Teichmüller harmonic map flow into nonpositively curved targets

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Abstract

The Teichmüller harmonic map flow deforms both a map from a closed surface M into an arbitrary closed Riemannian manifold, and a constant curvature metric on M, so as to reduce the energy of the map as quickly as possible [12]. The flow then tries to converge to a branched minimal immersion when it can [12, 14]. The only thing that can stop the flow is a finite-time degeneration of the metric on M where one or more collars are pinched. In this paper we show that finite-time degeneration cannot happen in the case that the target has nonpositive sectional curvature. In particular, when combined with earlier results [12, 14], this shows that the flow will decompose an arbitrary such map into a collection of branched minimal immersions.

1 Introduction

Given a smooth closed orientable surface $M := M_\gamma$ of genus $\gamma \geq 2$ and a smooth compact Riemannian manifold $N = (N, G)$ of any dimension, we can imagine taking a gradient flow of the harmonic map energy

$$E(u, g) := \frac{1}{2}\int_M |du|_g^2 dv_g,$$

simultaneously for both $u : M \to N$ a map and g a hyperbolic (constant Gauss curvature -1) metric on M. More precisely, given a fixed parameter $\eta > 0$, the Teichmüller harmonic map flow, introduced in [12], is the flow defined by

$$\frac{\partial u}{\partial t} = \tau_g(u); \qquad \frac{\partial g}{\partial t} = \frac{\eta^2}{4} Re(P_g(\Phi(u, g))), \tag{1.1}$$

where $\tau_g(u)$ represents the tension field of u (i.e. $\operatorname{tr} \nabla du$), P_g represents the L^2-orthogonal projection from the space of quadratic differentials on (M, g) onto the space of *holomorphic* quadratic differentials, and $\Phi(u, g)$ represents the Hopf differential – see [12] for further information. The flow decreases the energy according to

$$\frac{dE}{dt} = -\int_M \left[|\tau_g(u)|^2 + \left(\frac{\eta}{4}\right)^2 |Re(P_g(\Phi(u, g)))|^2\right]. \tag{1.2}$$

Given any initial data $(u_0, g_0) \in H^1(M, N) \times \mathcal{M}_{-1}$, with $\mathcal{M}_{-1}$ the set of smooth hyperbolic metrics on M, we know [13] that a (weak) solution of (1.1) exists on a maximal interval $[0, T)$, smooth except possibly at finitely many times, and that $T < \infty$ only if the flow of metrics degenerates in moduli space as $t \nearrow T$, that is if the length $\ell(g(t))$ of the shortest closed geodesic in $(M, g(t))$ converges to zero as $t \nearrow T$. In the case that $T = \infty$, a description of the asymptotics of the flow was given in [14] (following on from [12]). Loosely speaking, it was shown that the surface $(M, g(t))$ can degenerate into finitely many lower genus surfaces, with the map $u(t)$ subconverging (modulo bubbling) to branched minimal immersions (or constant maps) on each of these components.

That theory immediately begs the question of whether the flow exists for all time (and thus enjoys this asymptotic convergence to minimal surfaces) or whether on the contrary, $\ell(g(t))$ can decay to zero in finite time, in which case a 'collar' in the surface $(M, g(t))$ must pinch in finite time (see e.g. [12]).

In this paper we show that in the case that the target (N, G) has nonpositive curvature, the Teichmüller harmonic map flow is very well behaved, with a *smooth* solution existing for all time, given arbitrary initial data, and no bubbling occurring at infinite time. The flow then directly decomposes an arbitrary map into a collection of branched minimal immersions.

The theory of the classical harmonic map flow originated in the seminal paper of Eells and Sampson [3] in which the hypothesis of nonpositive curvature was also present, and the essential idea that this hypothesis can help control the evolution of the supremum of the energy density will be useful to us here, although we cannot deduce boundedness of the energy density any more. The main challenge in our work is to prevent the degeneration of collars (which makes no sense in the classical case) and most of the techniques we develop here are far removed from [3]. Our main result could be stated as:

Theorem 1.1. *Suppose M, (N, G) and $\mathcal{M}_{-1}$ are as above, with (N, G) having nonpositive sectional curvature. Given any initial data $(u_0, g_0) \in C^\infty(M, N) \times \mathcal{M}_{-1}$, there exists a smooth solution $(u(t), g(t))$ to (1.1), for $t \in [0, \infty)$.*

The proof of Theorem 1.1 will be somewhat involved, but at the coarsest level, it will turn out that the rate of collapse of a collar will be controlled by a weighted energy

$$\hat{I} = \frac{1}{2} \int_M \frac{|du|^2(x)}{[\mathrm{inj}_g(x)]^2} dv_g(x),$$

(where $\mathrm{inj}_g(x)$ is the injectivity radius of (M, g) at x) and a key part of this work will revolve around obtaining an upper bound for $\hat{I}$ over finite time intervals. Of course, the objective is to prevent energy from gathering on thin collars where the injectivity radius is small. Our argument to deal with this involves an analysis that is reminiscent of the theory of neck analysis for almost harmonic maps [10, 8, 20], except our estimates must deal with the case where the energy on the collar is *not* small, and where the tension field can be *large*. We are also unable to use Hopf differential estimates to relate angular energy with radial energy on the collar, forcing us to deviate substantially from existing techniques.

Coupling our result with the asymptotic description of the flow from [14], and using the nonpositive curvature once more, we will prove:

Theorem 1.2. *In the situation of Theorem 1.1, there exist a sequence of times $t_n \to \infty$, an integer $1 \leq k \leq 3(\gamma - 1)$ and a hyperbolic punctured surface (Σ, h, c) with $2k$ punctures (i.e. a closed Riemann surface $(\hat{\Sigma}, \hat{c})$, possibly disconnected, that has been punctured $2k$ times and then equipped with a compatible complete hyperbolic metric h) such that the following holds.*

1. *The surfaces $(M, g(t_n), c(t_n))$ converge to the surface (Σ, h, c) by collapsing k simple closed geodesics σ_n^j in the sense of Proposition A.3; in particular there is a sequence of diffeomorphisms $f_n : \Sigma \to M \setminus \cup_{j=1}^k \sigma_n^j$ such that*

$$f_n^* g(t_n) \to h \text{ and } f_n^* c(t_n) \to c \text{ smoothly locally,}$$

where $c(t)$ denotes the complex structure of $(M, g(t))$.

2. *The maps $f_n^* u(t_n) := u(t_n) \circ f_n$ converge to a limit u_∞ strongly in $W^{2,2}_{loc}(\Sigma)$.*
3. *The limit $u_\infty : \Sigma \to N$ extends to a smooth branched minimal immersion (or constant map) on each component of the compactification $(\hat{\Sigma}, \hat{c})$ of (Σ, c) obtained by filling in each of the $2k$ punctures.*

Remark 1.3. We do not claim that the image of u_∞ need be connected. Indeed, we will show elsewhere that suitable limits of the images of collapsing collars in $(M, g(t_n))$ can contain nontrivial curves that are not accounted for by u_∞.

Although it does not require the main innovations of this paper, when the target is negatively curved *and* the initial map u_0 is incompressible, a particularly clean conclusion follows using the work in [12], including [12, Remark 3.4] (see also [18] and [17]).

Corollary 1.4. *Suppose (M, g_0) is a closed hyperbolic surface and (N, G) is a compact manifold with nonpositive sectional curvature. Suppose moreover that $u_0 : M \to N$ is any smooth incompressible map. Then there exists a hyperbolic metric $\bar{g}$ on M and a smooth branched minimal immersion $\bar{u} : (M, \bar{g}) \to (N, G)$ homotopic to u_0.*

Indeed, there exist a global smooth solution (u, g) of (1.1) for all $t \geq 0$, with (u_0, g_0) as initial data, together with sequences of times $t_n \to \infty$ and diffeomorphisms $f_n : M \to M$ isotopic to the identity such that

1. *$f_n^*[g(t_n)] \to \bar{g}$ smoothly, and*
2. *$u(t_n) \circ f_n \to \bar{u}$ in $W^{2,2}(M, N)$, and in particular, in $C^0(M, N)$.*

Remark 1.5. As we will discuss elsewhere, in contrast to the theory of harmonic maps (Hartman [6]) there can exist multiple branched minimal immersions within the same homotopy class of maps (even with disjoint image) even when the curvature of the target is strictly negative. In this case, the corresponding domain metrics must represent different points in Teichmüller space.

Remark 1.6. Although we state our results for compact target manifolds, the proof extends to somewhat more general situations that are important for applications we have in mind. For example, if N is noncompact but the image of u_0 lies within the sublevel set of a proper convex function on N, then the theory extends, with the image of the flow remaining within this sublevel set.

The hypothesis of nonpositive curvature for the target will be used in two main ways. A familiar consequence of this hypothesis is the nonexistence of so-called *bubbles*, i.e. nonconstant harmonic maps $S^2 \mapsto (N, G)$, which using well-understood principles (e.g. [16, 19, 2] etc.) prevents the energy of an almost-harmonic map from being too concentrated in isolated regions, which in turn allows us to make dramatically improved estimates in our collar analysis and proves that bubbling singularities cannot occur in the flow. An unconventional feature of our work is that the nonpositive curvature hypothesis will also allow us ultimately to control the weighted energy $\hat{I}$ mentioned earlier. In particular, it will allow us to control the full weighted energy in terms of an angular weighted energy in situations where the available information on the Hopf differential is not sufficiently strong to perform this task. Instead we proceed via a dynamical argument rather than by analysing the flow at individual time slices.

This paper is organised as follows. In Section 2 we prove a formula for $\frac{d\ell}{dt}$ as we move g in a general direction $Re(P_g(\Psi))$ (Lemmata 2.2 and 2.3) and assemble the proof of our main Theorem 1.1 based on growth estimates for the weighted energy (Lemma 2.4). In Section 3 we control the angular energy on collars, the main result being Lemma 3.1. In Section 4 we develop our understanding of holomorphic quadratic differentials in order to prove the formula for $\frac{d\ell}{dt}$ given in Lemma 2.2. In Section 5 we establish that the full weighted energy can grow at most exponentially fast (Lemma 2.4), the key result being Lemma 5.1, and hence will remain bounded over finite time intervals, as required in Section 2.

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2 Ruling out collar degeneration

In this section, we assemble the proof of Theorem 1.1, giving a global smooth solution of our flow. The starting point is the weak solution constructed in [13], that exists until such a time T that the length of the shortest closed geodesic converges to zero. Most of our attention for the remainder of the paper will be devoted to controlling the evolution of weighted energy along thin collars, which will in turn give the necessary control on the evolution of the length of the shortest closed geodesic to prevent it from converging to zero in finite time. However, we begin with some more familiar arguments that establish that the weak solution is in fact a smooth solution for $t \in [0, T)$. As is well understood from [19, 13], the only obstruction to a weak solution being smooth is the development of singularities at which one can perform a standard rescaling procedure to extract a bubble, i.e. a nonconstant harmonic map from S^2 to the target (N, G) (cf. [14] and [19]). However, no such bubble can exist in our situation because of the following standard fact.

Lemma 2.1. *If (N, G) is a Riemannian manifold of nonpositive sectional curvature, then every harmonic map $S^2 \to (N, G)$ is a constant map.*

Proof. By lifting to the universal cover, we may assume that (N, G) is simply connected. The squared distance function $d^2(x_0, \cdot)$ to any fixed point $x_0 \in M$ is a strictly convex function because of the nonpositive curvature [1], and any harmonic map from any closed Riemannian manifold into any Riemannian manifold supporting a convex function is necessarily constant because the composition of the map and the convex function must be subharmonic [5]. $\square$

Therefore, in the context of Theorem 1.1, we can be sure of the existence of a smooth solution of the Teichmüller harmonic map flow until any such time that a simple closed geodesic pinches to a point, and our task is to prove that this cannot happen in finite time.

2.1 Basics of Teichmüller theory

Fix a closed surface M of genus $\gamma \geq 2$, and consider the space $\mathcal{M}_{-1}$ of metrics g on M of constant Gauss curvature -1. It is well understood (see for example [14] and Lemma A.1) that (M, g) decomposes into a *thick part* consisting of all points at which the injectivity radius is at least arsinh(1), and a finite collection of disjoint *collar regions* $\mathcal{C}_j$ for $1 \leq j \leq k$. Each collar region has at its centre a simple closed geodesic σ_j of length ℓ_j. (See Lemma A.1.)

Tangent vectors in $\mathcal{M}_{-1}$ at g can be decomposed into the sum of a Lie derivative term $\mathcal{L}_X g$ (corresponding to a change in parametrisation of the same metric) and a term of the form $Re(\Theta)$, where Θ lies in the $(3\gamma - 3)$-dimensional complex vector space $\mathcal{H}(M, g)$ of holomorphic quadratic differentials (see e.g. [21]). Therefore, we can represent a smooth path in Teichmüller space as a smooth family $g(t)$ of metrics in $\mathcal{M}_{-1}$ for which $\frac{\partial g}{\partial t} = Re(\Theta(t))$ for some smooth family $\Theta(t) \in \mathcal{H}(M, g(t))$. In the next section we will need to understand how the lengths ℓ_j of the geodesics σ_j evolve as $g(t)$ evolves in this way.

2.2 Proof of the main theorem ruling out collar degeneration

Our basic set-up for Theorem 1.1 is that we have a one-parameter family of metrics $g(t)$ evolving under the equation $\frac{\partial g}{\partial t} = Re(P_g(\Psi(t)))$, with $\Psi(t) = \frac{\eta^2}{4}\Phi(u,g)$, and we need to control the evolution of the length of the shortest closed geodesic in order to prevent it from decreasing to zero in finite time, which would correspond to the degeneration of a collar. Our basic result in this direction is the following, which may be of independent interest. We work with respect to the complex coordinate $dz := ds + id\theta$, where (s,θ) are cylindrical coordinates on the collar (see Lemma A.1) .

Lemma 2.2. *Given a closed surface M of genus $\gamma \geq 2$, there exists $\tilde{\ell} \in (0, 2\,\mathrm{arsinh}(1))$ and $C < \infty$ depending only on γ such that the following is true. Suppose $g(t)$ is a one-parameter family of metrics in $\mathcal{M}_{-1}$ for t in a neighbourhood of 0 such that at $t = 0$, we have*

$$\frac{\partial g}{\partial t} = Re(P_g(\Psi)) \quad \textit{for some} \quad \Psi \in \mathcal{Q}_{L^2}(M, g(0)),$$

and we have a collar $\mathcal{C}$ in $(M, g(0))$ around a simple closed geodesic of length $\ell < \tilde{\ell}$. Then

$$\frac{d\ell}{dt} \sim -\frac{\ell^2}{16\pi^3} Re\langle \Psi, dz^2 \rangle_{L^2(\mathcal{C},g)},$$

at $t = 0$, in the sense that

$$\left| \frac{d\ell}{dt} + \frac{\ell^2}{16\pi^3} Re\langle \Psi, dz^2 \rangle_{L^2(\mathcal{C},g)} \right| \leq C\ell^2 \|\Psi\|_{L^1(M,g)}.$$

Note that the content of this lemma revolves around the fact that Ψ need not be holomorphic but rather can be any element of the infinite dimensional space $\mathcal{Q}_{L^2}(M,g)$ of measurable quadratic differentials with finite L^2 norm. To fix normalisations, take an arbitrary local complex coordinate $z = x + iy$ and write $g = \rho^2(dx^2 + dy^2)$, so $\Psi = \psi(dx + idy)^2$ for some locally defined complex valued L^2 function ψ. We are then normalising so that

$$\|\Phi\|^2_{L^2(M,g)} = \int_M |\psi|^2 |dz^2|^2 dv_g = 4 \int_M \rho^{-2} |\psi|^2 dx \wedge dy.$$

Formulae for the first and even second derivatives of ℓ of a quite different flavour to ours can be found in [4, 22] and the references therein.

One could get a feel for Lemma 2.2 by using it to reprove the incompleteness of Teichmüller space (see Section 4.4). Lemma 2.2 will be proved in Section 4, based on an analysis of the space of holomorphic quadratic differentials. We use it now in the special case of the Teichmüller harmonic map flow, to prove:

Lemma 2.3. *Let (u, g) be a solution of the Teichmüller harmonic map flow (1.1), with non-increasing energy $E(u(t), g(t)) \leq E_0$, defined on a surface of genus at least two and on a time interval $[0, T)$. Given a collar $\mathcal{C}$ in (M, g) at time t, with central geodesic σ of length $\ell < \tilde{\ell}$ (where $\tilde{\ell}$ comes from Lemma 2.2) we have*

$$\left| \frac{d}{dt} \log(\ell) + \frac{\eta^2}{16\pi^3} \cdot \ell \int_{\mathcal{C}} (|u_s|^2 - |u_\theta|^2) \rho^{-2} ds d\theta \right| \leq C\ell\eta^2 E_0, \tag{2.1}$$

with $C < \infty$ depending only on the genus γ. In particular, the evolution of ℓ is controlled in terms of a weighted energy

$$I := \int_{\mathcal{C}} e(u,g)\rho^{-2} dv_g,$$

where $e(u,g) = \frac{1}{2}(|u_s|^2 + |u_\theta|^2)\rho^{-2}$ is the energy density and ρ is the conformal factor defined in Lemma A.1, in the sense that

$$\left| \frac{d}{dt} \log \ell \right| \leq C\ell[I + E_0],$$

with C depending only on γ and the coupling constant η.

Proof. Under Teichmüller harmonic map flow, the metric g evolves according to

$$\frac{\partial g}{\partial t} = \frac{\eta^2}{4} Re(P_g(\Phi(u,g))),$$

where $\Phi(u,g) = (|u_s|^2 - |u_\theta|^2 - 2i\langle u_s, u_\theta\rangle)dz^2$ on the collar. Therefore in the language of Lemma 2.2, we have

$$Re\langle \Psi, dz^2\rangle = \frac{\eta^2}{4} \int_{\mathcal{C}} (|u_s|^2 - |u_\theta|^2)|dz^2|^2 \rho^2 ds d\theta = \eta^2 \int_{\mathcal{C}} (|u_s|^2 - |u_\theta|^2)\rho^{-2} ds d\theta,$$

by (A.9). Note also that $\|\Phi\|_{L^1} \leq 4E_0$, so

$$\|\Psi\|_{L^1(M,g)} \leq \eta^2 E_0.$$

Therefore Lemma 2.2 implies that

$$-\frac{d}{dt} \log \ell \sim \frac{\eta^2}{16\pi^3} \cdot \ell \int_{\mathcal{C}} (|u_s|^2 - |u_\theta|^2)\rho^{-2} ds d\theta \tag{2.2}$$

up to an error that is bounded by (2.1) for C depending only on the genus γ. □

We see from Lemma 2.3 that we could deduce that $\ell(g(t))$ does not decrease to zero in finite time if we could prove that ℓI remains bounded over arbitrary compact time intervals. In fact, we will prove the stronger statement that I itself remains bounded. This will imply that the smooth solution of the Teichmüller harmonic map flow discussed above must exist for all time, which will complete the proof of Theorem 1.1.

