Local mollification of Riemannian metrics using Ricci flow, and Ricci limit spaces

Miles Simon and Peter M. Topping

28 June 2017

Abstract

Given a three-dimensional Riemannian manifold containing a ball with an explicit lower bound on its Ricci curvature and positive lower bound on its volume, we use Ricci flow to perturb the Riemannian metric on the interior to a nearby Riemannian metric still with such lower bounds on its Ricci curvature and volume, but additionally with uniform bounds on its full curvature tensor and all its derivatives. The new manifold is near to the old one not just in the Gromov-Hausdorff sense, but also in the sense that the distance function is uniformly close to what it was before, and additionally we have Hölder/Lipschitz equivalence of the old and new manifolds.

One consequence is that we obtain a local bi-Hölder correspondence between Ricci limit spaces in three dimensions and smooth manifolds. This is more than a complete resolution of the three-dimensional case of the conjecture of Anderson-Cheeger-Colding-Tian, describing how Ricci limit spaces in three dimensions must be homeomorphic to manifolds, and we obtain this in the most general, locally non-collapsed case. The proofs build on results and ideas from recent papers of Hochard and the current authors.

Contents

1 Introduction

Since its introduction in 1982 by Hamilton, Ricci flow has become central to our efforts to understand the topology of manifolds that are either low dimensional, or that satisfy a curvature condition. See for example [13, 16, 2, 3]. Loosely speaking, the strategy is to allow Ricci flow to evolve such a manifold into one that we can recognise. In this paper we investigate the use of Ricci flow in a different direction. We study its use as a local mollifier of Riemannian metrics, taking an initial metric that is coarsely controlled on some local region, and giving back a smooth metric satisfying good estimates on a slightly smaller region.

We will apply our theory in order to understand Ricci limit spaces, i.e. Gromov-Hausdorff limits of sequences of manifolds with a uniform lower bound on their Ricci tensors, and a uniform positive lower bound on the volume of one unit ball.

Our main Ricci flow result is the following theorem, which applies to Riemannian 3-manifolds that are not assumed to be complete. The result without the uniform conclusions on Ricci curvature, volume and distance follows from recent work of Hochard [15], and our proof crucially uses some of his ideas as we describe later. We will use our uniform conclusions on Ricci curvature, volume and distance in applications to Ricci limit spaces.

Theorem 1.1 (Mollification theorem)**.** *Suppose that (M^3, g_0) is a Riemannian manifold, $x_0 \in M$, $B_{g_0}(x_0, 1) \subset\subset M$, with geometry controlled by*

$$\begin{cases} \mathrm{Ric}_{g_0} \geq -\alpha_0 < 0 & \text{on } B_{g_0}(x_0,1) \\ \mathrm{VolB}_{g_0}(x_0,1) \geq v_0 > 0 \end{cases} \tag{1.1}$$

Then for any $\varepsilon \in (0, 1/10)$, there exist $T, v, \alpha, c_0 > 0$ depending only on α_0, v_0 and ε, and a Ricci flow $g(t)$ defined for $t \in [0, T]$ on the slightly smaller ball $\mathfrak{B} := B_{g_0}(x_0, 1-\varepsilon)$, with $g(0) = g_0$ on $\mathfrak{B}$, such that for each $t \in [0, T]$, $B_{g(t)}(x_0, 1-2\varepsilon) \subset\subset \mathfrak{B}$ where the Ricci flow is defined, and

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha & \text{on } \mathfrak{B} \\ \mathrm{VolB}_{g(t)}(x_0, 1-2\varepsilon) \geq v > 0. \end{cases} \tag{1.2}$$

