)$ for the value where $H(r(t), t) = \pi$, we see that we must have

$$\liminf_{t\uparrow T} r(t) = 0, \tag{5.1}$$

since otherwise we could rule out blowup by comparison with one of the barriers of Chang and Ding [3]. In particular, this implies that $H(\cdot, T)$ takes values only in $[0, \pi]$. Moreover, by our hypotheses on h, $E(H(\cdot, T))$ can be only a little over 2π, the bubble having absorbed 4π of energy.

From time T onwards, we construct a flow just as we did in Theorem 1.1, but instead of starting with $g(r)$ (corresponding to the map $u_0 : D \to S^2$) we start with $H(r, T)$ (corresponding to the weak limit of the Struwe solution as $t \uparrow T$). In other words, we take an increasing sequence of functions ψ_n defined by

$$\psi_n(r) = \min\left\{2\tan^{-1}(nr), H(r, T)\right\}. \tag{5.2}$$

We may then carry out a similar limiting procedure to construct a reverse bubbling flow starting where the Struwe solution ended at time $t = T$.

Some extra complications arise from using $H(\cdot, T)$ instead of g in the construction. First, we no longer have $E(u_n(t)) < 6\pi$, so we rule out the existence of singularities in the flows u_n by using the maximum principle instead of energy considerations (as described at that point in Section 4). Second, parts (ii) and (iii) of Lemma 4.3 require modification. If we let $\widehat{H}(r, t)$ represent the Struwe solution with initial conditions represented by $H(\cdot, T)$ then part (ii) will now be

$$\Psi_n(r, t) \leq \widehat{H}(r, t). \tag{5.3}$$

Part (iii) also requires modification since we can no longer be sure to be able to find $\tau > 0$ such that $f(r, \tau) \geq \psi_n(r)$ for all $r \in [0, 1]$. Instead, we claim that for any $\eta > 0$, we have $\Psi_n(r, t + \eta) \leq f(r, t)$ for all $t > 0$ *provided* we allow the μ in the definition of f to depend on η. To do this, we argue that for sufficiently large μ depending on η, we have[2] $\pi - \theta(r, \mu) > \widehat{H}(r, \eta) \geq \Psi_n(r, \eta)$ for $r \in (0, 1]$, and therefore (as before) we have $f(r, \tau) \geq \Psi_n(r, \eta)$ for sufficiently small $\tau > 0$. By the maximum principle, we then have

$$\Psi_n(r, t + \eta) \leq f(r, t + \tau) \leq f(r, t) \tag{5.4}$$

for all $t > 0$ as desired. Finally, having modified part (iii) of Lemma 4.3, we must update the appropriate part of the proof of Lemma 4.4. For t_0 as in the proof, we set $\eta = t_0/2$,

[2]Note that $(\partial\widehat{H}/\partial r)(0, \eta) < 0$ (by working, for example, directly from (1.2)) and $\widehat{H}(r, \eta) \in [0, \pi)$ for $r \in (0, \pi]$.

and choose now the $\delta \in (0,1)$ of the proof to be sufficiently small so that $f(r, t_0/2) \ll \varepsilon_0$ for $r \in [0, \delta]$. We then have, using (5.4),

$$\Psi_n(r,t) \le f(r, t-\eta) \le f\left(r, \frac{t_0}{2}\right) \ll \varepsilon_0 \tag{5.5}$$

for $t \in [t_0, T]$ and $r \in [0, \delta]$, which substitutes (4.12) of the proof of Lemma 4.4.

Note that both the bubble which formed and the bubble we added were degree one maps $S^2 \to S^2$.

Acknowledgment

After we finished this work, we heard that related results have been obtained independently by Bertsch, Dal Passo, and van der Hout—see [1].

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