Lemma 2.4. *Suppose M, (N, G) and $\mathcal{M}_{-1}$ are as above, with (N, G) having nonpositive sectional curvature. Then for any $E_0 > 0$ there exists a constant $C < \infty$ such that the following holds true. Let (u, g) be any solution of the Teichmüller harmonic map flow (1.1), defined on an interval $[0, T)$, with initial energy $E(u(0), g(0)) \leq E_0$. Then for any time $t \in [0, T)$ such that $(M, g(t))$ contains a collar with central geodesic of length $\ell < \bar{\ell}$ (the number in Lemma 2.2) we can estimate the weighted energy defined above by*

$$I := \int_{\mathcal{C}} e(u,g)\rho^{-2} dv_g \leq Ce^{Ct}(1 + (\mathrm{inj}_{g(0)} M)^{-2}),$$

where $\mathrm{inj}_g M := \inf_{x \in M} \mathrm{inj}_g(x)$.

We remark that at each point of such a collar the injectivity radius $\mathrm{inj}_g(p)$ and the conformal factor $\rho(p)$ are of comparable size, see (A.1), so bounding I on each such collar is indeed equivalent to bounding the global weighted energy $\hat{I}$ briefly mentioned before.

We will prove Lemma 2.4 in Section 5, but before that, in Section 3 we will derive estimates on a weighted *angular* energy alone – see Lemma 3.1. This is ironic given that a careful reading of (2.1) shows that large angular energy appears to help *prevent* degeneration of the neck. However, the proof of Lemma 2.4 will bootstrap angular energy estimates to full energy estimates. One might expect this to follow using the Hopf differential $\Phi(u, g)$, which in some sense measures the difference between angular energy and full energy. However, although estimates on the Hopf differential do follow from the flow equations, the obvious ones are not strong enough for our purposes, and so instead we use a dynamic argument via the parabolic Bochner formula in the proof of Lemma 2.4. That part uses crucially the nonpositivity of the target as well as the smallness of the angular energy. The earlier control on the angular energy requires the nonexistence of bubbles in the target, which is guaranteed using the nonpositivity of the curvature.

2.3 Asymptotics in the case of nonpositively curved targets

In this section we make the final observations required to prove Theorem 1.2.

Theorem 1.1 already gives global smooth existence for the Teichmüller harmonic map flow, and [14, Theorem 1.1] describes the decomposition of the flow into branched minimal immersions with the required level of convergence except at a finite set S of points in M at which bubbling occurs; indeed, while [14, Theorem 1.1] only states convergence of $f_n^* u(t_n)$ in each $W^{1,p}_{loc}$, $p < \infty$, in the proof the sequences are chosen so that $\|\tau_g(u)(t_n)\|_{L^2} \to 0$, which, when combined with the convergence of the metrics and the local $W^{1,p}$ convergence away from S, implies that $\Delta_h(f_n^* u(t_n) - u_\infty)$ converges to zero (strongly) in $L^2_{loc}(\Sigma \setminus S)$, thus giving the desired local H^2 convergence.

The only remaining step is to prove that the set S is empty. If it were not, then at each point in S one could perform a standard rescaling procedure to extract a bubble, i.e. a nonconstant harmonic map from S^2 to the target (N, G) (cf. [14] and [19]). However, no such bubble can exist because of Lemma 2.1, so the proof is complete.

3 Controlling the angular energy

Our goal in this section is to control a weighted angular energy that is similar to the weighted energy I from Lemma 2.3, but only considers θ derivatives. More precisely, we prove the following key lemma.

Lemma 3.1. *For any $E_0 < \infty$ and compact Riemannian manifold N not supporting any bubbles, there exists $C < \infty$ such that for any $\ell \in (0, 2\,\mathrm{arsinh}(1))$ and map $u : (\mathcal{C}(\ell), g) \to N$ from a hyperbolic collar, with energy $E(u) \leq E_0$, the angular energy is controlled according to*

$$I^{(\theta)} := \int_{\mathcal{C}(\ell)} \rho^{-2} |u_\theta|^2 d\theta ds \leq C(1 + \|\tau_g(u)\|^2_{L^2(\mathcal{C}(\ell), g)}).$$

Of course, without the weighting coefficient ρ^{-2}, the left-hand side would be the normal angular energy, and would thus be bounded. When additionally the tension can be controlled, we will show that the angular energy decays exponentially along the collar, and this decay dominates the growth of ρ^{-2} towards the centre of the collar.

We will give the proof of Lemma 3.1 towards the end of Section 3.2 once we have developed some preliminary theory.

3.1 Controlling the concentration of energy

In this section we elaborate on the well-known principles that regions of concentrated energy in almost harmonic maps (i.e. maps with small tension field) can be blown up to yield bubbles, and that concentrated energy poses the only obstruction to getting higher order estimates. (Recall from the introduction that a bubble is a nonconstant harmonic map from S^2 to N.) See (for example) Corollary 3.5 for a consequence of these principles.

Given $s \in \mathbb{R}$ and $\Lambda \in (0,\infty)$, we define the cylinder $\mathscr{C}_\Lambda(s)$ to be $(s-\Lambda, s+\Lambda) \times S^1$. This will either be equipped with the standard cylindrical metric or (part of) a hyperbolic collar metric, as appropriate.

Lemma 3.2. *Given a compact Riemannian manifold N not supporting any bubbles, and constants $\varepsilon_1 > 0$ and $E_0 < \infty$, there exist $\tilde{\Lambda} \in (1,\infty)$ and $K < \infty$ such that the following holds. Given any smooth map $u : \mathscr{C}_{\tilde{\Lambda}}(0) \to N$ with total energy $E(u; \mathscr{C}_{\tilde{\Lambda}}(0)) \leq E_0$, we have that*

$$E(u; B_{r_0}(p)) < \varepsilon_1 \qquad \textit{for all } p \in \{0\} \times S^1$$

for all $r_0 \leq (1 + K\|\tau(u)\|_{L^2(\mathscr{C}_{\tilde{\Lambda}}(0))})^{-1}$.

We stress that all quantities in the lemma above are computed with respect to the flat metric $ds^2 + d\theta^2$, despite the fact that we will often apply it to cylinders on which we have a hyperbolic metric.

Proof. We proceed by contradiction: if the lemma were false, then there would exist $\varepsilon_1 > 0$, $E_0 \in (0,\infty)$ and a point $p \in \{0\} \times S^1$, and a sequence of smooth maps $u_i : \mathscr{C}_i(0) \to N$ with $E(u; \mathscr{C}_i(0)) \leq E_0$, such that

$$E(u_i; B_{r_i}(p)) \geq \varepsilon_1 \tag{3.1}$$

for $r_i = (1 + i\|\tau(u_i)\|_{L^2(\mathscr{C}_i(0))})^{-1} \in (0,1]$. First we consider the case that after passing to a subsequence we can arrange that $\|\tau(u_i)\|_{L^2(\mathscr{C}_i(0))} \to 0$. In this case, we can perform a standard bubbling analysis to the sequence u_i in order to extract a bubble. More precisely, we can pass to a subsequence and extract a local weak $W^{1,2}$-limit $u_\infty : (-\infty,\infty) \times S^1 \to N$, which is harmonic, and by the Sacks-Uhlenbeck removable singularity theorem [16] we can add two points at infinity in the cylinder $(-\infty,\infty) \times S^1$ and extend u_∞ to a harmonic map $S^2 \mapsto N$. Since there exist no bubbles by hypothesis, u_∞ must be a constant map. But then the bubbling theory, combined with (3.1) tells us that we can blow up the maps u_i about an appropriate sequence of points $p_i \in B_{r_i}(p)$ and extract a nonconstant harmonic limit $\tilde{u}_\infty : \mathbb{R}^2 \to N$ in $W^{2,2}_{loc}(\mathbb{R}^2, N)$, which can be extended (by adding a point at infinity) to a nonconstant harmonic map $S^2 \mapsto N$, i.e. a bubble, which by hypothesis cannot exist. We have arrived at a contradiction in the case that the tension decays to zero (for a subsequence).

The remaining case is that $\|\tau(u_i)\|_{L^2(\mathscr{C}_i(0))}$ is bounded below by some positive constant, uniformly in i, which forces $r_i \to 0$. In this case, we blow up each map u_i by a factor r_i, and end up with a sequence of maps $\tilde{u}_i$ on fatter and fatter cylinders, so that $E(\tilde{u}_i, B_1) \geq \varepsilon_1$ for each i, and so that

$$\|\tau(\tilde{u}_i)\|_{L^2} = r_i\|\tau(u_i)\|_{L^2(\mathscr{C}_i(0))} = \frac{1-r_i}{i} \to 0$$

as $i \to \infty$. A similar bubbling argument to before allows us to extract a bubble, giving a contradiction in this case too. □

We will combine the lemma above with the following standard regularity estimate in which B_r represents the disc of radius $r > 0$ in the flat plane.

Lemma 3.3 (cf. [12, Lemma 3.3]). *Given a compact target N, there exists $\varepsilon_0 > 0$ and $C < \infty$ such that for any $r > 0$, and smooth map $u : B_r \to N$ with $E(u; B_r) \leq \varepsilon_0$, we have*

$$\int_{B_{r/2}} |\nabla^2 u|^2 + |\nabla u|^4 \leq C \left(\frac{E(u; B_r)}{r^2} + \|\tau(u)\|^2_{L^2(B_r)} \right).$$

Note that the estimate for $\|\nabla^2 u\|^2_{L^2}$ is the standard one. The estimate for $\|\nabla u\|^4_{L^4}$ follows by applying Sobolev to $|\nabla u|^2$ to yield

$$\int_{B_{r/2}} |\nabla u|^4 \leq CE(u; B_{r/2}) \left[\int_{B_{r/2}} |\nabla^2 u|^2 + r^{-2} E(u; B_{r/2}) \right],$$

and bounding the first $E(u; B_{r/2})$ by ε_0.

The combination of Lemma 3.3 and Lemma 3.2 will yield:

Corollary 3.4. *For any $E_0 < \infty$, and compact Riemannian manifold N not supporting any bubbles, there exist $\Lambda \in (2, \infty)$ and $C < \infty$ such that the following holds. Given any smooth map $u : \mathscr{C}_\Lambda(0) \to N$ with total energy $E(u; \mathscr{C}_\Lambda(0)) \leq E_0$, we have that*

$$\int_{\mathscr{C}_1(0)} |\nabla^2 u|^2 + |\nabla u|^4 \leq C \left(E(u; \mathscr{C}_2(0)) + \|\tau(u)\|^2_{L^2(\mathscr{C}_\Lambda(0))} \right).$$

Proof. For our given target N, let ε_1 be the ε_0 from Lemma 3.3 and feed ε_1 into Lemma 3.2 together with our E_0, to obtain in particular the constants $\tilde{\Lambda}$ and K. We set $\Lambda := \tilde{\Lambda} + 1$. For u as in the corollary, we then define $r_0 = (1 + K\|\tau(u)\|_{L^2(\mathscr{C}_\Lambda(0))})^{-1}$ so that we will be able to apply Lemma 3.2 on cylinders $\mathscr{C}_{\tilde{\Lambda}}(s)$ for $s \in [-1, 1]$. We now cover $\mathscr{C}_1(0)$ by balls $B_{r_0/2}(p_i)$ of radius $r_0/2$, with $p_i \in \mathscr{C}_1(0)$; moreover, we can achieve this with no more than Cr_0^{-2} balls, and so that each point in $\mathscr{C}_2(0)$ is covered no more than C times by the balls $B_{r_0}(p_i)$, for universal C. Adding the estimates from Lemma 3.3 for each of these balls, we obtain

$$\begin{aligned}\int_{\mathscr{C}_1(0)} |\nabla^2 u|^2 + |\nabla u|^4 &\leq \sum_i \int_{B_{r_0/2}(p_i)} |\nabla^2 u|^2 + |\nabla u|^4 \\ &\leq C \sum_i \left(\frac{E(u; B_{r_0}(p_i))}{r_0^2} + \|\tau(u)\|^2_{L^2(B_{r_0}(p_i))} \right) \\ &\leq C \left(\frac{E(u; \mathscr{C}_2(0))}{r_0^2} + \|\tau(u)\|^2_{L^2(\mathscr{C}_2(0))} \right).\end{aligned} \tag{3.2}$$

Using the formula above for r_0, we deduce

$$\int_{\mathscr{C}_1(0)} |\nabla^2 u|^2 + |\nabla u|^4 \leq CE(u; \mathscr{C}_2(0))(1 + K\|\tau(u)\|_{L^2(\mathscr{C}_\Lambda(0))})^2 + C\|\tau(u)\|^2_{L^2(\mathscr{C}_2(0))},$$

and hence

$$\int_{\mathscr{C}_1(0)} |\nabla^2 u|^2 + |\nabla u|^4 \leq C \left(E(u; \mathscr{C}_2(0)) + \|\tau(u)\|^2_{L^2(\mathscr{C}_\Lambda(0))} \right) \tag{3.3}$$

as desired, since C is allowed to depend on E_0. □

Although we will need the corollary above in the given form, we will also be able to simplify its conclusion using the following:

Corollary 3.5. *For any $E_0 < \infty$, $\varepsilon_2 > 0$, and compact Riemannian manifold N not supporting any bubbles, there exist $\Lambda \in (3, \infty)$ and $C < \infty$ such that the following holds. Given any smooth map $u : \mathscr{C}_\Lambda(0) \to N$ with total energy $E(u; \mathscr{C}_\Lambda(0)) \leq E_0$, we have that*

$$E(u; \mathscr{C}_2(0)) \leq \varepsilon_2 + C\|\tau(u)\|^2_{L^2(\mathscr{C}_\Lambda(0))},$$

and

$$\int_{\mathscr{C}_1(0)} |\nabla^2 u|^2 + |\nabla u|^4 \leq \varepsilon_2 + C\|\tau(u)\|^2_{L^2(\mathscr{C}_\Lambda(0))}.$$

Proof. First, we pick $C_0 < \infty$ large enough so that for any $r \in (0, 1]$, the cylinder $\mathscr{C}_2(0)$ can be covered by $C_0 r^{-2}$ balls in $(-\infty, \infty) \times S^1$ of radii r, and with centres in $\mathscr{C}_2(0)$. We may then define $\varepsilon_1 := \frac{\varepsilon_2}{2C_0}$, and appeal to Lemma 3.2 to obtain $\tilde{\Lambda}$ and K. We fix $\Lambda := \tilde{\Lambda} + 2$, and consider a map $u : \mathscr{C}_\Lambda(0) \to N$. By setting $r_0 := (1 + K\|\tau(u)\|_{L^2(\mathscr{C}_\Lambda(0))})^{-1}$, we are then able to cover $\mathscr{C}_2(0)$ by $C_0 {r_0}^{-2}$ balls $B_{r_0}(p_i) \subset (-\infty, \infty) \times S^1$ with $p_i \in \mathscr{C}_2(0)$, and we may apply Lemma 3.2 for each i to obtain bounds $E(u; B_{r_0}(p_i)) < \varepsilon_1$. Summing these estimates yields

$$\begin{aligned} E(u; \mathscr{C}_2(0)) &\leq \sum_i E(u; B_{r_0}(p_i)) < C_0 {r_0}^{-2} \varepsilon_1 \\ &= C_0(1 + K\|\tau(u)\|_{L^2(\mathscr{C}_\Lambda(0))})^2 \frac{\varepsilon_2}{2C_0} \\ &\leq \varepsilon_2 + C\|\tau(u)\|^2_{L^2(\mathscr{C}_\Lambda(0))}, \end{aligned} \tag{3.4}$$

which is the first part of the corollary. By combining what we have proved with Corollary 3.4, allowing Λ (and C) to increase and ε_2 to decrease as necessary, we deduce the second part of the corollary. □

3.2 Estimates on the angular energy

In this section we prove Lemma 3.1, telling us that the weighted angular energy is controlled in terms of the tension field.

As usual, we will be implicitly using the collar lemma A.1 and its notation. In particular given a hyperbolic collar $(\mathcal{C}(\ell), g)$, and writing $\mathcal{C} = (-X(\ell), X(\ell)) \times S^1$, we let $X_\delta = X_\delta(\ell)$ be the number given in (A.8) for which the δ-thin part of the collar is described (in collar coordinates) by $(-X_\delta, X_\delta) \times S^1$.

The main ingredient is the following result, which forces the angular energy on unit-length chunks of our long cylinder to decay exponentially as we move in from the ends of the cylinder, at least when the tension field is suitably small.

Proposition 3.6. *Let N be a compact target that supports no bubbles. Then given any $E_0 < \infty$ there exist numbers $C < \infty$, $\Lambda < \infty$ and $\delta \in (0, \operatorname{arsinh}(1))$ such that for any $\ell \in (0, 2\operatorname{arsinh}(1))$, map u from the hyperbolic collar $(\mathcal{C}(\ell), g)$ to N with total energy $E(u; \mathcal{C}(\ell)) \leq E_0$, and $s_0 \in (-X_\delta(\ell), X_\delta(\ell))$ we have $\mathscr{C}_\Lambda(s_0) \subset \mathcal{C}(\ell)$ and*

$$\int_{s_0 - \frac{1}{2}}^{s_0 + \frac{1}{2}} \int_{S^1} |u_\theta|^2 d\theta ds \leq C \cdot e^{-(X(\ell) - |s_0|)} + C \cdot \int_{-X_\delta(\ell)}^{X_\delta(\ell)} e^{-|s - s_0|} \rho^2(s) \cdot \|\tau_g(u)\|^2_{L^2(\mathscr{C}_\Lambda(s), g)} ds.$$

In the situation that u is a map with *small* tension and small energy, there is some history of results that show exponential decay of energy along cylinders, starting with Hadamard's three circle theorem, and including [10, 8, 20]. The key to such results is typically to derive a second order differential inequality for the angular energy over circles $\{s\} \times S^1$ and then apply the maximum principle. Our present situation is more complicated because we don't make any *a priori* assumptions of smallness of energy or tension, which prevents us from deriving angular energy estimates on such circles. Instead, we derive a second order 'delay' differential inequality for an angular energy averaged over short lengths of cylinder, defined by:

$$\Theta(s_0) := \int_{-X(\ell)}^{X(\ell)} \int_{S^1} \varphi^4(s-s_0)|u_\theta(s,\theta)|^2 d\theta ds,$$

where $\varphi \in C_0^\infty((-1,1),[0,1])$ is a cut-off function with $\varphi \equiv 1$ on $[-\frac{1}{2},\frac{1}{2}]$, and $|s_0| \leq X(\ell)-1$.

Lemma 3.7. *Let N be a compact target that supports no bubbles. Then given any $E_0 < \infty$ there exist numbers $\delta \in (0, \operatorname{arsinh}(1))$, $C_1 \in [0,\infty)$ and $\Lambda < \infty$ such that for any $\ell \in (0, 2\operatorname{arsinh}(1))$, any map $u : (\mathcal{C}(\ell), g) \to N$ with total energy $E(u;\mathcal{C}(\ell)) \leq E_0$, and any $s \in (-X_\delta(\ell), X_\delta(\ell))$, we have $\mathscr{C}_\Lambda(s) \subset \mathcal{C}(\ell)$, and the differential inequality*

$$\Theta''(s) - \frac{3}{2}\Theta(s) + \frac{1}{8}\Big[\Theta(s-\frac{1}{2}) + \Theta(s+\frac{1}{2})\Big] \geq -C_1\rho^2(s)\|\tau_g(u)\|^2_{L^2(\mathscr{C}_\Lambda(s),g)} \tag{3.5}$$

is satisfied for the angular energy function Θ associated with u.