Moreover, the curvature tensor and all its derivatives are controlled according to

$$\begin{cases} |\mathrm{Rm}|_{g(t)} \leq c_0/t \\ \left|\nabla^k \mathrm{Rm}\right|_{g(t)} \leq C(k, \alpha_0, v_0, \varepsilon)/t^{1+\frac{k}{2}} \end{cases} \tag{1.3}$$

on $\mathfrak{B}$ for all $t \in (0, T]$ and $k \in \mathbb{N}$. If we fix $s \in [0, T]$ and $x, y \in B_{g(s)}(x_0, \frac{1}{2} - 2\varepsilon)$, then $x, y \in B_{g(t)}(x_0, \frac{1}{2} - \varepsilon)$ for all $t \in [0, T]$ so the infimum length of curves connecting x and y within $(\mathfrak{B}, g(t))$ is realised by a geodesic within $B_{g(t)}(x_0, 1-2\varepsilon) \subset \mathfrak{B}$ where the Ricci flow is defined. Moreover, for any $t \in [0, T]$, and such x and y, we have

$$d_{g_0}(x, y) - \beta\sqrt{c_0 t} \leq d_{g(t)}(x, y) \leq e^{\alpha t} d_{g_0}(x, y), \tag{1.4}$$

where $\beta = \beta(n) \geq 1$, and the Hölder estimate

$$d_{g_0}(x, y) \leq \gamma \left[d_{g(t)}(x, y)\right]^{\frac{1}{1+4c_0}}$$

where $\gamma < \infty$ depends only on c_0.

Remark 1.2. We clarify that within a Riemannian manifold (M, g), containing a point x, we write $\mathrm{VolB}_g(x, r)$ for the volume (with respect to g) of a ball $B_g(x, r)$ centred at x of radius r (with respect to g). Later we allow $r \in \mathbb{R}$, and if $r \leq 0$, the ball is empty and the volume zero.

There is an implicit subtlety in considering Riemannian distances on incomplete Riemannian manifolds that we avoid in the theorem above through the assertions that

$$x, y \in B_{g(t)}(x_0, \tfrac{1}{2} - \varepsilon) \quad \text{and} \quad B_{g(t)}(x_0, 1-2\varepsilon) \subset\subset \mathfrak{B},$$

as we explain in the following general remark (with $R = 1 - 2\varepsilon$).

Remark 1.3. If we have a Riemannian manifold (N, g), and we are considering a ball $B_g(x_0, R) \subset\subset N$, then we can make sense of the distance between two points $x, y \in B_g(x_0, R)$ in two ways – either as the infimum length of connecting paths that remain within $B_g(x_0, R)$, or as the infimum length of such paths that may stray anywhere within N. In general, the latter distance will be shorter. However, we observe that if we are sure that x and y lie within the *half* ball $B_g(x_0, R/2)$, then these two distances must agree, and there exists a minimising geodesic between x and y that remains within $B_g(x_0, R)$.

Locally defined Ricci flows were considered by D.Yang, assuming volume bounds from below, supercritical integral bounds on the initial Ricci curvature and a smallness condition on the $L^{n/2}$ norm of the initial Riemannian curvature tensor. See [21, Theorem 9.2]. One could also draw a comparison with the theory of Ricci flows on manifolds with boundary, see the work of Gianniotis [12] and the references therein. They are also considered in the work of Hochard [15].

In this paper we apply the local Ricci flow we construct in Theorem 1.1 to understand the geometry of so-called Ricci limit spaces, i.e. the metric spaces that arise as limits of manifolds with a uniform lower Ricci bound, and a positive uniform lower bound on the volume of *one* unit ball. Such limits need not be smooth; it is easy to see that a metric cone can arise as such a limit, even in two dimensions, and similarly we can even have a dense set of cone points (e.g. [5, Example 0.29]). All such cone points are singular points, defined to be the points in the limit for which there exists a tangent cone that is not Euclidean space. Nevertheless, the following theorem establishes that in (up to) three dimensions, Ricci limit spaces are locally bi-Hölder to smooth manifolds.

Theorem 1.4 (Ricci limits in 3D are bi-Hölder homeomorphic to smooth manifolds locally)**.** *Suppose that (M_i^3, g_i) is a sequence of (not necessarily complete) Riemannian manifolds such that for all i we have $x_i \in M_i$, $B_{g_i}(x_i,1) \subset\subset M_i$, and*

$$\begin{cases} \mathrm{Ric}_{g_i} \geq -\alpha_0 < 0 & \text{on } B_{g_i}(x_i,1) \\ \mathrm{VolB}_{g_i}(x_i,1) \geq v_0 > 0. \end{cases} \tag{1.5}$$

Then there exist a smooth three-dimensional manifold without boundary $\mathcal{M}$, containing a point x_0, and a metric[1] $d_0 : \mathcal{M} \times \mathcal{M} \to [0,\infty)$ generating the same topology as $\mathcal{M}$ such that $B_{d_0}(x_0, 1/10) \subset\subset \mathcal{M}$ and so that after passing to a subsequence the compact metric spaces $(\overline{B_{g_i}(x_i,1/10)}, d_{g_i})$ Gromov-Hausdorff converge to $(\mathcal{B}, d_0)$, where $\mathcal{B} = \overline{B_{d_0}(x_0,1/10)}$.

Moreover there exists a sequence of smooth maps $\varphi_i : \mathcal{M} \to B_{g_i}(x_i, 1/2) \subset M_i$, mapping x_0 to x_i, diffeomorphic onto their images, such that

$$d_{g_i}(\varphi_i(x), \varphi_i(y)) \to d_0(x,y) \qquad \text{uniformly for } (x,y) \in \mathcal{B} \times \mathcal{B},$$

as $i \to \infty$, and for all $\eta \in (0, 1/100)$,

$$B_{g_i}(x_i, 1/10 - \eta) \subset \varphi_i(\mathcal{B}) \subset B_{g_i}(x_i, 1/10 + \eta), \tag{1.6}$$

for sufficiently large i.

Finally, for any smooth *Riemannian metric g on $\mathcal{M}$, the identity map $(\mathcal{B}, d_g) \to (\mathcal{B}, d_0)$ is Hölder continuous. Conversely, the identity map $(\mathcal{B}, d_0) \to (\mathcal{B}, d_g)$ is Lipschitz continuous.*

Here, and elsewhere in this paper, we assume implicitly that all manifolds are connected.

The assertion concerning the maps φ_i is not only implying the previous Gromov-Hausdorff convergence, even pointed Gromov-Hausdorff convergence, but is also saying that we may essentially take the Gromov-Hausdorff approximations to be smooth.

To clarify, when we write $(\overline{B_{g_i}(x_i,1/10)}, d_{g_i})$, we are referring to the metric d_{g_i} defined on the whole of (M_i, g_i), which is subsequently restricted to $\overline{B_{g_i}(x_i,1/10)}$, which need not agree with the infimum of lengths of connecting paths that remain within this 1/10-ball itself, cf. Remark 1.3.

[1] in the sense of metric spaces

The proof of this theorem, given in Section 6, relies on being able to flow the sequence (M_i, g_i) locally using the Mollification theorem 1.1. In particular, the lower Ricci bounds of that theorem will be crucial, and these in turn rely on the work in [19].

As indicated earlier, the regular set of a Ricci limit space is the set of points at which all tangent cones are Euclidean space (here $\mathbb{R}^3$). There is a bigger space, the ε-regular set $\mathcal{R}_\varepsilon$ that is roughly the set of points at which the limit is within ε of being Euclidean on sufficiently small scales, in a scaled sense. In the example above with a possibly dense set of cone points, $\mathcal{R}_\varepsilon$ would include the cone points that have sufficiently small cone angle. It is a theorem of Cheeger-Colding [6, Theorem 5.14] that for sufficiently small $\varepsilon > 0$, the interior of $\mathcal{R}_\varepsilon$ is bi-Hölder homeomorphic to a smooth manifold. In particular, in the special case that the singular set is empty, their theorem implies bi-Hölder equivalence.