Accepting this lemma for the moment, we can give the:

Proof of Proposition 3.6. For the given target N and energy upper bound E_0, let $\delta > 0$, C_1 and Λ be as in Lemma 3.7. We fix arbitrary $\ell \in (0, 2\operatorname{arsinh}(1))$, although if $\ell \geq 2\delta$ we have $(-X_\delta, X_\delta) = \emptyset$, so there is nothing to prove.

We then consider a map $u : (\mathcal{C}(\ell), g) \to N$ with total energy $E(u;\mathcal{C}(\ell)) \leq E_0$. Let

$$L(f) = f''(s) - \frac{3}{2}f(s) + \frac{1}{8}\big[f(s+\frac{1}{2}) + f(s-\frac{1}{2})\big]$$

be the operator describing the left-hand side of the differential inequality (3.5) satisfied by the angular energy function Θ. We observe that while L is not a classical differential operator, the usual comparison principle for ODE still applies. More precisely, let $f, \tilde{f} \in C^2([-X_\delta, X_\delta])$ be any two functions such that

$$L(f) \leq L(\tilde{f}) \text{ on } (-X_\delta + \frac{1}{2}, X_\delta - \frac{1}{2}) \text{ with } f \geq \tilde{f} \text{ on } [-X_\delta, -X_\delta + \frac{1}{2}] \cup [X_\delta - \frac{1}{2}, X_\delta].$$

Then $f \geq \tilde{f}$ on all of $[-X_\delta, X_\delta]$; indeed, if $f - \tilde{f}$ were to achieve a negative minimum at some $s_0 \in (-X_\delta + \frac{1}{2}, X_\delta - \frac{1}{2})$, then

$$(f-\tilde{f})''(s_0) \leq \frac{3}{2}\min(f-\tilde{f}) - \frac{1}{8}\big[(f-\tilde{f})(s_0+\frac{1}{2}) + (f-\tilde{f})(s_0-\frac{1}{2})\big] \leq (\frac{3}{2}-\frac{1}{4})\min(f-\tilde{f}) < 0$$

would lead to a contradiction.

In order to bound the angular energy function Θ, we compare it with solutions of a slightly modified equation, namely of

$$f'' - f = -C_1 \cdot G \qquad \text{on } [-X_\delta, X_\delta], \tag{3.6}$$

where $G(s) := \rho^2(s)\|\tau_g(u)\|^2_{L^2(\mathscr{C}_\Lambda(s),g)}$ is the function on the right-hand side of (3.5), modulo the constant $-C_1$.

Recall that any solution of (3.6) can be described by

$$f_{A,B}(s) := A\cdot e^{s-X_\delta} + B\cdot e^{-s-X_\delta} + \frac{C_1}{2}\int_{-X_\delta}^{X_\delta} e^{-|s-q|}G(q)dq, \quad A,B\in\mathbb{R}. \tag{3.7}$$

Because $C_1 \geq 0$, if $A,B>0$ then the functions $f_{A,B}$ are positive and

$$\frac{1}{8}\left(f_{A,B}(s+\frac{1}{2}) + f_{A,B}(s-\frac{1}{2})\right) \leq \frac{e^{\frac{1}{2}}}{4}\cdot f_{A,B}(s) < \frac{1}{2}f_{A,B}(s)$$

for any $s\in[-X_\delta+\frac{1}{2}, X_\delta-\frac{1}{2}]$. Consequently, $L(f_{A,B}) \leq -C_1G(s) \leq L(\Theta)$. Since we always have $\Theta(s)\leq 2E_0$ for any s, we may apply the comparison theorem with $A,B=2E_0e$, say, to conclude that

$$\Theta(s) \leq f_{A,B}(s) = 2E_0\cdot(e^{s-X_\delta+1}+e^{-s-X_\delta+1}) + \frac{C_1}{2}\int_{-X_\delta}^{X_\delta} e^{-|s-q|}G(q)dq.$$

Since $X(\ell)-X_\delta(\ell)\leq\frac{C}{\delta}$ is bounded independently of ℓ, by Proposition A.2, we thus obtain the claim of Proposition 3.6. □

The proof above relied on Lemma 3.7, claiming a second order differential inequality for the locally smoothed angular energy $\Theta(s)$.

Proof of Lemma 3.7. Defining, for $s\in(-X(\ell),X(\ell))$,

$$\vartheta(s) := \int_{\{s\}\times S^1} |u_\theta|^2,$$

we may compute

$$\vartheta'(s) = 2\int_{\{s\}\times S^1} u_\theta\cdot u_{s\theta},$$

and

$$\begin{aligned}\vartheta''(s) &= 2\int_{\{s\}\times S^1} |u_{s\theta}|^2 + u_\theta\cdot u_{ss\theta}\\ &= 2\int_{\{s\}\times S^1} |u_{s\theta}|^2 - u_{\theta\theta}\cdot u_{ss}.\end{aligned} \tag{3.8}$$

Meanwhile, the tension τ of u with respect to the flat cylindrical metric is given in terms of the second fundamental form $A(\cdot)$ of the target $(N,G)\hookrightarrow\mathbb{R}^{N_0}$ by

$$\tau = u_{ss}+u_{\theta\theta}+A(u)(u_s,u_s)+A(u)(u_\theta,u_\theta)$$

and thus

$$\vartheta''(s) = 2\int_{\{s\}\times S^1} |u_{s\theta}|^2 + |u_{\theta\theta}|^2 - u_{\theta\theta}\cdot\tau + u_{\theta\theta}\cdot[A(u)(u_s,u_s)+A(u)(u_\theta,u_\theta)].$$

We develop the penultimate term using integration by parts:

$$\left|\int u_{\theta\theta}\cdot[A(u)(u_s,u_s)]\right| = \left|\int u_\theta\cdot[A(u)(u_s,u_s)]_\theta\right| \leq C_N\int |u_\theta|^2|u_s|^2 + |u_{s\theta}||u_s||u_\theta|$$

and then apply Young's inequality to estimate

$$\vartheta''(s) \geq (2-\frac{1}{4})\int_{\{s\}\times S^1} |u_{s\theta}|^2+|u_{\theta\theta}|^2 - C\int_{\{s\}\times S^1}|\tau|^2 - C_N\int_{\{s\}\times S^1}|u_\theta|^2|u_s|^2+|u_\theta|^4. \tag{3.9}$$

Let us now assume that s_0 is not too close to the ends of the cylinder, more precisely that $|s_0| \leq X(\ell) - 1$, so that we can change variables ($\tilde{s} = s - s_0$) and write

$$\Theta(s_0) := \int \varphi^4(\tilde{s})\vartheta(\tilde{s} + s_0)d\tilde{s}.$$

Differentiating twice and using (3.9), we find that

$$\begin{aligned}\Theta''(s_0) &= \int \varphi^4(\tilde{s})\vartheta''(\tilde{s} + s_0)d\tilde{s} = \int \varphi^4(s - s_0)\vartheta''(s)ds \\ &\geq \int \varphi^4(s - s_0)\left((2 - \frac{1}{4})(|u_{s\theta}|^2 + |u_{\theta\theta}|^2) - C|\tau|^2 - C_N\left(|u_\theta|^2|u_s|^2 + |u_\theta|^4\right)\right)d\theta ds.\end{aligned} \tag{3.10}$$

Considering now s_0 to be fixed, we set $\tilde{\varphi}(s) := \varphi(s - s_0)$. Allowing C to depend on N, we can then write

$$\Theta''(s_0) \geq \int \tilde{\varphi}^4\left((2 - \frac{1}{4})(|u_{s\theta}|^2 + |u_{\theta\theta}|^2) - C\left(|\tau|^2 + |u_\theta|^2|u_s|^2 + |u_\theta|^4\right)\right). \tag{3.11}$$

In order to control the final term, we use the Sobolev inequality over the entire collar to find that

$$\begin{aligned}\int \tilde{\varphi}^4|u_\theta|^4 &= \|\tilde{\varphi}^2|u_\theta|^2\|_{L^2}^2 \\ &\leq C\|\tilde{\varphi}^2|u_\theta|^2\|_{W^{1,1}}^2 \\ &\leq C\left[\left(\int \tilde{\varphi}^2|u_\theta|^2\right)^2 + \left(\int \tilde{\varphi}^2(|u_{s\theta}| + |u_{\theta\theta}|)|u_\theta|\right)^2 + \left(\int \tilde{\varphi}|u_\theta|^2\right)^2\right] \\ &\leq C\left[\left(\int_{spt\tilde{\varphi}} |u_\theta|^2\right)^2 + \left(\int_{spt\tilde{\varphi}} |u_\theta|^2\right)\left(\int \tilde{\varphi}^4(|u_{s\theta}|^2 + |u_{\theta\theta}|^2)\right)\right].\end{aligned} \tag{3.12}$$

In order to control the right-hand side in terms of Θ, we require some extra powers of $\tilde{\varphi}$ in the integrands. We achieve this by observing that if s is within the support of $\tilde{\varphi}$, then we have $s \in [s_0 - 1, s_0 + 1]$, and hence

$$1 \leq \tilde{\varphi}^4(s - \frac{1}{2}) + \tilde{\varphi}^4(s + \frac{1}{2}). \tag{3.13}$$

Therefore, assuming now that $|s_0| \leq X(\ell) - 3/2$ (to prevent the integrals falling off the end of $\mathcal{C}(\ell)$) we have

$$\int_{spt\tilde{\varphi}} |u_\theta|^2 \leq \Theta(s_0 - \frac{1}{2}) + \Theta(s_0 + \frac{1}{2}). \tag{3.14}$$

Applied to only one factor of $\int_{spt\tilde{\varphi}} |u_\theta|^2$ in (3.12), one consequence of this estimate is

$$\int \tilde{\varphi}^4|u_\theta|^4 \leq C\left(\Theta(s_0 - \frac{1}{2}) + \Theta(s_0 + \frac{1}{2})\right)\left[E(u;\mathscr{C}_1(s_0)) + \int_{\mathscr{C}_1(s_0)} (|u_{s\theta}|^2 + |u_{\theta\theta}|^2).\right] \tag{3.15}$$

Hence by Corollary 3.5, for arbitrary ε_2 (to be picked later) there exists $\Lambda_{\varepsilon_2} < \infty$ such that if $|s_0| \leq X(\ell) - \Lambda_{\varepsilon_2}$, then

$$\int \tilde{\varphi}^4 |u_\theta|^4 \leq C\varepsilon_2 \left(\Theta(s_0 - \frac{1}{2}) + \Theta(s_0 + \frac{1}{2})\right) + C_{\varepsilon_2} \|\tau(u)\|^2_{L^2(\mathscr{C}_{\Lambda_{\varepsilon_2}}(s_0))} \tag{3.16}$$

where C is not allowed to depend on ε_2, but C_{ε_2} is. The estimate above will be used to control the final term in (3.11). We now wish to control the penultimate term, and thus estimate, using again (3.12), but now applying (3.14) to all factors of $\int_{spt\tilde{\varphi}} |u_\theta|^2$

$$\begin{aligned}
\int \tilde{\varphi}^4 |u_\theta|^2 |u_s|^2 &\leq \left(\int \tilde{\varphi}^4 |u_\theta|^4\right)^{\frac{1}{2}} \left(\int \tilde{\varphi}^4 |u_s|^4\right)^{\frac{1}{2}} \\
&\leq C \left(\Theta(s_0 - \frac{1}{2}) + \Theta(s_0 + \frac{1}{2})\right) \left(\int \tilde{\varphi}^4 |u_s|^4\right)^{\frac{1}{2}} \\
&\quad + C \left(\Theta(s_0 - \frac{1}{2}) + \Theta(s_0 + \frac{1}{2})\right)^{\frac{1}{2}} \left(\int \tilde{\varphi}^4 |u_s|^4\right)^{\frac{1}{2}} \left(\int \tilde{\varphi}^4 (|u_{s\theta}|^2 + |u_{\theta\theta}|^2)\right)^{\frac{1}{2}} \\
&\leq C \left(\Theta(s_0 - \frac{1}{2}) + \Theta(s_0 + \frac{1}{2})\right) \left[\left(\int \tilde{\varphi}^4 |u_s|^4\right)^{\frac{1}{2}} + \left(\int \tilde{\varphi}^4 |u_s|^4\right)\right] \\
&\quad + \frac{1}{4} \int \tilde{\varphi}^4 (|u_{s\theta}|^2 + |u_{\theta\theta}|^2).
\end{aligned} \tag{3.17}$$

To control the part in square brackets, we use Young's inequality to estimate

$$\left(\int \tilde{\varphi}^4 |u_s|^4\right)^{\frac{1}{2}} \leq \sqrt{\varepsilon_2} + \frac{1}{\sqrt{\varepsilon_2}} \int \tilde{\varphi}^4 |u_s|^4$$

(with the same value of ε_2 as above, still to be chosen) and then use Corollary 3.5 to control

$$\int \tilde{\varphi}^4 |u_s|^4 \leq \int_{\mathscr{C}_1(s_0)} |u_s|^4 \leq \varepsilon_2 + C_{\varepsilon_2} \|\tau(u)\|^2_{L^2(\mathscr{C}_{\Lambda_{\varepsilon_2}}(s_0))},$$

Λ_{ε_2} as before, and we again require $|s_0| \leq X(\ell) - \Lambda_{\varepsilon_2}$ so that $\mathscr{C}_{\Lambda_{\varepsilon_2}}(s_0) \subset \mathcal{C}(\ell)$. Therefore (3.17) becomes

$$\begin{aligned}
\int \tilde{\varphi}^4 |u_\theta|^2 |u_s|^2 &\leq C \left(\Theta(s_0 - \frac{1}{2}) + \Theta(s_0 + \frac{1}{2})\right) \left[\sqrt{\varepsilon_2} + C_{\varepsilon_2} \|\tau(u)\|^2_{L^2(\mathscr{C}_{\Lambda_{\varepsilon_2}}(s_0))}\right] \\
&\quad + \frac{1}{4} \int \tilde{\varphi}^4 (|u_{s\theta}|^2 + |u_{\theta\theta}|^2) \\
&\leq C\sqrt{\varepsilon_2} \left(\Theta(s_0 - \frac{1}{2}) + \Theta(s_0 + \frac{1}{2})\right) + C_{\varepsilon_2} \|\tau(u)\|^2_{L^2(\mathscr{C}_{\Lambda_{\varepsilon_2}}(s_0))} \\
&\quad + \frac{1}{4} \int \tilde{\varphi}^4 (|u_{s\theta}|^2 + |u_{\theta\theta}|^2)
\end{aligned} \tag{3.18}$$

where C_{ε_2} is allowed to depend on both E_0 and ε_2 but again, C does not depend on ε_2. Combining (3.11) with our estimates (3.16) and (3.18), and choosing $\varepsilon_2 > 0$ sufficiently small, gives us

$$\begin{aligned}
\Theta''(s_0) &+ \frac{1}{8} \left(\Theta(s_0 - \frac{1}{2}) + \Theta(s_0 + \frac{1}{2})\right) \\
&\geq \int \tilde{\varphi}^4 \left((2 - \frac{1}{2})(|u_{s\theta}|^2 + |u_{\theta\theta}|^2)\right) - C\|\tau(u)\|^2_{L^2(\mathscr{C}_{\Lambda_{\varepsilon_2}}(s_0))},
\end{aligned} \tag{3.19}$$

and by Wirtinger's inequality (i.e. the Poincaré inequality in one dimension)

$$\int_{\{s\}\times S^1} |u_{\theta\theta}|^2 \geq \int_{\{s\}\times S^1} |u_\theta|^2,$$

and so

$$\Theta''(s_0) + \frac{3}{2}\Theta(s_0) + \frac{1}{8}\left(\Theta(s_0 - \frac{1}{2}) + \Theta(s_0 + \frac{1}{2})\right) \geq -C\|\tau(u)\|^2_{L^2(\mathscr{C}_{\Lambda_{\varepsilon_2}}(s_0))}. \tag{3.20}$$

Now ε_2, and hence $\Lambda = \Lambda_{\varepsilon_2}$ has been fixed, we choose $\delta > 0$ sufficiently small so that whenever $s_0 \in (-X_\delta, X_\delta)$ we automatically have $|s_0| \leq X(\ell) - \Lambda$. Note that unless $\ell > 0$ is small enough, the claim will be vacuous because the collar $\mathcal{C}(\ell)$ will contain no points of injectivity radius less than δ and that for $s \in \mathscr{C}_\Lambda(s_0)$, we have

$$\rho(s) \leq C\rho(s_0),$$

thanks to (A.5) so that

$$\|\tau(u)\|^2_{L^2(\mathscr{C}_\Lambda(s_0))} = \|\rho\,\tau_g(u)\|^2_{L^2(\mathscr{C}_\Lambda(s_0),g)} \leq \rho(s_0)^2\|\tau_g(u)\|^2_{L^2(\mathscr{C}_\Lambda(s_0),g)},$$

g the hyperbolic metric on the collar, from which we conclude the lemma. □

Equipped with Proposition 3.6, we are finally in a position to prove the main estimate for the weighted angular energy.

Proof of Lemma 3.1. For the given, N, E_0 and map u, apply Proposition 3.6 to give C, Λ and δ such that

$$\int_{s_0-\frac{1}{2}}^{s_0+\frac{1}{2}}\int_{S^1} |u_\theta|^2 d\theta ds \leq C\cdot e^{-(X(\ell)-|s_0|)} + C\cdot\int_{-X_\delta}^{X_\delta} e^{-|s-s_0|}\rho^2(s)\cdot\|\tau_g(u)\|^2_{L^2(\mathscr{C}_\Lambda(s),g)}ds$$

for all $s_0 \in (-X_\delta, X_\delta)$. Multiplying by $\rho^{-2}(s_0)$ and integrating, we find that

$$\begin{aligned}\int_{-X_\delta}^{X_\delta}\int_{S^1}\rho^{-2}(s)|u_\theta|^2(s,\theta)d\theta ds &\leq C\int_{-X_\delta}^{X_\delta}\rho^{-2}(s_0)\int_{s_0-1/2}^{s_0+1/2}\int_{S^1}|u_\theta|^2(s,\theta)d\theta\,ds\,ds_0\\ &\leq C\int_{-X_\delta}^{X_\delta}\rho^{-2}(s_0)e^{-(X(\ell)-|s_0|)}ds_0\\ &\quad + C\int_{-X_\delta}^{X_\delta}\int_{-X_\delta}^{X_\delta} e^{-|s-s_0|}\rho^{-2}(s_0)\rho^2(s)\|\tau_g(u)\|^2_{L^2(\mathscr{C}_\Lambda(s),g)}ds\,ds_0,\end{aligned} \tag{3.21}$$

where we have appealed to (A.5) to see that for $s \in (s_0 - \frac{1}{2}, s_0 + \frac{1}{2})$, we have $\rho^{-2}(s) \leq C\rho^{-2}(s_0)$, or equivalently $\rho(s_0) \leq C\rho(s)$.

Claim: For any $q \in [-X(\ell), X(\ell)]$, the function $f(s) := \rho^{-2}(s)e^{-2|s-q|/3}$ for $s \in [-X(\ell), X(\ell)]$ is maximised when $s = q$. That is,

$$\rho^{-2}(s)e^{-2|s-q|/3} \leq \rho^{-2}(q). \tag{3.22}$$

To see the claim, take logarithms, and differentiate, estimating using (A.4) that for $s < q$ we have $(\log f)' \geq -\frac{2}{\pi} + \frac{2}{3} > 0$, whereas for $s > q$ we have $(\log f)' \leq +\frac{2}{\pi} - \frac{2}{3} < 0$ as required.

Now, the first of the final two terms on the right-hand side of (3.21) is bounded, independently of ℓ, since by the claim we have

$$\rho^{-2}(s_0)e^{-(X(\ell)-|s_0|)} \leq \rho^{-2}(X(\ell))e^{-(X(\ell)-|s_0|)/3} \leq \pi^2 e^{-(X(\ell)-|s_0|)/3}$$

for $\ell \in (0, 2\,\mathrm{arsinh}(1))$ by (A.2).

To handle the final term on the right-hand side of (3.21), we can apply the claim again to find that

$$\int_{-X_\delta}^{X_\delta} e^{-|s-s_0|}\rho^{-2}(s_0)ds_0 \le \rho^{-2}(s)\int_{-X_\delta}^{X_\delta} e^{-|s-s_0|/3}ds_0 \le C\rho^{-2}(s),$$

and hence we can improve (3.21) to

$$\begin{aligned}\int_{-X_\delta}^{X_\delta}\int_{S^1}\rho^{-2}(s)|u_\theta|^2(s,\theta)d\theta ds &\le C + C\int_{-X_\delta}^{X_\delta}\|\tau_g(u)\|^2_{L^2(\mathscr{C}_\Lambda(s),g)}ds \\ &\le C\left(1+\|\tau_g(u)\|^2_{L^2(\mathcal{C}(\ell),g)}\right),\end{aligned} \tag{3.23}$$

because $\mathscr{C}_\Lambda(s) \subset \mathcal{C}(\ell)$ for all $s \in (-X_\delta, X_\delta)$.

To complete the proof, note that on the δ-thick part of the collar, described by $\{(s,\theta) : |s| \in [X_\delta, X(\ell))\}$, the weight function is bounded by $\rho^{-2} \le \rho^{-2}(X_\delta) \le \pi^2\delta^{-2}$, compare (A.6), so we may estimate

$$\left(\int_{-X(\ell)}^{-X_\delta} + \int_{X_\delta}^{X(\ell)}\right)\int_{S^1}\rho^{-2}(s)|u_\theta|^2(s,\theta)d\theta ds \le 2\pi^2\delta^{-2}E(u)$$

in terms of the total energy $E(u) \le E_0$.

Combining this with (3.23) completes the proof. □

4 Paths in Teichmüller space

Our goal in this section is to prove Lemma 2.2. The main issue is to understand in detail the structure of the space of holomorphic quadratic differentials as the underlying metric degenerates by pinching one or more collars. This topic has been addressed by many other authors (see for example [25]). Our approach is particularly adapted to get refined estimates that help us understand the projection P_g. We assume the Collar lemma A.1.

4.1 Structure of the space of holomorphic quadratic differentials

We will need to understand the properties of holomorphic quadratic differentials on hyperbolic surfaces, in particular in regions where the injectivity radius $\mathrm{inj}_g(p)$ is small. One fundamental fact of hyperbolic surface theory, cf. [7], Proposition IV.4.2, is that for any $0 < \delta < \mathrm{arsinh}(1)$, the δ-thin part of the surface, consisting of all points at which the injectivity radius is less than δ, is given by a finite union of disjoint hyperbolic cylinders of finite length around closed geodesics, which are explicitly described by the Collar lemma A.1 of Keen-Randol. (Recall that these closed geodesics have length ℓ, which we assume always to be less than $2\,\mathrm{arsinh}(1)$.) Away from these collars, and more generally on the subset δ-thick(M,g) of points p with injectivity radius $\mathrm{inj}_g(p) \ge \delta$, where $\delta > 0$, holomorphic quadratic differentials are well controlled. The most basic estimate, following from standard estimates for holomorphic functions on discs, is that for $\Phi \in \mathcal{H}(M,g)$, $p \in [1,\infty)$ and $\delta > 0$, we have

$$\|\Phi\|_{L^\infty(\delta\text{-thick}(M,g))} \le C_\delta\|\Phi\|_{L^p(M,g)} \tag{4.1}$$

with C_δ depending only on δ, cf. Lemma A.8 in [14]. While this estimate is sufficient for most of the paper, it is useful at times to observe that indeed

$$\|\Phi\|_{L^\infty(\delta\text{-thick}(M,g))} \leq C_\delta \|\Phi\|_{L^p(\frac{\delta}{2}\text{-thick}(M,g))} \tag{4.2}$$

for any $0 < \delta < \operatorname{arsinh}(1)$, still with a constant C_δ depending only on δ.

The Fourier decomposition of holomorphic quadratic differentials Φ on each hyperbolic collar $(\mathcal{C}, g)$

$$\Phi = \left(\sum_{n=-\infty}^{\infty} b_n e^{ns} e^{in\theta} \right) \cdot dz^2, \quad b_n \in \mathbb{C} \tag{4.3}$$

where $dz = ds + id\theta$ as before, gives an $L^2(\mathcal{C}, g)$-orthogonal decomposition of each such Φ into its principal part $b_0 dz^2 = b_0(\Phi)dz^2 = b_0(\Phi, \mathcal{C})dz^2$ and its *collar decay* part $\omega^\perp(\Phi, \mathcal{C}) := \Phi - b_0(\Phi)dz^2$ which, by [14, Lemma 2.2], satisfies the key estimate

$$\|\omega^\perp(\Phi, \mathcal{C})\|_{L^\infty(\delta\text{-thin}(\mathcal{C},g))} \leq C\delta^{-2} e^{-\frac{\pi}{\delta}} \|\omega^\perp(\Phi, \mathcal{C})\|_{L^2(\delta_0\text{-thick}(\mathcal{C},g))}, \tag{4.4}$$

for some universal constants $C < \infty$ and $\delta_0 \in (0, \operatorname{arsinh}(1))$, and any $\delta \in (0, \delta_0]$. (Here we are writing $\delta\text{-thin}(\mathcal{C}, g) := \mathcal{C} \cap \delta\text{-thin}(M, g)$, and similarly for δ-thick.)

On the other hand, when $\Phi \in \mathcal{H}(M, g)$ we can also control the right-hand side of (4.4), and even the L^∞ norm, as follows. Since we have made an orthogonal decomposition of Φ, we know that $\langle \Phi, dz^2 \rangle_{L^2(\mathcal{C},g)} = b_0(\Phi)\|dz^2\|^2_{L^2(\mathcal{C})}$, and hence

$$|b_0(\Phi, \mathcal{C})| \leq \frac{\|\Phi\|_{L^1(\mathcal{C})}\|dz^2\|_{L^\infty(\mathcal{C})}}{\|dz^2\|^2_{L^2(\mathcal{C})}} \leq C\ell\|\Phi\|_{L^1(\mathcal{C})},$$

for universal C, by (A.9).

Consequently, noting that, by (A.6), $\|dz^2\|_{L^\infty(\delta_0\text{-thick}(\mathcal{C},g))} \leq 2\pi^2\delta_0^{-2}$ is bounded above independently of $\ell \in (0, 2\operatorname{arsinh}(1))$, and recalling (4.1), we have by the triangle inequality

$$\begin{aligned} \|\omega^\perp(\Phi, \mathcal{C})\|_{L^\infty(\delta_0\text{-thick}(\mathcal{C},g))} &\leq \|\Phi\|_{L^\infty(\delta_0\text{-thick}(\mathcal{C},g))} + \|b_0(\Phi)dz^2\|_{L^\infty(\delta_0\text{-thick}(\mathcal{C},g))} \\ &\leq C\|\Phi\|_{L^1(M,g)} \end{aligned} \tag{4.5}$$

for universal $C < \infty$. Using the fact that collars have uniformly bounded area as $\ell \downarrow 0$ (see [14, Lemma A.5]), the norm on the right-hand side of (4.4) is controlled by the left-hand side of (4.5), and we find that

$$\|\omega^\perp(\Phi, \mathcal{C})\|_{L^\infty(\delta\text{-thin}(\mathcal{C},g))} \leq C\delta^{-2} e^{-\frac{\pi}{\delta}} \|\Phi\|_{L^1(M,g)}, \tag{4.6}$$

for universal $C < \infty$, and any $\delta \in (0, \delta_0]$. Thus, deep within a collar, only the principal part of a holomorphic quadratic differential is significant.

In the special case that $\delta = \delta_0$, estimate (4.6) can be added to (4.5) to give a bound on the whole collar of

$$\|\omega^\perp(\Phi, \mathcal{C})\|_{L^\infty(\mathcal{C},g)} \leq C\|\Phi\|_{L^1(M,g)}, \tag{4.7}$$

for a universal constant $C < \infty$.

Remark that the above estimates could also be derived based on the observation that $\omega^\perp$ and $b_0 dz^2$ are orthogonal also on subcylinders of the collar, in particular on the δ_0-thick-part. Indeed, combined with (4.4) and (4.2) this gives the slight improvement of (4.6) that

$$\|\omega^\perp(\Phi, \mathcal{C})\|_{L^\infty(\delta\text{-thin}(\mathcal{C},g))} \leq C\delta^{-2} e^{-\frac{\pi}{\delta}} \|\Phi\|_{L^1(\frac{\delta_0}{2}\text{-thick}(M,g))}, \tag{4.8}$$

still with universal C and any $\delta \in (0, \delta_0]$, which will be needed in the proof of Lemma 4.5 later on.

Following [14], given a closed hyperbolic surface (M, g) on which we identify k collars $\mathcal{C}_1, \ldots, \mathcal{C}_k$, we can then define the subspace

$$W := \{\Theta \in \mathcal{H}(M, g) : b_0(\Theta, \mathcal{C}_j)dz^2 = 0 \text{ for every } j \in \{1 \ldots k\}\} \tag{4.9}$$

of all holomorphic quadratic differentials with principal part equal to zero on each of the k collars.

Remark 4.1. When we return to viewing the space of holomorphic quadratic differentials $\mathcal{H}(M, g)$ on (M, g) as tangent vectors in $\mathcal{M}_{-1}$ at g, via the isomorphism $\Phi \mapsto Re(\Phi)$, the subspace W will have a simple interpretation. Indeed, if the lengths of the geodesics at the centre of the collars $\mathcal{C}_1, \ldots, \mathcal{C}_k$ are given as $\ell_1, \ldots, \ell_k$ near to g in $\mathcal{M}_{-1}$, then W will represent the kernel of $(\partial\ell_1, \ldots, \partial\ell_k)$ within $\mathcal{H}(M, g)$. See also Remarks 4.11 and 4.8.

The following lemma is part of [14, Lemma 2.4].

Lemma 4.2 (Decomposition of $\mathcal{H}(M, g_n)$: Definition of W_n)**.** *Given a closed surface M of genus $\gamma \geq 2$, there exists a universal constant C such that the following is true. Suppose g_n is a sequence of hyperbolic metrics on M, and apply the differential geometric form of the Deligne-Mumford compactness theorem A.3 in order to pass to a subsequence and obtain precisely k collars $\mathcal{C}_n^j$ degenerating. After omitting finitely many terms, the subspace*

$$W_n := \left\{\Theta \in \mathcal{H}(M, g_n) : b_0(\Theta, \mathcal{C}_n^j)dz^2 = 0 \textit{ for every } j \in \{1 \ldots k\}\right\} \tag{4.10}$$

of holomorphic quadratic differentials that have vanishing principal part on every collar $\mathcal{C}_n^j$, $j \in \{1 \ldots k\}$, is of (complex) dimension $3(\gamma - 1) - k$, and elements $w \in W_n$ decay rapidly along the collars in the sense that for sufficiently small $\delta > 0$ (so that the δ-thin part of (M, g_n) lies within $\cup_{j=1}^k \mathcal{C}_n^j$) independent of n, we have

$$\|w\|_{L^\infty(\delta\text{-}thin(M,g_n))} \leq C \cdot \delta^{-2} e^{-\pi/\delta} \|w\|_{L^2(M,g_n)}. \tag{4.11}$$

Furthermore, the spaces W_n converge to $\mathcal{H}(\Sigma, h)$, the space of integrable holomorphic quadratic differentials on the noncompact limit surface, in the sense that for every $w \in \mathcal{H}(\Sigma, h)$ there exists a sequence $w_n \in W_n$ with

$$f_n^* w_n \to w \textit{ smoothly locally on } \Sigma \textit{ and } \|w\|_{L^2(\Sigma,h)} = \lim_{n\to\infty} \|w_n\|_{L^2(M,g_n)}. \tag{4.12}$$

Note that (4.11) also follows from (4.6), albeit with a constant C now depending on the genus. Note also that a combination of (4.11) and (4.1) tells us that we can also control

$$\|w\|_{L^\infty(M,g_n)} \leq C \cdot \|w\|_{L^2(M,g_n)}, \tag{4.13}$$

for all $w \in W_n$, or, via (4.6), alternatively with the L^1 norm on the right-hand side (with C depending only on the $\liminf_{n\to\infty}$ of the injectivity radius on $M \backslash \cup_{j=1}^k \mathcal{C}_n^j$, i.e. with C independent of n).

Corollary 4.3. *In the setting of Lemma 4.2, there exists a constant $C < \infty$ such that for sufficiently large n, and for any $\Psi \in \mathcal{Q}_{L^2}(M, g_n)$, we have*

$$\|P_{g_n}^{W_n}(\Psi)\|_{L^\infty(M,g_n)} \leq C\|\Psi\|_{L^1(M,g_n)}. \tag{4.14}$$

Proof. For sufficiently large n, (4.13) implies

$$\|P_{g_n}^{W_n}(\Psi)\|_{L^\infty(M,g_n)} \leq C\|P_{g_n}^{W_n}(\Psi)\|_{L^2(M,g_n)}.$$

On the other hand, by definition,

$$\|P_{g_n}^{W_n}(\Psi)\|_{L^2(M,g_n)}^2 = \langle \Psi, P_{g_n}^{W_n}(\Psi)\rangle \leq \|\Psi\|_{L^1(M,g_n)}\|P_{g_n}^{W_n}(\Psi)\|_{L^\infty(M,g_n)}.$$

Combining these two estimates gives the result. □

Before we discuss the structure of the L^2-orthogonal complement $W_n^\perp$, we briefly discuss the size of both the principal and the collar decay part of holomorphic quadratic differentials on the δ-*thick* part of a collar, for *fixed* $\delta \in (0, \text{arsinh}(1))$. Since we have made an orthogonal decomposition we know that for any $\Phi \in \mathcal{H}(M,g)$ and any collar $\mathcal{C} = \mathcal{C}(\ell)$ in (M,g), we have

$$|b_0(\Phi,\mathcal{C})| \leq \|dz^2\|_{L^2(\mathcal{C},g)}^{-1}\|\Phi\|_{L^2(M,g)} = \left(\frac{32\pi^5}{\ell^3} - \frac{16\pi^4}{3} + O(\ell^2)\right)^{-1/2}\|\Phi\|_{L^2(M,g)}, \quad (4.15)$$

by (A.9). Since $|dz^2| = 2\rho^{-2}$ is bounded above (and below) on δ-thick$(\mathcal{C},g)$ independently of ℓ, this means in particular that for any L^2-unit element $\Phi \in \mathcal{H}(M,g)$ we have

$$|b_0(\Phi,\mathcal{C})dz^2| \leq C\ell^{3/2} \quad (4.16)$$

on the δ-thick part of the collar, where C depends only on δ.

Meanwhile, on this thick part of $\mathcal{C}(\ell)$ the *collar decay* part of a unit element can be of order 1, i.e. huge compared with the principal part.

The following lemma guarantees that such a difference in size does not occur for elements of the orthogonal complement of W_n; we obtain a basis of $W_n^\perp$ with each element concentrated on one of the degenerating collars and there having principal part almost as large as it can be according to (4.15), and with the remaining collar decay part now only of order $O(\ell^{3/2})$, i.e. of the same order as the principal part on the thick part of the surface.