Theorem 1.4 has various corollaries, for example:

Corollary 1.5 (Conjecture of Anderson, Cheeger, Colding, Tian, 3D case)**.** *Suppose that (M_i^3, g_i) is a sequence of complete Riemannian manifolds such that for all i we have $y_i \in M_i$, and*

$$\begin{cases} \mathrm{Ric}_{g_i} \geq -\alpha_0 < 0 & \textit{throughout } M_i \\ \mathrm{VolB}_{g_i}(y_i, 1) \geq v_0 > 0. \end{cases} \tag{1.7}$$

Then there exist a three-dimensional topological manifold M and a metric $d : M \times M \to [0, \infty)$ generating the same topology as M and making (M, d) a complete metric space, such that after passing to a subsequence, we have

$$(M_i, d_{g_i}, y_i) \to (M, d, y_0)$$

in the pointed Gromov-Hausdorff sense, for some $y_0 \in M$. The charts of M can be taken to be bi-Hölder with respect to d.

For the definition of pointed Gromov-Hausdorff convergence, see [4, Definition 8.1.1].

Corollary 1.5 solves the conjecture made in [6], Conjecture 0.7 for three-dimensional manifolds. See also, for example [7, Remark 1.19] and [5, Remark 10.23]. A related conjecture of Anderson, see [1, Conjecture 2.3] and [6, Conjecture 0.8], constrains the singular set when one assumes in addition that the Ricci curvature is bounded from *above*. In three dimensions, that conjecture would reduce to the singular set being empty, which would imply the bi-Hölder equivalence that we have proved. Another situation in which the singular set can be shown to be empty in three-dimensions is when one assumes, in addition, L^p control on the curvature tensor, for appropriately large p [7]. In contrast, our work is allowing cone points in the limit, and indeed cone points that are not close to being Euclidean.

Of course, the existence of some Gromov-Hausdorff limit, after passing to a subsequence, is an immediate consequence of Gromov compactness. Our results says that this limit can be assumed to be homeomorphic to a smooth manifold. This is in contrast to the higher dimensional case, as can be seen by blowing down the Eguchi-Hanson (Ricci flat) metric, which will converge to the quotient of $\mathbb{R}^4$ by the map $x \mapsto -x$, which is *not* homeomorphic to any manifold. The conjecture for higher dimensions is that a codimension 4 subset of the limit space should be homeomorphic to a topological manifold.

Note that the noncollapsedness condition in the corollary above is that the volume of *one* unit ball is uniformly bounded below. It is expected that it is *not* possible to start the Ricci flow globally with such a manifold. If we weaken the claim of the corollary and assume a uniform lower bound on the volume of *all* unit balls, i.e. even as we move the centre-point around, then one now *can* start the Ricci flow globally (cf. Theorem 1.7 or [15]) and the result would follow from a combination of [19] and [15]. Raphael Hochard has informed us that a future revision of his paper [15] will contain a complete proof of this latter result. His current preprint proceeds without establishing uniform lower Ricci bounds. Earlier results of the first author have established this result under additional hypotheses [17].

We will give the proof of Corollary 1.5 in Section 7. The same proof will work if we merely assume that for each $r > 0$, the Ricci curvature is uniformly bounded below on balls $B_{g_i}(y_i, r)$, with the bound depending on r but being independent of i.

The existence part of the Mollification theorem 1.1 will be deduced by rescaling the following main local (in space) existence theorem. Some important ideas in the proof arise first in the work of Hochard [15], where related results without the uniform Ricci control were proved.

Theorem 1.6 (Local existence theorem in 3D). *Suppose $s_0 \geq 4$. Suppose further that (M^3, g_0) is a Riemannian manifold, $x_0 \in M$, $B_{g_0}(x_0, s_0) \subset\subset M$, and*

$$\begin{cases} \mathrm{Ric}_{g_0} \geq -\alpha_0 < 0 & \text{on } B_{g_0}(x_0, s_0) \\ \mathrm{VolB}_{g_0}(x, 1) \geq v_0 > 0 & \text{for all } x \in B_{g_0}(x_0, s_0 - 1). \end{cases} \tag{1.8}$$

Then there exist $T, \alpha, c_0 > 0$ depending only on α_0 and v_0, and a Ricci flow $g(t)$ defined for $t \in [0, T]$ on $B_{g_0}(x_0, s_0 - 2)$, with $g(0) = g_0$ where defined, such that

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha & \text{on } B_{g_0}(x_0, s_0 - 2) \\ |\mathrm{Rm}|_{g(t)} \leq c_0/t & \text{on } B_{g_0}(x_0, s_0 - 2) \end{cases} \tag{1.9}$$

for all $t \in (0, T]$.