Lemma 4.4 (Decomposition of $\mathcal{H}(M,g_n)$: Decomposition of $W_n^\perp$)**.** *Suppose we are in the setting of Lemma 4.2, and have identified the subspace W_n of complex dimension $3(\gamma-1)-k$. Then there exist a constant $C \in (0,\infty)$, and for all $\delta > 0$ a further constant $C_\delta \in (0,\infty)$, such that for sufficiently large n we can find an L^2-orthonormal basis $\{\Omega_n^j\}$, $1 \leq j \leq k$ of the L^2-orthogonal complement $W_n^\perp$ of W_n in $\mathcal{H}(M,g_n)$, with one element concentrated on each collar $\mathcal{C}_n^j$ in the sense that for each $j \in \{1,\ldots,k\}$,*

$$\|\Omega_n^j\|_{L^\infty(M\setminus\mathcal{C}_n^j,g_n)} \leq C(\ell_n^j)^{3/2}, \quad (4.17)$$

with the principal parts on the other collars $\mathcal{C}_n^i$ controlled by the stronger bounds

$$|b_0(\Omega_n^j,\mathcal{C}_n^i)| \leq C(\ell_n^j)^{3/2}(\ell_n^i)^3, \quad i \neq j. \quad (4.18)$$

On the one collar $\mathcal{C}_n^j$ where Ω_n^j concentrates it is essentially given as a constant multiple of dz^2 in the sense that

$$\|\Omega_n^j - \beta_n^j dz^2\|_{L^\infty(\mathcal{C}_n^j,g_n)} \leq C(\ell_n^j)^{3/2} \text{ where } \left|\frac{(\ell_n^j)^{3/2}}{(32\pi^5)^{1/2}} - \beta_n^j\right| \leq C(\ell_n^j)^{9/2}, \quad (4.19)$$

for $\beta_n^j dz^2 = b_0(\Omega_n^j,\mathcal{C}_n^j)dz^2$ the principal part on $\mathcal{C}_n^j$ and ℓ_n^j the length of the central geodesic in $(\mathcal{C}_n^j,g_n)$. Moreover, for each $j \in \{1,\ldots,k\}$, we have

$$\|\Omega_n^j\|_{L^1(M,g_n)} \leq C(\ell_n^j)^{1/2}, \quad (4.20)$$

$$\|\Omega_n^j\|_{L^\infty(M,g_n)} \leq C(\ell_n^j)^{-1/2}, \quad (4.21)$$

and

$$\|\Omega_n^j\|_{L^\infty(\delta\text{-thick}(M,g_n))} \leq C_\delta(\ell_n^j)^{3/2}. \quad (4.22)$$

This lemma will be derived from the following lemma which is an improvement of [15, Lemma 2.6 and Corollary 2.7] and which gives an explicit definition of basis elements concentrating on collars, but does not guarantee orthogonality

Lemma 4.5. *Suppose we are in the setting of Lemma 4.2, and have identified the subspace* W_n *of complex dimension* $3(\gamma-1)-k$*. Then there exist a constant* $C \in (0,\infty)$*, and for all* $\delta > 0$ *a further constant* $C_\delta \in (0,\infty)$*, such that for sufficiently large* n*, the* L^2*-unit holomorphic quadratic differentials* $\widetilde{\Omega}_n^1, \dots, \widetilde{\Omega}_n^k \in W_n^\perp$ *characterised by*

$$b_0(\widetilde{\Omega}_n^j, \mathcal{C}_n^i) = 0 \text{ for every } i \neq j \text{ while } b_0(\widetilde{\Omega}_n^j, \mathcal{C}_n^j) > 0 \tag{4.23}$$

concentrate on the respective collars $\mathcal{C}_n^1, \dots, \mathcal{C}_n^k$ *in the sense that for each* $j \in \{1, \dots, k\}$*, we have*

$$\|\widetilde{\Omega}_n^j\|_{L^\infty(M \setminus \mathcal{C}_n^j, g_n)} \leq C(\ell_n^j)^{3/2}, \tag{4.24}$$

while on this collar $\widetilde{\Omega}_n^j$ *is essentially given as a constant multiple of* dz^2 *in the sense that*

$$\|\widetilde{\Omega}_n^j - \tilde{\beta}_n^j dz^2\|_{L^\infty(\mathcal{C}_n^j, g_n)} \leq C(\ell_n^j)^{3/2}, \tag{4.25}$$

for the principal part $\tilde{\beta}_n^j dz^2 = b_0(\widetilde{\Omega}_n^j, \mathcal{C}_n^j)dz^2$ *of* $\widetilde{\Omega}_n^j$ *satisfying*

$$1 - C(\ell_n^j)^3 \leq \tilde{\beta}_n^j \|dz^2\|_{L^2(\mathcal{C}_n^j, g_n)} \leq 1. \tag{4.26}$$

Moreover, the elements $\widetilde{\Omega}_n^j$ *are almost orthogonal,*

$$\langle \widetilde{\Omega}_n^j, \widetilde{\Omega}_n^i \rangle_{L^2(M,g)} \leq C(\ell_n^j)^{3/2}(\ell_n^i)^{3/2}, \qquad i \neq j \tag{4.27}$$

and for each $j \in \{1, \dots, k\}$*, we have*

$$\|\widetilde{\Omega}_n^j\|_{L^1(M,g_n)} \leq C(\ell_n^j)^{1/2}, \tag{4.28}$$

$$\|\widetilde{\Omega}_n^j\|_{L^\infty(M,g_n)} \leq C(\ell_n^j)^{-1/2}, \tag{4.29}$$

and

$$\|\widetilde{\Omega}_n^j\|_{L^1(\delta\text{-thick}(M,g_n))} \leq C_\delta(\ell_n^j)^{3/2}. \tag{4.30}$$

Remark 4.6. Note that by virtue of (A.9) and (4.26), the constants $\tilde{\beta}_n^j$ are constrained by

$$\left| \tilde{\beta}_n^j - \frac{(\ell_n^j)^{3/2}}{(32\pi^5)^{1/2}} \right| \leq C(\ell_n^j)^{9/2}, \tag{4.31}$$

in particular

$$\left\| \widetilde{\Omega}_n^j - \frac{(\ell_n^j)^{3/2}}{(32\pi^5)^{1/2}} dz^2 \right\|_{L^\infty(\mathcal{C}_n^j, g_n)} \leq C(\ell_n^j)^{3/2}. \tag{4.32}$$

Remark 4.7. With more effort, the dependencies of the constants C, C_δ in Lemma 4.5 can be dramatically improved, although we do not require such refinements in this paper. For example, by working directly, one can show that the constants in (4.28) and (4.29) can be chosen to be universal (cf. [15]). Improved dependencies in Lemma 4.5 then yield improved dependencies in Lemma 4.4.

Remark 4.8. For n so large that the collars have been almost pinched, the basis $\{\Omega_n^j\}$ of $W_n^\perp$ will be very similar to the basis $\{\widetilde{\Omega}_n^j\}$. A further similar basis in this limit, see Wolpert [25], can be given in terms of the scaled Weil-Petersson gradients $\{-\sqrt{\frac{\pi}{2\ell_n^j}}\nabla \ell_n^j\}$. In contrast, our basis $\{\widetilde{\Omega}_n^j\}$ can be viewed as the dual basis to $\{\partial \ell_n^j\}$, with each element scaled to unit length. See also Remarks 4.11 and 4.1.

Remark 4.9. Although we will not need them, we remark that based on (4.25) and (4.24) we get refinements of (4.24) and (4.25) at the *central geodesics* of a collar. That is, for $i \neq j$, we have

$$\|\widetilde{\Omega}_n^j\|_{L^\infty(\sigma_n^i, g_n)} \leq C(\ell_n^j)^{3/2} \left((\ell_n^i)^{-2} e^{-\frac{2\pi}{\ell_n^i}} \right), \tag{4.33}$$

and for each i,

$$\|\widetilde{\Omega}_n^i - \tilde{\beta}_n^i dz^2\|_{L^\infty(\sigma_n^i, g_n)} \leq C(\ell_n^i)^{-1/2} e^{-\frac{2\pi}{\ell_n^i}}. \tag{4.34}$$

Proof of Lemma 4.5. As Lemma 4.2 has told us, after having omitted finitely many terms, the subspace $W_n^\perp$ has dimension k. Therefore, for each fixed $j \in \{1, \dots, k\}$, there is a unique element $\widetilde{\Omega}_n^j$ of $W_n^\perp$ with $\|\widetilde{\Omega}_n^j\|_{L^2(M,g_n)} = 1$ satisfying (4.23). The key step in the proof of Lemma 4.5 is to show that claim (4.30) holds true for this basis of $W_n^\perp$. Assume instead that there exists a number $\bar{\delta} > 0$ such that after passing to a subsequence

$$\Lambda_n := (\ell_n^j)^{-3/2} \|\widetilde{\Omega}_n^j\|_{L^1(\bar{\delta}\text{-thick}(M,g_n))} \to \infty \quad \text{ as } n \to \infty, \tag{4.35}$$

for some $1 \leq j \leq n$, say for $j = 1$.

We now choose a sequence $\delta_n \in (0, \bar{\delta}]$ with $\delta_n \to 0$ so that still

$$\Lambda_n \cdot \delta_n \to \infty \text{ as } n \to \infty \tag{4.36}$$

and set

$$\lambda_n := \|\widetilde{\Omega}_n^1\|_{L^1(\delta_n\text{-thick}(M,g_n))} \geq \Lambda_n (\ell_n^1)^{3/2}. \tag{4.37}$$

We then consider the normalised sequence

$$\widehat{\Omega}_n := (\lambda_n)^{-1} \cdot \widetilde{\Omega}_n^1,$$

so

$$\|\widehat{\Omega}_n\|_{L^1(\delta_n\text{-thick}(M,g_n))} = 1, \tag{4.38}$$

and prove the following two claims which obviously contradict each other.

Claim 1: [Not everything disappears down the collar.] There exists a number $\delta > 0$ such that for all n sufficiently large

$$\|\widehat{\Omega}_n\|_{L^1(\delta\text{-thick}(M,g_n))} \geq \frac{1}{2}. \tag{4.39}$$

Claim 2: [Everything disappears down the collar.] After passing to a subsequence and pulling back by the diffeomorphisms $f_i : \Sigma \to M \setminus \cup_{j=1}^k \sigma_i^j$ given by the Deligne-Mumford compactness theorem A.3, we have

$$f_n^* \widehat{\Omega}_n \to 0 \text{ smoothly locally on } \Sigma, \tag{4.40}$$

which implies

$$\lim_{n\to\infty} \|\widehat{\Omega}_n\|_{L^1(\delta\text{-thick}(M,g_n))} = 0 \quad \text{ for every } \delta > 0.$$

Proof of Claim 1. Recall that for sufficiently small $\delta > 0$, the δ-thin part of (M, g_n) is given as a union of disjoint subsets of the degenerating collars $\mathcal{C}_n^i$. In view of the normalisation (4.38), we thus need to show that (after possibly reducing $\delta > 0$)

$$\sum_{i=1}^k \|\widehat{\Omega}_n\|_{L^1\left(\delta\text{-thin}(\mathcal{C}_n^i) \setminus \delta_n\text{-thin}(\mathcal{C}_n^i)\right)} \leq \frac{1}{2} \tag{4.41}$$

for all n sufficiently large. By definition, the principal part of Ω_n^1, and thus of $\widehat{\Omega}_n$, on $\mathcal{C}_n^i$ vanishes for all $i \neq 1$, so that (4.8) applies, resulting for n large in an estimate of

$$\begin{aligned}\|\widehat{\Omega}_n\|_{L^1(\delta\text{-thin}(\mathcal{C}_n^i)\setminus\delta_n\text{-thin}(\mathcal{C}_n^i))} &\leq C\|\widehat{\Omega}_n\|_{L^\infty(\delta\text{-thin}(\mathcal{C}_n^i))} \leq C\delta^{-2}e^{-\pi/\delta}\|\widehat{\Omega}_n\|_{L^1(\frac{\delta_0}{2}\text{-thick}(M,g_n))} \\ &\leq \frac{1}{4k}\|\widehat{\Omega}_n\|_{L^1(\delta_n\text{-thick}(M,g_n))} = \frac{1}{4k}\end{aligned} \tag{4.42}$$

provided $\delta > 0$ is initially chosen small enough. The same estimate is of course valid also for the collar decay part $\omega^\perp(\widehat{\Omega}_n, \mathcal{C}_n^1)$.

In order to prove (4.41) it thus remains to control the contribution of the principal part to the L^1 norm on δ-thin$(\mathcal{C}_n^1) \setminus \delta_n$-thin$(\mathcal{C}_n^1)$. Here we crucially use the assumption (4.35) which means that the principal part of $\widetilde{\Omega}_n^1$ is small compared to $\widetilde{\Omega}_n^1$ on the $\bar{\delta}$-thick part of the surface. More precisely, by the a priori bound (4.16) valid for all elements of $\mathcal{H}$, we know that

$$\frac{\|b_0(\widetilde{\Omega}_n^1, \mathcal{C}_n^1)dz^2\|_{L^1(\bar{\delta}\text{-thick}(\mathcal{C}_n^1))}}{\|\widetilde{\Omega}_n^1\|_{L^1(\bar{\delta}\text{-thick}(M,g_n))}} \leq \frac{C_{\bar{\delta}}(\ell_n^1)^{3/2}}{\Lambda_n(\ell_n^1)^{3/2}} \to 0 \text{ as } n \to \infty.$$

The relation (4.36) now guarantees that this property is preserved for our sequence $\delta_n \to 0$: Indeed, by Proposition A.2 and (A.9), we have

$$\|dz^2\|_{L^1(\delta_n\text{-thick}(\mathcal{C}_n^1))} = 8\pi(X(\ell_n^1) - X_{\delta_n}(\ell_n^1)) \leq C\delta_n^{-1},$$

and combining this estimate with (4.16), (4.36) and (4.37) yields

$$\frac{\|b_0(\widetilde{\Omega}_n^1, \mathcal{C}_n^1)dz^2\|_{L^1(\delta_n\text{-thick}(\mathcal{C}_n^1))}}{\|\widetilde{\Omega}_n^1\|_{L^1(\delta_n\text{-thick}(M,g_n))}} = \frac{|b_0(\widetilde{\Omega}_n^1)|.\|dz^2\|_{L^1(\delta_n\text{-thick}(\mathcal{C}_n^1))}}{\lambda_n} \leq \frac{C(\ell_n^1)^{3/2}}{\lambda_n\delta_n} \leq \frac{C}{\Lambda_n\delta_n} \to 0 \tag{4.43}$$

as $n \to \infty$. But the left-hand side is precisely the L^1 norm of the principal part of the normalised $\widehat{\Omega}_n$ on δ_n-thick$(\mathcal{C}_n^1)$, so that

$$\|\widehat{\Omega}_n\|_{L^1(\delta\text{-thin}(\mathcal{C}_n^i)\setminus\delta_n\text{-thin}(\mathcal{C}_n^i))} \to 0$$

is certainly less than $\frac{1}{4}$ for n large, completing the proof of (4.41), i.e. Claim 1. □

Proof of Claim 2. Observe that while the L^1-norms of $\widehat{\Omega}_n$ are in general unbounded, our normalisation (4.38) implies that for every $\delta > 0$

$$\limsup_{n\to\infty} \|\widehat{\Omega}_n\|_{L^1(\delta\text{-thick}(M,g_n))} \leq 1.$$

To begin with, we remark that such a bound turns out to be sufficient to extract a subsequence in n so that $f_n^*\widehat{\Omega}_n$ converges smoothly locally to a limit $\widehat{\Omega}_\infty \in \mathcal{H}(\Sigma, h)$, f_n the diffeomorphisms of the Deligne-Mumford compactness theorem A.3; indeed, as $f_n^*\widehat{\Omega}_n$ are holomorphic with respect to the complex structures $f_n^*c_n \to c$ their C^k norm on balls of a given radius are bounded in terms of their L^1 norms on slightly larger balls, resulting in (uniform in n) bounds of

$$\|f_n^*\widehat{\Omega}_n\|_{C^k(f_n^*(\delta\text{-thick}(M,g_n)))} \leq C_{k,\delta}\|\widehat{\Omega}_n\|_{L^1(\delta/2\text{-thick}(M,g_n))} \leq C_{k,\delta}$$

for every $\delta > 0$, $k \in \mathbb{N}$. Combined with the theorem of Arzela-Ascoli, this then allows us to extract a subsequence to give smooth local convergence $f_n^*\widehat{\Omega}_n \to \widehat{\Omega}_\infty$ to a holomorphic quadratic differential on Σ; for details of this argument we refer to Section A.3 of [14], in particular to Lemma A.9 and its proof. Since additionally

$$\|\widehat{\Omega}_\infty\|_{L^1(\delta\text{-thick}(\Sigma,h))} = \lim_{n\to\infty} \|\widehat{\Omega}_n\|_{L^1(\delta\text{-thick}(M,g_n))} \leq 1 \text{ for any } \delta > 0, \tag{4.44}$$

see [14, Lemma A.7], $\widehat{\Omega}_\infty$ is indeed an element of $\mathcal{H}(\Sigma,h)$, the space of all holomorphic quadratic differentials on the noncompact limit surface with finite $L^1(\Sigma,h)$ norm, or by [14, Lemma A.11] equivalently, with finite $L^2(\Sigma,h)$-norm. As we claim that $\widehat{\Omega}_\infty$ is identically zero we therefore need to prove that

$$\langle w, \widehat{\Omega}_\infty\rangle_{L^2(\Sigma,h)} = 0 \text{ for every } w \in \mathcal{H}(\Sigma,h). \tag{4.45}$$

Given $w \in \mathcal{H}(\Sigma,h)$ we let $w_n \in W_n$ be an approximating sequence as in (4.12) and use that, by definition, $\widehat{\Omega}_n$ is an element of $W_n^\perp$ so that for any $\delta > 0$

$$\begin{aligned}|\langle w, \widehat{\Omega}_\infty\rangle_{L^2(\Sigma,h)}| &\leq |\langle w, \widehat{\Omega}_\infty\rangle_{L^2(\delta\text{-thick}(\Sigma,h))}| + |\langle w, \widehat{\Omega}_\infty\rangle_{L^2(\delta\text{-thin}(\Sigma,h))}| \\ &\leq |\lim_{n\to\infty}\langle w_n, \widehat{\Omega}_n\rangle_{L^2(\delta\text{-thick}(M,g_n))}| + \|w\|_{L^2(\delta\text{-thin}(\Sigma,h))}\cdot\|\widehat{\Omega}_\infty\|_{L^2(\delta\text{-thin}(\Sigma,h))} \\ &= |\lim_{n\to\infty}\langle w_n, \widehat{\Omega}_n\rangle_{L^2(\delta\text{-thin}(M,g_n))}| + R(\delta)\end{aligned} \tag{4.46}$$

where we observe that $R(\delta) = \|w\|_{L^2(\delta\text{-thin}(\Sigma,h))}\cdot\|\widehat{\Omega}_\infty\|_{L^2(\delta\text{-thin}(\Sigma,h))} \to 0$ as $\delta \to 0$ since both w and $\widehat{\Omega}_\infty$ have finite $L^2(\Sigma,h)$ norm.

On the other hand, we recall that for $\delta > 0$ sufficiently small, the δ-thin part of (M,g_n) is given as the union of the δ-thin parts of the degenerating collars $\mathcal{C}_n^i$, and stress once more that the collar decay part of any holomorphic quadratic differential is L^2-orthogonal to dz^2 on arbitrary subcylinders of the collars. Since the principal part of w_n on *all* degenerating collars $\mathcal{C}_n^i$ is zero, the inner product in (4.46) is given only in terms of the collar decay parts of $\widehat{\Omega}_n$ so, by (4.8) and (4.11), for n sufficiently large (so that $\delta_n < \frac{\delta_0}{2}$)

$$\begin{aligned}|\langle w_n, \widehat{\Omega}_n\rangle_{L^2(\delta\text{-thin}(M,g_n))}| &\leq \sum_{i=1}^k \|w_n\|_{L^2(\delta\text{-thin}(\mathcal{C}_n^i))}\cdot\|\omega^\perp(\widehat{\Omega}_n,\mathcal{C}_n^i)\|_{L^2(\delta\text{-thin}(\mathcal{C}_n^i))} \\ &\leq C\big(\delta^{-2}e^{-\pi/\delta}\big)^2\sum_{i=1}^k \|w_n\|_{L^2(M,g_n)}\cdot\|\widehat{\Omega}_n\|_{L^1(\frac{\delta_0}{2}\text{-thick}(M,g_n))} \\ &\leq C\big(\delta^{-2}e^{-\pi/\delta}\big)^2.\end{aligned} \tag{4.47}$$

Combined with (4.46) we thus find that

$$|\langle w, \widehat{\Omega}_\infty\rangle_{L^2(\Sigma,h)}| \leq R(\delta) + C\delta^{-4}e^{-2\pi/\delta} \to 0 \text{ as } \delta \to 0$$

so that the limit $\widehat{\Omega}_\infty$ obtained above must be zero, proving not only (4.40), but at the same time, by (4.44), also the claimed L^1-bounds. □

This completes the proof of the key estimate (4.30).

As for the remaining parts, (4.24) follows from (4.30), (4.2) and (4.8), if we choose $\delta \in (0,\delta_0]$ to be sufficiently small so that the δ-thin part of (M,g_n) is contained within $\cup_{j=1}^k \mathcal{C}_n^j$ for sufficiently large n.

Estimate (4.25) follows from (4.8), (4.30) and (4.31).

To establish (4.26), we compute

$$\begin{aligned}1 = \|\widetilde{\Omega}_n^j\|^2_{L^2(M,g_n)} &= \|\widetilde{\Omega}_n^j\|^2_{L^2(M\setminus\mathcal{C}_n^j,g_n)} + \|\widetilde{\Omega}_n^j\|^2_{L^2(\mathcal{C}_n^j,g_n)} \\ &= \|\widetilde{\Omega}_n^j\|^2_{L^2(M\setminus\mathcal{C}_n^j,g_n)} + \|\tilde{\beta}_n^j dz^2\|^2_{L^2(\mathcal{C}_n^j,g_n)} + \|\widetilde{\Omega}_n^j - \tilde{\beta}_n^j dz^2\|^2_{L^2(\mathcal{C}_n^j,g_n)}.\end{aligned} \tag{4.48}$$

This implies the second inequality of (4.26), but also, when combined with (4.24) and (4.25), it gives

$$1 - \|\tilde{\beta}^j_n dz^2\|^2_{L^2(\mathcal{C}^j_n, g_n)} \le C(\ell^j_n)^3.$$

To show (4.27) we use the orthogonality of the principal and collar decay parts as well as that $b_0(\widetilde{\Omega}^i_n, \mathcal{C}^j_n) = 0$ for $i \neq j$ to see that the inner product depends only on terms that are small according to (4.25) and (4.24), namely

$$\begin{aligned} \langle \widetilde{\Omega}^i_n, \widetilde{\Omega}^j_n \rangle_{L^2(M,g_n)} &= \langle \widetilde{\Omega}^i_n - \tilde{\beta}^i_n dz^2, \widetilde{\Omega}^j_n \rangle_{L^2(\mathcal{C}^i_n, g_n)} + \langle \widetilde{\Omega}^i_n, \widetilde{\Omega}^j_n - \tilde{\beta}^j_n dz^2 \rangle_{L^2(\mathcal{C}^j_n, g_n)} \\ &\quad + \langle \widetilde{\Omega}^i_n, \widetilde{\Omega}^j_n \rangle_{L^2(M \setminus (\mathcal{C}^j_n \cup \mathcal{C}^i_n), g_n)} \end{aligned} \tag{4.49}$$

which implies (4.27).

Finally, both (4.28) and (4.29) follow from (4.24), (4.25), (4.31) and (A.9). □

Proof of Lemma 4.4. The orthonormal bases $\{\Omega^j_n\}$ will arise as slight adjustments of the unit vectors $\{\widetilde{\Omega}^j_n\}$ from Lemma 4.5. To simplify notations, we fix n and drop the subscript n. By the estimate (4.27) we can adjust $\{\widetilde{\Omega}^j\}$ to an orthonormal basis $\{\Omega^j\}$ inductively using Gram-Schmidt, setting $\Omega^1 = \widetilde{\Omega}^1$ and

$$\Omega^j := \left[\widetilde{\Omega}^j - \sum_{i=1}^{j-1} \langle \widetilde{\Omega}^j, \Omega^i \rangle \Omega^i \right] \lambda_j^{-1},$$

for $j = 2, \ldots, k$, where $\lambda_j := \|\widetilde{\Omega}^j - \sum_{i=1}^{j-1} \langle \widetilde{\Omega}^j, \Omega^i \rangle \Omega^i\|_{L^2(M,g)}$. Based on (4.27) we may then prove by induction that for $j = 2, \ldots, k$, we may write

$$\Omega^j = \widetilde{\Omega}^j \lambda_j^{-1} + \sum_{i=1}^{j-1} c_{ji} \widetilde{\Omega}^i, \tag{4.50}$$

with

$$\begin{aligned} |c_{ji}| &\le C(\ell^j \ell^i)^{3/2} && \text{if } k \ge j > i \ge 1 \\ 1 - \lambda_j^2 &= \sum_{i=1}^{j-1} (\langle \widetilde{\Omega}^j, \Omega^i \rangle)^2 \le C(\ell^j)^3 \sum_{i=1}^{j-1} (\ell^i)^3 && \text{if } j \in \{2, \ldots, k\}. \end{aligned} \tag{4.51}$$

Because of (4.50) and (4.51), we see that (4.17), (4.18), (4.19), (4.20), (4.21) and (4.22) hold for the orthonormal basis $\{\Omega_j\}$ because of the analogous statements we proved for Lemma 4.5. For example, to prove (4.17), we estimate

$$\begin{aligned} \|\Omega^j\|_{L^\infty(M \setminus \mathcal{C}^j, g)} &= \|\widetilde{\Omega}^j \lambda_j^{-1} + \sum_{i=1}^{j-1} c_{ji} \widetilde{\Omega}^i\|_{L^\infty(M \setminus \mathcal{C}^j, g)} \\ &\le C\|\widetilde{\Omega}^j\|_{L^\infty(M \setminus \mathcal{C}^j, g)} + \sum_{i=1}^{j-1} |c_{ji}|.\|\widetilde{\Omega}^i\|_{L^\infty(M \setminus \mathcal{C}^j, g)} \\ &\le C(\ell^j)^{3/2} + C \sum_{i=1}^{j-1} (\ell^j \ell^i)^{3/2} (\ell^i)^{-1/2} \\ &\le C(\ell^j)^{3/2}, \end{aligned} \tag{4.52}$$

while (4.18) follows immediately from (4.50), (4.51), (4.31) and the definition of $\widetilde{\Omega}^i$. To prove (4.19) we use (4.51) first to obtain the bound on the principal part $\beta^j dz^2 = \lambda_j^{-1} \tilde{\beta}^j dz^2$

$$\left| \beta^j - \frac{(\ell^j)^{3/2}}{(32\pi^5)^{1/2}} \right| \le \left| \tilde{\beta}^j - \frac{(\ell^j)^{3/2}}{(32\pi^5)^{1/2}} \right| + |\tilde{\beta}^j|.|\lambda_j^{-1} - 1| \le C(\ell^j)^{9/2}$$

from (4.31), and then to derive the bound on the collar decay part

$$\begin{aligned}\|\Omega^j-\beta^j dz^2\|_{L^\infty(\mathcal{C}^j,g)} &\leq \|\lambda_j^{-1}\left(\widetilde{\Omega}^j-\tilde{\beta}^j dz^2\right)\|_{L^\infty(\mathcal{C}^j,g)}+\sum_{i=1}^{j-1}|c_{ji}|.\|\widetilde{\Omega}^i\|_{L^\infty(\mathcal{C}^j,g)} \\ &\leq C(\ell^j)^{3/2}+C(\ell^j)^{3/2}\sum(\ell^i)^{3/2}(\ell^i)^{3/2}\leq C(\ell^j)^{3/2}\end{aligned} \tag{4.53}$$

from the corresponding bound (4.25) on $\tilde{\Omega}^j$ as well as from (4.24).

To prove (4.20) we use (4.50), (4.51) and (4.28). Recalling how we proved (4.29), we see that (4.21) follows immediately (for example) from (4.17), (4.19) and (A.9), while (4.22) follows from (4.30) combined with (4.50) and (4.51). □

4.2 A formula for the projection of general quadratic differentials onto $\mathcal{H}$ on thin regions of hyperbolic surfaces

Based on the properties of holomorphic quadratic differentials derived in the previous section we can now prove:

Proposition 4.10. *There exists $\delta_0\in(0,\text{arsinh}(1))$ universal such that given a closed surface M of genus $\gamma\geq 2$, there exist $\tilde{\ell}\in(0,2\delta_0)$ and $C<\infty$ depending only on γ such that the following is true. Suppose g is a hyperbolic metric on M and that we have a collar $\mathcal{C}=\mathcal{C}(\ell)$ in (M,g) with $\ell<\tilde{\ell}$. Then the projection $P_g(\Psi)$ of any quadratic differential $\Psi\in\mathcal{Q}_{L^2}(M,g)$ onto the space of holomorphic quadratic differentials is described on this collar by*

$$P_g(\Psi)\sim\frac{\ell^3}{32\pi^5}\langle\Psi,dz^2\rangle_{L^2(\mathcal{C},g)}dz^2 \tag{4.54}$$

in the sense that the principal part $b_0dz^2:=b_0(P_g(\Psi),\mathcal{C})dz^2$ of $P_g(\Psi)$ on $\mathcal{C}$ satisfies

$$\left|b_0-\frac{\ell^3}{32\pi^5}\langle\Psi,dz^2\rangle_{L^2(\mathcal{C})}\right|\leq C\cdot\ell^3\|\Psi\|_{L^1(M,g)} \tag{4.55}$$

while the remaining part decays rapidly along the collar, satisfying

$$\|P_g(\Psi)-b_0dz^2\|_{L^\infty(\delta\text{-}thin(\mathcal{C},g))}\leq C\|\Psi\|_{L^1(M,g)}\cdot\delta^{-2}e^{-\pi/\delta}\ \textit{for every}\ \delta\in(0,\delta_0]. \tag{4.56}$$

Furthermore

$$\|P_g(\Psi)\|_{L^1(M,g)}\leq C\|\Psi\|_{L^1(M,g)}. \tag{4.57}$$

Proof. We argue by contradiction that the proposition is true with δ_0 the value from Section 4.1. Suppose the main statement of the lemma is false, i.e. that there exist a closed surface M of genus $\gamma\geq 2$, a sequence of metrics $g_n\in\mathcal{M}_{-1}$, a sequence of elements $\Psi_n\in\mathcal{Q}_{L^2}(M,g_n)$ and a sequence of collars $\mathcal{C}_n=\mathcal{C}(\ell_n)$ in (M,g_n) with $\ell_n\to 0$, such that at least one of the estimates

$$|b_0(P_{g_n}(\Psi_n),\mathcal{C}_n)-\frac{\ell_n^3}{32\pi^5}\langle\Psi_n,dz^2\rangle_{L^2(\mathcal{C}_n)}|\leq n\cdot\ell_n^3\|\Psi_n\|_{L^1(M,g_n)} \tag{4.58}$$

or

$$\|\omega^\perp(P_{g_n}(\Psi_n),\mathcal{C}_n)\|_{L^\infty(\delta\text{-thin}(\mathcal{C}_n))}\leq n\|\Psi_n\|_{L^1(M,g_n)}\cdot\delta^{-2}e^{-\pi/\delta}\ \text{for every}\ \delta\in(0,\delta_0] \tag{4.59}$$

is violated for each n. We will show that in fact, both are satisfied, for a subsequence, even with the coefficients n replaced with a large constant C.

We may apply the differential geometric form of the Deligne-Mumford Compactness Theorem A.3 to (M, g_n) and assume, after passing to a subsequence, that we have k collars degenerating, where k is a finite number bounded in terms of the genus γ of M.

We now use Lemmata 4.2 and 4.4 to decompose the space $\mathcal{H}(M, g_n) = W_n \oplus W_n^\perp$ of holomorphic quadratic differentials on (M, g_n), and write each $P_{g_n}(\Psi_n)$ as a sum

$$P_{g_n}(\Psi_n) := w_n + \sum_{i=1}^{k} \langle \Psi_n, \Omega_n^i \rangle \Omega_n^i, \tag{4.60}$$

where $w_n = P_{g_n}^{W_n}(\Psi_n) \in W_n$ and Ω_n^1 is the basis element in $W_n^\perp$ corresponding to the collar $\mathcal{C}_n := \mathcal{C}_n^1$ we are analysing. The essential idea is that out of these $k+1$ terms, only the Ω_n^1 term will contribute substantially to the restriction of $P_{g_n}(\Psi_n)$ to the thin part of the collar $\mathcal{C}_n$.

First of all, we recall from Corollary 4.3 that w_n is bounded

$$\|w_n\|_{L^\infty(M)} \leq C \|\Psi_n\|_{L^1(M)} \tag{4.61}$$

so, since it has vanishing principal part on each degenerating collar, (4.6) yields

$$\begin{aligned} \|w_n\|_{L^\infty(\delta\text{-thin}(\mathcal{C}_n))} = \|\omega^\perp(w_n)\|_{L^\infty(\delta\text{-thin}(\mathcal{C}_n))} &\leq C\delta^{-2} e^{-\pi/\delta} \|w_n\|_{L^1(M)} \\ &\leq C\delta^{-2} e^{-\pi/\delta} \|\Psi_n\|_{L^1(M)} \text{ for all } \delta \in (0, \delta_0], \end{aligned} \tag{4.62}$$

C independent of n. Here and in the following all norms are computed with respect to g_n and we abbreviate $b_0(\cdot) = b_0(\cdot, \mathcal{C}_n)$ and $\omega^\perp(\cdot) = \omega^\perp(\cdot, \mathcal{C}_n)$.

To analyse $\langle \Psi_n, \Omega_n^i \rangle \Omega_n^i$ for $i \neq 1$ we first use (4.21) to bound

$$|\langle \Psi_n, \Omega_n^i \rangle_{L^2(M)}| \leq C \cdot (\ell_n^i)^{-1/2} \|\Psi_n\|_{L^1(M)}. \tag{4.63}$$

Recall that the collars $(\mathcal{C}_n^i)_{i=1}^k$ are disjoint, so using (4.17) and the orthogonality of principal and collar decay part on subcollars, we obtain

$$\|\omega^\perp(\Omega_n^i)\|_{L^2(\delta_0\text{-thick}(\mathcal{C}_n))} \leq \|\Omega_n^i\|_{L^2(\delta_0\text{-thick}(\mathcal{C}_n))} \leq \|\Omega_n^i\|_{L^2(M \setminus \mathcal{C}_n^i)} \leq C(\ell_n^i)^{3/2},$$

for $i \neq 1$ (with δ_0 from (4.4)) which, combined with (4.4) and (4.63), gives that for every $i \neq 1$ and $\delta \in (0, \delta_0]$

$$\|\omega^\perp(\langle \Psi_n, \Omega_n^i \rangle \Omega_n^i)\|_{L^\infty(\delta\text{-thin}(\mathcal{C}_n))} \leq C \ell_n^i \delta^{-2} e^{-\pi/\delta} \|\Psi_n\|_{L^1(M)}, \tag{4.64}$$

again with C independent of n. Estimate (4.18) furthermore implies that also the contribution of the Ω_n^i to the principal part of $P_g \Psi$ on $\mathcal{C}_n$ is small if $i \neq 1$, namely using once more (4.63), we get

$$|b_0(\langle \Psi_n, \Omega_n^i \rangle \Omega_n^i)| \leq C \ell_n^i (\ell_n)^3 \|\Psi_n\|_{L^1(M)} \leq C (\ell_n)^3 \|\Psi_n\|_{L^1(M)}. \tag{4.65}$$

In order to evaluate the dominating term $\langle \Psi_n, \Omega_n^1 \rangle \Omega_n^1$, we first use (4.17) and (4.19) from Lemma 4.4, and abbreviate $\alpha := 1/(32\pi^5)^{1/2}$ to estimate

$$\begin{aligned} \left| \langle \Psi_n, \Omega_n^1 \rangle_{L^2(M)} - \alpha \ell_n^{3/2} \langle \Psi_n, dz^2 \rangle_{L^2(\mathcal{C}_n)} \right| &\leq |\langle \Psi_n, \Omega_n^1 \rangle_{L^2(M \setminus \mathcal{C}_n)}| + \left| \langle \Psi_n, \Omega_n^1 - b_0(\Omega_n^1) dz^2 \rangle_{L^2(\mathcal{C}_n)} \right| \\ &\quad + |b_0(\Omega_n^1) - \alpha \ell_n^{3/2}| \cdot |\langle \Psi_n, dz^2 \rangle_{L^2(\mathcal{C}_n)}| \\ &\leq C \ell_n^{3/2} \|\Psi_n\|_{L^1(M)}, \end{aligned} \tag{4.66}$$

using (A.9). In particular, by (A.9), or by working directly from (4.21), we have

$$|\langle\Psi_n,\Omega_n^1\rangle_{L^2(M)}| \leq C\ell_n^{-1/2}\|\Psi_n\|_{L^1(M)}. \tag{4.67}$$

A combination of (4.4) and (4.19) allows us to estimate

$$\begin{aligned}\|\omega^\perp(\Omega_n^1)\|_{L^\infty(\delta\text{-thin}(\mathcal{C}_n))} &\leq C\delta^{-2}e^{-\pi/\delta}\|\omega^\perp(\Omega_n^1)\|_{L^2(\delta_0\text{-thick}(\mathcal{C}_n))}\\ &\leq C\delta^{-2}e^{-\pi/\delta}\|\omega^\perp(\Omega_n^1)\|_{L^\infty(\delta_0\text{-thick}(\mathcal{C}_n))}\\ &\leq C\delta^{-2}e^{-\pi/\delta}\ell_n^{3/2}\end{aligned}$$

for $\delta\in(0,\delta_0]$. We can combine this with (4.67) to find that the collar decay part is small:

$$\|\omega^\perp(\langle\Psi_n,\Omega_n^1\rangle\Omega_n^1)\|_{L^\infty(\delta\text{-thin}(\mathcal{C}_n))} \leq C\ell_n\delta^{-2}e^{-\pi/\delta}\|\Psi_n\|_{L^1(M)} \text{ for all } \delta\in(0,\delta_0],$$

C independent of n, which in view of (4.62) and (4.64) means that (4.59) is fulfilled for all n sufficiently large.

On the other hand, combined with (4.19) and (4.67), estimate (4.66) also implies that

$$\begin{aligned}&|b_0(\langle\Psi_n,\Omega_n^1\rangle\Omega_n^1) - \alpha^2\ell_n^3\langle\Psi_n,dz^2\rangle_{L^2(\mathcal{C}_n)}|\\ &\quad\leq \left|\langle\Psi_n,\Omega_n^1\rangle(b_0(\Omega_n^1)-\alpha\ell_n^{3/2})\right| + \left|\alpha\ell_n^{3/2}\langle\Psi_n,\Omega_n^1\rangle - \alpha^2\ell_n^3\langle\Psi_n,dz^2\rangle_{L^2(\mathcal{C}_n)}\right|\\ &\quad\leq C\ell_n^{-1/2}\|\Psi_n\|_{L^1(M)}\ell_n^{9/2} + \alpha\ell_n^{3/2}C\ell_n^{3/2}\|\Psi\|_{L^1(M)}\\ &\quad\leq C\ell_n^3\|\Psi_n\|_{L^1(M)}.\end{aligned} \tag{4.68}$$

Since any other contribution to the principal part of $P_g\Psi$ is bounded by (4.65) this implies that also (4.58) is fulfilled for all sufficiently large n, leading to a contradiction to our assumption.

Finally, to prove the last estimate (4.57) stated in the lemma, we can argue by contradiction as above, and combine the bound (4.61) on w_n with the estimate

$$\|\langle\Omega_n^i,\Psi_n\rangle\Omega_n^i\|_{L^1(M,g)} \leq C\|\Psi_n\|_{L^1(M,g)}$$

resulting from (4.20) and (4.21), to conclude that

$$\|P_g(\Psi)\|_{L^1(M,g)} \leq C\|\Psi\|_{L^1(M,g)}$$

holds true for any $\Psi\in\mathcal{Q}_{L^2(M,g)}$ with a constant depending only on the genus.