The curvature upper and lower bounds in this theorem will give us good control on the distance function, via a refinement of [19, Lemma 3.4], valid in any dimension, that is given in Lemma 3.1 below. In particular, although $g(t)$ is not assumed to be complete, this control will guarantee that the Ricci flow is 'big enough'.

Theorem 1.6 is stated in a way that allows us to apply it on larger and larger balls within a *complete* Riemannian manifold (M^3, g_0), to give local Ricci flows with enough regularity to give compactness. That is, a subsequence will converge to a Ricci flow starting with (M^3, g_0) that enjoys the same geometric control as the initial data. The existence of a Ricci flow without the uniform control on Ricci curvature and volume, and without the control on how the distance function can increase, was first proved by Hochard [15], and the following result could be proved indirectly by combining the results of [15] and [19].

Theorem 1.7 (Global existence theorem in 3D). *Suppose that (M^3, g_0) is a complete Riemannian manifold with the properties that*

$$\begin{cases} \mathrm{Ric}_{g_0} \geq -\alpha_0 < 0 \\ \mathrm{VolB}_{g_0}(x, 1) \geq v_0 > 0 & \text{for all } x \in M. \end{cases} \tag{1.10}$$

Then there exist $T, v, \alpha, c_0 > 0$ depending only on α_0 and v_0, and a smooth, complete Ricci flow $g(t)$ defined for $t \in [0, T]$ on M, with $g(0) = g_0$, such that

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha \\ \mathrm{VolB}_{g(t)}(x, 1) \geq v > 0 & \text{for all } x \in M \\ |\mathrm{Rm}|_{g(t)} \leq c_0/t & \text{throughout } M \end{cases} \tag{1.11}$$

for all $t \in (0, T]$. Moreover, for any $0 \leq t_1 \leq t_2 \leq T$, and any $x, y \in M$, we have

$$d_{g(t_1)}(x, y) - \beta\sqrt{c_0}(\sqrt{t_2} - \sqrt{t_1}) \leq d_{g(t_2)}(x, y) \leq e^{\alpha(t_2 - t_1)} d_{g(t_1)}(x, y), \tag{1.12}$$

where $\beta = \beta(n)$.

We prove Theorem 1.7 in Section 8.

Finally, as a variation of the ideas above, we record that it is possible to run the Ricci flow starting with appropriate rough data.

Theorem 1.8 (Ricci flow from a uniformly noncollapsed 3D Ricci limit space)**.** *Suppose that (M_i^3, g_i) is a sequence of complete Riemannian manifolds such that for all i we have $x_i \in M_i$, and*

$$\begin{cases} \mathrm{Ric}_{g_i} \geq -\alpha_0 < 0 & \text{throughout } M_i \\ \mathrm{VolB}_{g_i}(x,1) \geq v_0 > 0 & \text{for all } x \in M_i. \end{cases} \tag{1.13}$$

Then there exists a smooth manifold M, a point $x_\infty \in M$, a complete Ricci flow $g(t)$ on M for $t \in (0,T]$, where $T > 0$ depends only on α_0 and v_0, and a continuous distance metric d_0 on M such that $d_{g(t)} \to d_0$ locally uniformly as $t \downarrow 0$, and after passing to a subsequence in i we have that (M_i, d_{g_i}, x_i) converges in the pointed Gromov-Hausdorff sense to (M, d_0, x_∞). Furthermore, the Ricci flow satisfies the estimates (1.11) and (1.12) for some $\alpha, v, c_0 > 0$ depending only on α_0 and v_0.