□

4.3 Proof of the general formula for $\frac{d\ell}{dt}$, Lemma 2.2

We can now prove Lemma 2.2 based on the formula of the projection derived in the previous section.

Proof of Lemma 2.2. Let M be a closed surface of genus $\gamma\geq 2$ and let $g(t)$ be a one-parameter family in $\mathcal{M}_{-1}$ such that $\partial_t g|_{t=0} = Re(P_g(\Psi))$ for some $\Psi\in\mathcal{Q}_{L^2}(M,g(0))$, and assume that $(M,g(0))$ contains a collar $\mathcal{C}$ around a simple closed geodesic σ of length $\ell<\tilde{\ell}$, the number of Proposition 4.10.

At $t=0$, writing $g_{\theta\theta} = g\left(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\right)$, we have

$$\ell = \int_\sigma g_{\theta\theta}^{1/2}d\theta = 2\pi g_{\theta\theta}^{1/2},$$

i.e. $g_{\theta\theta} = \left(\frac{\ell}{2\pi}\right)^2$. At $t = 0$, σ is a geodesic so we have

$$\begin{aligned}\frac{d\ell}{dt} &= \int_\sigma \left(\frac{1}{2}\left[g_{\theta\theta}\right]^{-1/2}\frac{\partial g_{\theta\theta}}{\partial t}\right)d\theta = \frac{\pi}{\ell}\int_\sigma \frac{\partial g_{\theta\theta}}{\partial t}d\theta = \frac{\pi}{\ell}\int_\sigma Re(P_g(\Psi))\left(\tfrac{\partial}{\partial\theta},\tfrac{\partial}{\partial\theta}\right)d\theta \\ &= \frac{\pi}{\ell}\int_\sigma Re(b_0(P_g(\Psi))dz^2)\left(\tfrac{\partial}{\partial\theta},\tfrac{\partial}{\partial\theta}\right) + Re(\omega^\perp(P_g(\Psi)))\left(\tfrac{\partial}{\partial\theta},\tfrac{\partial}{\partial\theta}\right)d\theta \\ &= -\frac{2\pi^2}{\ell}Re(b_0(P_g(\Psi))),\end{aligned} \tag{4.69}$$

where we split $P_g(\Psi) = b_0(P_g(\Psi))dz^2 + \omega^\perp(P_g(\Psi))$ into its principal part and its collar decay part, and notice that the latter integrates to zero by (4.3).

Proposition 4.10 tells us that

$$\left|Re(b_0(P_g(\Psi))) - \frac{\ell^3}{32\pi^5}Re\langle\Psi, dz^2\rangle_{L^2(\mathcal{C})}\right| \le C\ell^3\|\Psi\|_{L^1},$$

so indeed, by (4.69),

$$\left|\frac{d\ell}{dt} + \frac{\ell^2}{16\pi^3}Re\langle\Psi, dz^2\rangle_{L^2(\mathcal{C})}\right| \le C(\ell^{-1}\ell^3)\|\Psi\|_{L^1} \le C\ell^2\|\Psi\|_{L^1}.$$

□

Remark 4.11. From the proof we immediately see that for *holomorphic* quadratic differentials Ψ, we have

$$\frac{d\ell}{dt} = -\frac{2\pi^2}{\ell}Re(b_0(\Psi)). \tag{4.70}$$

4.4 Incompleteness of Teichmüller space

In order to illustrate the use of Lemma 2.2, we show the well-known fact (e.g. Wolpert [23]) that Teichmüller space equipped with the Weil-Petersson metric is incomplete. Indeed, if we pick a metric g on any closed surface M of genus at least 2, with a collar $\mathcal{C}$ having $\ell < \tilde{\ell}$, then we may deform it as in Lemma 2.2 taking Ψ to be dz^2 on $\mathcal{C}$ and zero elsewhere. In this case, the distance $s(t)$ travelled through Teichmüller space is, by definition (with one choice of normalisation of the Weil-Petersson metric)

$$\frac{ds}{dt} = \frac{1}{4}\|P_g(\Psi)\|_{L^2(M)} \le \frac{1}{4}\|\Psi\|_{L^2(M)} = \frac{1}{4}\|dz^2\|_{L^2(\mathcal{C})} = \left(\frac{2\pi^5}{\ell^3}\right)^{1/2}(1 + O(\ell^3)),$$

as a result of (A.9). Meanwhile, by Lemma 2.2, we have

$$\left|\frac{d\ell}{dt} + \frac{\ell^2}{16\pi^3}\|dz^2\|^2_{L^2(\mathcal{C})}\right| \le C\ell^2\|dz^2\|_{L^1(\mathcal{C})},$$

and hence, by (A.9)

$$\left|\frac{d\ell}{dt} + \frac{2\pi^2}{\ell}\right| \le C\ell.$$

Combining these facts, we find that

$$\frac{d\ell^{1/2}}{ds} \le -\left(\frac{1}{2\pi}\right)^{1/2} + O(\ell^2),$$

and thus we can pinch a collar by moving a distance no more than $(2\pi\ell)^{1/2} + O(\ell^{5/2})$ in Teichmüller space. We do not claim this to be optimal in any way; already our results would allow us to show a stronger upper bound of $(2\pi\ell)^{1/2} + O(\ell^{7/2})$, but indeed it was proven by Wolpert in [25] that $\ell^{\frac{1}{2}}$ is convex and consequently this distance is bounded from above by $(2\pi\ell)^{1/2}$ itself, with a lower bound of $dist \geq (2\pi\ell)^{1/2} + O(\ell^{5/2})$ established in the same paper.

Remark that combining the (essentially explicit) upper bound (4.15) on the principal part of any unit holomorphic quadratic differential with (4.70) allows us to improve this lower bound to an estimate of the form

$$\text{dist} \geq (2\pi\ell)^{1/2}\left(1 - \frac{1}{84\pi}\ell^3 + O(\ell^5)\right). \tag{4.71}$$

In the more general case that a collection $\{\sigma_i\}_{i\in\mathcal{J}}$ of geodesics pinches, i.e. in which one considers the distance to the part (or stratum) of the boundary characterised by $\ell := \sum_{i\in\mathcal{J}} \ell_i = 0$, the lower bound in the estimate $(2\pi\ell)^{1/2} + O(\ell^{5/2}) \leq dist \leq (2\pi\ell)^{1/2}$ proven in [25] can be improved to (4.71) by a similar argument, using additionally that the degenerating collars are disjoint.

5 Controlling the weighted energy I

In this section, we finally prove the estimate on the full weighted energy

$$I = \int_{\mathcal{C}} e(u,g)\rho^{-2}dv_g$$

that we claimed in Lemma 2.4. Let (u,g) be any solution of the Teichmüller harmonic map flow (1.1) defined on an interval $[0,T)$ and let $t_0 \in [0,T)$ be such that $(M, g(t_0))$ contains a collar $\mathcal{C}_{t_0}$ around a simple closed geodesic $\sigma(t_0)$ of length $\ell(t_0) < \tilde{\ell}$, the number of Proposition 4.10. Since the evolution of $g(t)$ is smooth, for t close to t_0 also $(M, g(t))$ contains such a geodesic $\sigma(t)$ of length $\ell(t) < \tilde{\ell}$ with $\sigma(t)$ and the associated collar region $\mathcal{C}_t$ depending smoothly on t.

We may thus consider the evolution of the associated weighted energies I, or rather of a smoothed-out version of I given by

$$\mathcal{I}(t) := \int_{\mathcal{C}_t} e(u(t), g(t))\rho^{-2}(t)\varphi(\rho(t))dv_{g(t)}, \tag{5.1}$$

$\varphi \in C_0^\infty([0,2\delta),[0,1])$ a cut-off function with $\varphi \equiv 1$ on $[0,\delta]$, where we require $\delta > 0$ small enough such that $2\delta \leq \rho(X(\ell))$ for all $\ell \in (0, 2\,\text{arsinh}(1))$. Indeed, by (A.2), we can fix $\delta = \frac{1}{2\pi}$, which relieves any dependencies of constants on δ. Note that I and $\mathcal{I}$ are related in the sense that

$$0 \leq I - \mathcal{I} \leq \delta^{-2}E_0, \tag{5.2}$$

where E_0 is an upper bound on the total energy.

The main step in the proof of Lemma 2.4 is to show

Lemma 5.1. *Let (u,g) be a solution of (1.1) on a surface M of genus at least 2, for $t \in [0,T)$, into a target N of nonpositive sectional curvature, and let $t_0 \in [0,T)$ be such that $(M, g(t_0))$ contains a collar $\mathcal{C} = \mathcal{C}(\ell)$ with $\ell < \tilde{\ell}$, where $\tilde{\ell}$ is from Proposition 4.10. Then the weighted energy $\mathcal{I}$ defined by (5.1) satisfies*

$$\left|\frac{d}{dt}\log(1+\mathcal{I})\right| \leq C\left(1 + \|\tau_g(u)\|^2_{L^2(M,g)}\right) \tag{5.3}$$

for a constant C depending only on M, N, η and an upper bound E_0 on the initial energy.

Accepting this lemma for the moment, we can finally give the:

Proof of Lemma 2.4. Given (u, g) as in Lemma 2.4 and a time $t_0 \in [0, T)$ such that $(M, g(t_0))$ contains a collar around a geodesic $\sigma(t_0)$ of length $L_{g(t_0)}(\sigma(t_0)) < \tilde{\ell}$ we let $t_{min} \geq 0$ be the minimal number such that there is a smooth family of simple closed geodesics $(\sigma(t))_{t\in[t_{min},t_0]}$ in $(M, g(t))$ with $L_{g(t)}(\sigma(t)) < \tilde{\ell}$ for all $t \in (t_{min}, t_0]$.

If $t_{min} = 0$, we can initially bound the weighted energy $\mathcal{I}$ in terms of $E(u(0), g(0)) \leq E_0$ and $\ell_0 = L_{g(0)}(\sigma(0)) \geq 2\,\mathrm{inj}_{g(0)} M$ as

$$\mathcal{I}(0) \leq \big(\sup_{\mathcal{C}} \rho^{-2}\big) \int_{\mathcal{C}} e(u, g) dv_g \Big|_{t=0} \leq \left(\frac{2\pi}{\ell_0}\right)^2 E(u(0), g(0)) \leq C \cdot (\mathrm{inj}_{g(0)} M)^{-2},$$

with $C = C(E_0)$. Since the space-time integral of the squared tension is bounded by the initial energy, by (1.2), integration of (5.3) from $t = 0$ to t_0 gives the desired estimate

$$\mathcal{I}(t_0) \leq \exp\left[C \int_0^{t_0} \left(1 + \|\tau_g(u)\|^2_{L^2(M,g)}\right) dt\right] \cdot (\mathcal{I}(0) + 1) \leq e^{C(t_0+1)} \cdot (1 + (\mathrm{inj}_{g(0)} M)^{-2}) \quad (5.4)$$

first for $\mathcal{I}$, and then, by (5.2), also for the original weighted energy I.

On the other hand, if $t_{min} > 0$ then $L_{g(t_{min})}(\sigma(t_{min})) = \tilde{\ell}$ and thus $\mathcal{I}(t_{min}) \leq C\tilde{\ell}^{-2}E_0$ so integration of (5.3) from t_{min} to t_0 again proves Lemma 2.4. □

We now turn to the proof of Lemma 5.1.

To begin with, we derive a parabolic Bochner estimate in order to control the evolution of the energy density on degenerating collars.

Lemma 5.2. *Let (u, g) be any solution of (1.1) on a surface M of genus at least 2, and assume that the target manifold (N, G) has nonpositive sectional curvature. Let t be any time such that $(M, g(t))$ contains a collar $\mathcal{C}$ with central geodesic of length $\ell < \tilde{\ell}$ (for $\tilde{\ell}$ as in Proposition 4.10). Then the evolution of the energy density on this collar satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta_g\right) e(u, g) \leq C \cdot e(u, g) - \frac{1}{2} Re(b_0)\rho^{-4} \cdot (|u_s|^2 - |u_\theta|^2) + Im(b_0)\rho^{-4} \cdot \langle u_s, u_\theta\rangle, \quad (5.5)$$

for a constant $C < \infty$ depending only on M, N, η and an upper bound E_0 on the initial energy. Here $b_0 dz^2 = b_0(\frac{\eta^2}{4} P_g(\Phi(u, g)), \mathcal{C})dz^2$ denotes the principal part on this collar, (s, θ) are the usual collar coordinates on $(\mathcal{C}, g)$, as in Lemma A.1, and $dz = ds + id\theta$.

Remark 5.3. It is a consequence of Proposition 4.10, (A.9) and the definition $\Phi(u, g) = (|u_s|^2 - |u_\theta|^2 - 2i\langle u_s, u_\theta\rangle)dz^2$, that the principal part $b_0 dz^2$ of $\frac{\eta^2}{4} P_g(\Phi(u, g))$ appearing in the above Bochner formula is given by the weighted integrals

$$\begin{aligned} Re(b_0) &= \frac{\ell^3}{32\pi^5}\eta^2 \int_{\mathcal{C}} (|u_s|^2 - |u_\theta|^2)\rho^{-4} dv_g + r_1 \\ Im(b_0) &= -\frac{\ell^3}{16\pi^5}\eta^2 \int_{\mathcal{C}} \langle u_s, u_\theta\rangle \rho^{-4} dv_g + r_2, \end{aligned} \quad (5.6)$$

with error terms r_1, r_2 bounded by

$$|r_1| + |r_2| \leq C\ell^3 \cdot \|\Phi\|_{L^1} \leq C\ell^3 E_0,$$

C depending only on γ and η. In particular,

$$|Re(b_0)| \leq C\ell^3 \int_{\mathcal{C}} e(u, g)\rho^{-2}\varphi dv_g + C\ell^3 \leq C\ell^3(\mathcal{I} + 1) \quad (5.7)$$

while

$$|Im(b_0)| \le C\ell^3 \left((I^{(\theta)}\mathcal{I})^{1/2} + 1 \right), \tag{5.8}$$

contains the weighted angular energy $I^{(\theta)}$ controlled by Lemma 3.1 (C now also depending on E_0).

Proof of Lemma 5.2. The key assumption in the lemma is that the sectional curvature of the target is nonpositive. The Bochner formula for the classical harmonic map flow, as used in [3], thus implies that

$$(\partial_\varepsilon - \Delta_g)e(u(t+\varepsilon), g(t))|_{\varepsilon=0} \le C \cdot e(u,g),$$

and in that classical case one could deduce boundedness of the energy density over finite time intervals.

Let now $\mathcal{C}$ be a collar in (M,g) whose central geodesic has length $\ell < \tilde{\ell}$, the number of Proposition 4.10. The evolution of the metric on this collar is then described by

$$\begin{aligned} \partial_t g &= Re\left(\frac{\eta^2}{4} P_g(\Phi(u,g))\right) \\ &= Re(b_0)\cdot(ds^2 - d\theta^2) - Im(b_0)(ds\otimes d\theta + d\theta \otimes ds) + Re(\omega^\perp), \end{aligned} \tag{5.9}$$

$b_0 dz^2$ the principal part of $\frac{\eta^2}{4}P_g(\Phi(u,g))$ on $\mathcal{C}$, and $\omega^\perp$ its collar decay part which, by (4.7) and the final part of Proposition 4.10, is bounded by

$$\|\omega^\perp\|_{L^\infty(\mathcal{C},g)} \le C \cdot \|P_g(\Phi(u,g))\|_{L^1(M,g)} \le C\|\Phi(u,g)\|_{L^1(M,g)} \le C.$$

The change in the energy density $e(u,g) = \frac{1}{2}g^{ij}\partial_i u \partial_j u$ caused by the evolution of the metric is thus given as

$$\begin{aligned} \partial_\varepsilon e(u(t), g(t+\varepsilon))|_{\varepsilon=0} &= -\frac{1}{2}g^{ik}g^{jl}\partial_t g_{kl}\partial_i u \partial_j u = -\frac{1}{2}\frac{\partial g}{\partial t}(du,du) \\ &= -\rho^{-4}\left[\frac{1}{2}Re(b_0)\cdot(|u_s|^2 - |u_\theta|^2) - Im(b_0)\langle u_s, u_\theta\rangle\right] + R \end{aligned} \tag{5.10}$$

where $R = -\frac{1}{2}Re(\omega^\perp)(du,du)$, so

$$|R| = \left|\frac{1}{2}g^{ik}g^{jl}(Re(\omega^\perp))_{kl}\partial_i u \partial_j u\right| \le C\|\omega^\perp\|_{L^\infty(\mathcal{C},g)}e(u,g) \le Ce(u,g)$$

C depending only on M, N, η and E_0. This completes the proof of (5.5). □

The second tool we need for the proof of Lemma 5.1 is a formula for the evolution of the weight ρ^{-2} in (5.1) at general points of the collar. We recall, by (A.1), that the conformal factor ρ on a collar $\mathcal{C} = \mathcal{C}(\ell(t)) \subset (M, g(t))$ can be characterised in a coordinate-free way as

$$\rho(p,t) = \frac{\ell(t)}{2\pi\sinh(\ell(t)/2)}\cdot\sinh\left(\mathrm{inj}_{g(t)}(p)\right). \tag{5.11}$$

Lemma 5.4. *Let (u,g) be any smooth solution of (1.1) on a surface M of genus at least 2, for $t \in [0,T)$, and assume that at time $t_0 \in [0,T)$, the surface $(M, g(t_0))$ contains a collar $\mathcal{C}$ around a geodesic of length $\ell < \tilde{\ell}$, where $\tilde{\ell}$ is from Proposition 4.10. Then $\rho(p,t) = \rho(\ell(t), \mathrm{inj}_{g(t)}(p))$ evolves according to*

$$|\partial_t(\rho^2)(p) + Re(b_0)| \le C\cdot e^{-1/\rho(p)} \quad \textit{for all } p \in \mathcal{C} \tag{5.12}$$

where $b_0 dz^2 = b_0(\frac{\eta^2}{4}P_g(\Phi(u,g)), \mathcal{C})dz^2$ is the principal part on $\mathcal{C}$, and $C < \infty$ depends only on M, η and E_0.

Proof. In the lemma, the metric $g_0 := g(t_0)$ is being deformed in the direction $\partial_t g = Re(b_0 dz^2) + Re(\omega^\perp)$, where $\omega^\perp = \omega^\perp(\frac{\eta^2}{4} P_{g_0}(\Phi(t_0)), \mathcal{C})$. Heuristically, it is the first of these terms that is dominant. Indeed, if we consider an alternative, symmetric flow of hyperbolic metrics $\hat{g}(t)$ on $\mathcal{C}$ for t near t_0, with $\hat{g}(t_0) = g_0$ and $\partial_t \hat{g} = Re(b_0 dz^2)$ (one could write down such a flow explicitly) then the corresponding conformal factor $\hat{\rho}$ could be written at $q = (s_0, \theta_0) \in \mathcal{C}$ independently of the time-t collar coordinates as

$$L_{\hat{g}(t)}(\{s_0\} \times S^1) = 2\pi \hat{\rho}(q, t),$$

because of the symmetry of the deformation. In particular, we would have at $t = t_0$ that

$$\begin{aligned} \partial_t(\hat{\rho}^2)(q) &= 2\rho(s_0) \cdot \frac{1}{2\pi} \frac{\mathrm{d}}{\mathrm{d}t} L_{\hat{g}(t)}(\{s_0\} \times S^1) \\ &= \frac{\rho(s_0)}{2\pi} \int_{\{s_0\} \times S^1} \left((g_0)_{\theta\theta}\right)^{-1/2} \partial_t \hat{g}_{\theta\theta} d\theta \\ &= -Re(b_0). \end{aligned} \tag{5.13}$$

Another way of computing the derivative of the conformal factor $\hat{\rho}$, or indeed ρ, is via (5.11). Writing $F(x) = \frac{x}{2\pi \sinh(x/2)}$ so that $\rho(q, t) = F(\ell(t)) \sinh(\mathrm{inj}_{g(t)}(q))$, we compute at $t = t_0$

$$\partial_t \rho(q) = F'(\ell) \frac{d\ell}{dt} \sinh(\mathrm{inj}_{g_0}(q)) + F(\ell) \cosh(\mathrm{inj}_{g_0}(q)) \partial_t(\mathrm{inj}_{g(t)}(q)).$$

In order to compute $\partial_t(\mathrm{inj}_{g(t)}(q))$, we note that $\iota := \mathrm{inj}_{g_0}(q)$ can be realised as half the length of a unit speed geodesic $\sigma : [0, 2\iota] \to \mathcal{C}_\iota$ mapping the end points to q and wrapping once around the collar, where $\mathcal{C}_\iota := \{p \in \mathcal{C} \ : \ \mathrm{inj}_{g_0}(p) \leq \iota\}$ is the closure of ι-thin$(\mathcal{C}, g_0)$ when this thin part is nonempty. More generally, for t close to t_0,

$$\mathrm{inj}_{g(t)}(q) = \frac{1}{2} L_{g(t)}(\sigma(t))$$

for an appropriate smooth deformation $\sigma(t)$ of σ with fixed end points, and so

$$\partial_t \,\mathrm{inj}_{g(t)}(q) = \frac{1}{2} \partial_t L_{g(t)}(\sigma),$$

because σ is a geodesic and its deformation has fixed end points. But we can compute

$$\partial_t L_{g(t)}(\sigma) = \frac{1}{2} \int_0^{2\iota} \partial_t g(\dot{\sigma}, \dot{\sigma}),$$

and so assembling what we have seen, we get at $t = t_0$ that

$$\partial_t \rho^2(q) = 2\rho(q) F'(\ell) \frac{d\ell}{dt} \sinh(\mathrm{inj}_{g_0}(q)) + \frac{1}{2} \rho(q) F(\ell) \cosh(\mathrm{inj}_{g_0}(q)) \int_0^{2\iota} \partial_t g(\dot{\sigma}, \dot{\sigma}). \tag{5.14}$$

This formula equally well applies to the flow $\hat{g}(t)$, and so noting that $\frac{d\ell}{dt} = \frac{d\hat{\ell}}{dt}$ at $t = t_0$ (because the collar decay part $\omega^\perp$ does not contribute to $\frac{d\ell}{dt}$) we obtain from (5.13) that

$$-Re(b_0) = 2\rho(q) F'(\ell) \frac{d\ell}{dt} \sinh(\mathrm{inj}_{g_0}(q)) + \frac{1}{2} \rho(q) F(\ell) \cosh(\mathrm{inj}_{g_0}(q)) \int_0^{2\iota} Re(b_0 dz^2)(\dot{\sigma}, \dot{\sigma}). \tag{5.15}$$

This allows us to simplify (5.14) when applied to $g(t)$, to

$$\partial_t \rho^2(q) = -Re(b_0) + \frac{1}{2} \rho(q) F(\ell) \cosh(\mathrm{inj}_{g_0}(q)) \int_0^{2\iota} Re(\omega^\perp)(\dot{\sigma}, \dot{\sigma}), \tag{5.16}$$

and in particular, by (4.56) of Proposition 4.10 and the fact that $\|\Phi\|_{L^1(M,g_0)} \leq E_0$, we find that

$$\begin{aligned} \left|\partial_t \rho^2(q) + Re(b_0)\right| &\leq C\rho(q)\iota\|\omega^\perp\|_{L^\infty(\mathcal{C}_\iota)} \\ &\leq C\rho(q)\iota^{-1}e^{-\pi/\iota}\|\Phi\|_{L^1(M,g_0)} \\ &\leq C\rho(q)\iota^{-1}e^{-\pi/\iota} \\ &\leq Ce^{-1/\rho(q)} \end{aligned} \tag{5.17}$$

as desired, because $x \mapsto x^{-1}e^{-\pi/x}$ is increasing for $x \in (0, \pi)$, and $\iota \leq \pi\rho$ by (A.6). □

We can now finally estimate the evolution of weighted energy (5.1)

Proof of Lemma 5.1. Let (u, g) be a solution of (1.1) as in Lemma 5.1 and consider the evolution

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I} &= \int \partial_t(\rho^{-2}) \cdot e(u,g)\varphi dv_g + \int \rho^{-2}\partial_t(e(u,g)) \cdot \varphi dv_g + \int \rho^{-2}e(u,g)\partial_t(\varphi(\rho))dv_g \\ &= I_1 + I_2 + R_1 \end{aligned} \tag{5.18}$$

of the weighted energy $\mathcal{I}$ on a collar $\mathcal{C} = \mathcal{C}(\ell) \subset (M, g(t))$ with $\ell < \tilde{\ell}$, the number given by Proposition 4.10. We have used here that $\partial_t(dv_g) = \frac{1}{2}\operatorname{tr}(\frac{\partial g}{\partial t})dv_g \equiv 0$.

The key point in the proof is that the dominating terms in I_1 and in I_2 agree but appear with opposing signs and thus cancel, while all other terms either involve the angular energy, controlled by Lemma 3.1, or derivatives of the cut-off function.

On the one hand, Lemma 5.4 immediately implies that the first integral is given as

$$\begin{aligned} I_1 &= -\int \rho^{-4}\partial_t(\rho^2)e(u,g)\varphi dv_g \leq Re(b_0)\int \rho^{-4}e(u,g)\varphi dv_g + C\int \rho^{-4}e^{-1/\rho}e(u,g)\varphi dv_g \\ &\leq Re(b_0)\int \rho^{-4}e(u,g)\varphi dv_g + CE_0, \end{aligned} \tag{5.19}$$

where $b_0 dz^2$ is the principal part $b_0 = b_0(\frac{\eta^2}{4}P_g(\Phi(u,g)), \mathcal{C})$ corresponding to $\partial_t g$.

On the other hand, using the Bochner formula (5.5) we find that

$$\begin{aligned} I_2 &\leq -\frac{Re(b_0)}{2}\int (|u_s|^2 - |u_\theta|^2)\rho^{-6}\varphi dv_g + Im(b_0)\int \langle u_s, u_\theta\rangle \rho^{-6}\varphi dv_g \\ &\quad + \int \Delta_g(e(u,g))\rho^{-2}\varphi dv_g + C\int \rho^{-2}e(u,g)\varphi dv_g \\ &\leq -Re(b_0)\int e(u,g)\rho^{-4}\varphi dv_g + |Re(b_0)|\int_{\mathcal{C}} \rho^{-6}|u_\theta|^2 dv_g \\ &\quad + |Im(b_0)| \cdot \left(\int \rho^{-4}|u_s|^2\varphi dv_g\right)^{1/2} \cdot \left(\int_{\mathcal{C}} \rho^{-8}|u_\theta|^2 dv_g\right)^{1/2} \\ &\quad + \int e(u,g)\Delta_g(\rho^{-2}\varphi)dv_g + C\mathcal{I} \\ &\leq -Re(b_0)\int e(u,g)\rho^{-4}\varphi dv_g + C\ell^{-2}|Re(b_0)|I^{(\theta)} + C\ell^{-2}|Im(b_0)|\mathcal{I}^{1/2}(I^{(\theta)})^{1/2} \\ &\quad + C\mathcal{I} + R_2, \end{aligned} \tag{5.20}$$

where $R_2 = \int \Delta_g(\rho^{-2}\varphi) \cdot e(u,g)dv_g$ turns out to be a lower order term that we will analyse in (5.22) below. The first term compensates precisely for the main term obtained in (5.19) while, thanks to (5.7) and (5.8), the second and third terms are bounded by $C\ell I^{(\theta)}(\mathcal{I} + 1)$ and

$C\ell(I^{(\theta)}\mathcal{I}+1)$ respectively. Remark that we do not get a cancellation for these terms so it is crucial that the angular energy $I^{(\theta)}$ is a priori controlled owing to the analysis of Section 3 that could be carried out on time-slices.

Combining (5.19) and (5.20) with Lemma 3.1 and allowing C to depend on E_0, we find that

$$I_1 + I_2 \leq C \cdot (\ell I^{(\theta)} + 1)(\mathcal{I}+1) + R_2 \leq C(1 + \ell\|\tau_g(u)\|_{L^2}^2)(\mathcal{I}+1) + R_2 \tag{5.21}$$

which, up to estimates on the lower order terms

$$R_1 = \int \rho^{-2} e(u,g)\varphi' \partial_t(\rho) dv_g \quad \text{ and } \quad R_2 = \int \rho^{-2}\partial_{ss}(\rho^{-2}\varphi)\cdot e(u,g) dv_g$$

gives the estimate on $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}$ that we want to prove.

Since the derivative of the cut-off function is supported in the subset of the collar where $\rho \geq \delta$, we can bound R_1 using Lemma 5.4 as well as (5.7), by

$$|R_1| \leq C \int_{\{\rho\geq\delta\}} \rho^{-3} e(u,g)|\partial_t \rho^2| dv_g \leq C(|Re(b_0)| + 1) \leq C(\ell^3\mathcal{I}+1).$$

On the other hand, also $|\partial_s \rho^{-2}| \leq 2\rho^{-1}$, see (A.3), which is bounded on the support of $\partial_s\varphi(\rho)$, while $|\partial_{ss}(\rho^{-2})| \leq 2$ is bounded uniformly on the whole collar, so

$$|R_2| \leq CE_0 + 2\int \rho^{-2} e(u,g)\varphi dv_g \leq C(\mathcal{I}+1), \tag{5.22}$$

allowing C to depend on E_0, which completes the proof of the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I} \leq C(1+\ell\|\tau_g(u)\|_{L^2}^2)(\mathcal{I}+1) \leq C(1+\|\tau_g(u)\|_{L^2}^2)(\mathcal{I}+1)$$

stated in Lemma 5.1. □

A Appendix

We will need the following 'Collar lemma' throughout the paper.

Lemma A.1 (Keen-Randol [11]). *Let (M,g) be a closed hyperbolic surface and let σ be a simple closed geodesic of length ℓ. Then there is a neighbourhood around σ, a so-called collar, which is isometric to the cylinder*

$$\mathcal{C}(\ell) := (-X(\ell), X(\ell)) \times S^1$$

equipped with the metric $\rho^2(s)(ds^2 + d\theta^2)$ where

$$\rho(s) = \frac{\ell}{2\pi\cos(\frac{\ell s}{2\pi})} \qquad \text{and} \qquad X(\ell) = \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arctan\left(\sinh\left(\frac{\ell}{2}\right)\right)\right).$$

The geodesic σ then corresponds to the circle $\{(0,\theta) \mid \theta \in S^1\} \subset \mathcal{C}(\ell)$.

In this version of the collar lemma, the intrinsic distance w between the two ends of the collar is related to ℓ via

$$\sinh\frac{\ell}{2}\sinh\frac{w}{2} = 1,$$

which is sharp. In order to simplify the discussion of dependency of constants, and ensure that different collars do not intersect, we will only talk about collars with $0 < \ell < 2\operatorname{arsinh}(1)$ (cf. [14, Appendix A.2]).

We recall (cf. [14, Lemma A.5]) that the conformal factor ρ is related to the injectivity radius by the formula

$$\sinh(\operatorname{inj}(s,\theta)) \cdot \cos\left(\frac{\ell s}{2\pi}\right) = \sinh\left(\frac{\ell}{2}\right). \tag{A.1}$$

Note that at the ends of the collar we have

$$\rho(X) = \rho(-X) = \frac{\ell}{2\pi \tanh \frac{\ell}{2}} \sim \frac{1}{\pi} \qquad \text{for small } \ell > 0,$$

and

$$\operatorname{inj}(X,\theta) = \operatorname{inj}(-X,\theta) = \operatorname{arsinh}(\cosh(\frac{\ell}{2})) \sim \operatorname{arsinh}(1) \qquad \text{for small } \ell > 0.$$

Within the collar, for $s \in (-X(\ell), X(\ell))$, we have

$$\rho(s) \le \rho(X(\ell)) = \frac{\ell}{2\pi \tanh \frac{\ell}{2}} \in \left(\frac{1}{\pi}, \frac{\sqrt{2}\operatorname{arsinh}(1)}{\pi}\right), \tag{A.2}$$

for $\ell \in (0, 2\operatorname{arsinh}(1))$. Moreover, we can compute

$$\frac{d}{ds}\log\rho(s) = \frac{\ell}{2\pi}\tan\frac{\ell s}{2\pi}, \qquad \text{so} \qquad \left|\frac{d}{ds}\log\rho(s)\right| \le \rho(s) \tag{A.3}$$

and hence for $s \in (-X(\ell), X(\ell))$ and $\ell \in (0, 2\operatorname{arsinh}(1))$ we have

$$\left|\frac{d}{ds}\log\rho(s)\right| \le \frac{\ell}{2\pi \sinh \frac{\ell}{2}} \le \frac{1}{\pi}. \tag{A.4}$$

One consequence that we shall use several times is that for any $\Lambda > 0$ there exists $C \in (0, \infty)$ such that for any $\ell \in (0, 2\operatorname{arsinh}(1))$ and $s_0 \in (-X(\ell) + \Lambda, X(\ell) - \Lambda)$ (i.e. so that $\mathscr{C}_\Lambda(s_0) := (s_0 - \Lambda, s_0 + \Lambda) \times S^1 \subset \mathcal{C}(\ell)$) if such s_0 exists, we have

$$\frac{1}{C}\rho(s_0) \le \rho(s) \le C\rho(s_0) \tag{A.5}$$

for all $s \in \mathscr{C}_\Lambda(s_0)$.

Because $L_g(\{s_0\} \times S^1) = 2\pi\rho(s_0)$ we can always bound

$$\operatorname{inj}_g(p_0) \le \pi\rho(p_0). \tag{A.6}$$

For $\delta \in (0, \operatorname{arsinh}(1))$, the δ-thin part of a collar is given by the subcylinder

$$(-X_\delta(\ell), X_\delta(\ell)) \times S^1 \subseteq \mathcal{C}(\ell), \tag{A.7}$$

where

$$X_\delta(\ell) = \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arcsin\left(\frac{\sinh(\frac{\ell}{2})}{\sinh\delta}\right)\right) \tag{A.8}$$

for $\delta \ge \ell/2$, respectively zero for smaller values of δ.

Proposition A.2. *There exists universal $C \in (0, \infty)$ such that for every $\delta \in (0, \operatorname{arsinh}(1))$ and $0 < \ell \le 2\delta$, we have*

$$\frac{\pi}{\delta} - C \le X - X_\delta \le \frac{\pi^2}{2\delta}.$$

Proof. By definition of X and X_δ, we have

$$X - X_\delta = \frac{2\pi}{\ell}\left[\arcsin\left(\frac{\sinh(\frac{\ell}{2})}{\sinh\delta}\right) - \arctan\left(\sinh\left(\frac{\ell}{2}\right)\right)\right].$$

Using convexity of $\arcsin : [0,1] \to [0, \frac{\pi}{2}]$, we compute the required upper bound

$$X - X_\delta \le \frac{2\pi}{\ell}\arcsin\left(\frac{\sinh(\frac{\ell}{2})}{\sinh\delta}\right) \le \frac{2\pi}{\ell}\frac{\pi}{2}\frac{\sinh(\frac{\ell}{2})}{\sinh\delta} \le \frac{\pi^2}{\ell}\frac{\frac{\ell}{2}}{\delta} = \frac{\pi^2}{2\delta}.$$

On the other hand, by estimating $\arcsin\theta \ge \theta$ and $\arctan\theta \le \theta$ for $\theta \in [0,1]$, we have

$$X - X_\delta \ge \frac{2\pi}{\ell}\left[\frac{\sinh(\frac{\ell}{2})}{\sinh\delta} - \sinh\left(\frac{\ell}{2}\right)\right] \ge \pi\left[\frac{1}{\sinh\delta} - 1\right].$$

By estimating $\sinh\delta \le \delta + C\delta^3$ for $\delta \in (0, \operatorname{arsinh}(1))$ and some universal C, we have $(\sinh\delta)^{-1} \ge \frac{1}{\delta}(1 - C\delta^2)$ for some possibly different C, which completes the lower bound. □

We will use several times that working with respect to the hyperbolic metric, on a collar $\mathcal{C}$ as above,

$$|dz^2| = 2\rho^{-2}; \qquad \|dz^2\|_{L^\infty(\mathcal{C})} = \frac{8\pi^2}{\ell^2}; \qquad \|dz^2\|^2_{L^2(\mathcal{C})} = \frac{32\pi^5}{\ell^3} - \frac{16\pi^4}{3} + O(\ell^2), \tag{A.9}$$

as a short computation verifies.

To analyse sequences of degenerating hyperbolic surfaces we make repeatedly use of the differential geometric version of the Deligne-Mumford compactness theorem.

Proposition A.3. (Deligne-Mumford compactness, cf. [7].) *Let (M, g_i, c_i) be a sequence of closed hyperbolic Riemann surfaces of genus $\gamma \ge 2$ which degenerate in the sense that there is no uniform positive lower bound on $\operatorname{inj}_{g_i} M$. Then, after selection of a subsequence, (M, g_i, c_i) converges to a hyperbolic punctured Riemann surface (Σ, h, c), where Σ is obtained from M by removing a collection $\mathscr{E} = \{\sigma^j, j = 1, ..., k\}$ of k pairwise disjoint, homotopically nontrivial, simple closed curves on M and the convergence is to be understood as follows:*

For each i there exists a collection $\mathscr{E}_i = \{\sigma_i^j, j = 1, ..., k\}$ of pairwise disjoint simple closed geodesics on (M, g_i, c_i) with each σ_i^j homotopic to σ^j of length

$$\ell(\sigma_i^j) =: \ell_i^j \to 0 \text{ as } i \to \infty$$

and a diffeomorphism $f_i : \Sigma \to M \setminus \cup_{j=1}^k \sigma_i^j$ such that

$$(f_i)^* g_i \to h \text{ and } (f_i)^* c_i \to c \text{ in } C^\infty_{loc} \text{ on } \Sigma.$$

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