We give the proof of Theorem 1.8 in Section 9.

Acknowledgements: This work was supported by EPSRC grant number EP/K00865X/1.

2 Expanding and Shrinking Balls lemmata

In this section we record the two main local ball inclusion lemmas from [19]. In fact, the first is a refinement of the result from [19], which we will need in Section 6.

Lemma 2.1 (The expanding balls lemma, cf. [19, Lemma 3.1])**.** *Suppose $(M, g(t))$ is a Ricci flow for $t \in [-T, 0]$, $T > 0$, on a manifold M of any dimension. Suppose that $x_0 \in M$ and that $B_{g(0)}(x_0, R) \subset\subset M$ and $\mathrm{Ric}_{g(t)} \geq -K < 0$ on $B_{g(0)}(x_0, R) \cap B_{g(t)}(x_0, Re^{Kt}) \subset B_{g(t)}(x_0, R)$ for each $t \in [-T, 0]$. Then*

$$B_{g(0)}(x_0, R) \supset B_{g(t)}(x_0, Re^{Kt}) \tag{2.1}$$

for all $t \in [-T, 0]$.

Proof. It suffices to show that for $\alpha \in (0, 1)$ arbitrarily close to 1 we have

$$B_{g(0)}(x_0, R) \supset \overline{B_{g(t)}(x_0, R\alpha e^{\frac{Kt}{\alpha}})} \text{ for all } t \in [-T, 0]. \tag{2.2}$$

This assertion is clearly true for $t = 0$, and by smoothness, also for $t < 0$ sufficiently close to 0 (depending on α amongst other things). If (2.2) were not true, then we could let $t_0 \in [-T, 0]$ be the supremum of the times at which the inclusion fails, and by smoothness it will fail also at time t_0. Thus we can find $y \in \overline{B_{g(t_0)}(x_0, R\alpha e^{\frac{Kt_0}{\alpha}})}$ such that $d_{g(0)}(x_0, y) = R$. Pick a minimising unit speed geodesic γ, with respect to $g(t_0)$, connecting x_0 and y that lies within both $\overline{B_{g(t_0)}(x_0, R\alpha e^{\frac{Kt_0}{\alpha}})} \subset B_{g(t_0)}(x_0, Re^{Kt_0})$ and $\overline{B_{g(0)}(x_0, R)}$ where $\mathrm{Ric}_{g(t_0)} \geq -K$. We have

$$L_{g(t_0)}(\gamma) \leq R\alpha e^{\frac{Kt_0}{\alpha}}, \tag{2.3}$$

and by the flow equation

$$\left.\frac{d}{dt}\right|_{t=t_0} L_{g(t)}(\gamma) = -\int_\gamma \mathrm{Ric}(\dot\gamma, \dot\gamma) \leq K L_{g(t_0)}(\gamma). \tag{2.4}$$

Because t_0 is the supremum time at which the inclusion of (2.2) fails, for $t \in (t_0, 0]$ we have $L_{g(t)}(\gamma) > R\alpha e^{\frac{Kt}{\alpha}}$, and subtracting (2.3) gives

$$L_{g(t)}(\gamma) - L_{g(t_0)}(\gamma) > R\alpha(e^{\frac{Kt}{\alpha}} - e^{\frac{Kt_0}{\alpha}}).$$

Dividing by $t - t_0 > 0$ and taking a limit $t \downarrow t_0$ gives

$$\left.\frac{d}{dt}\right|_{t=t_0} L_{g(t)}(\gamma) \geq RKe^{\frac{Kt_0}{\alpha}} \geq \frac{1}{\alpha} K L_{g(t_0)}(\gamma),$$

which contradicts (2.4). □

Lemma 2.2 (The shrinking balls lemma, [19, Corollary 3.3]). *Suppose $(M, g(t))$ is a Ricci flow for $t \in [0, T]$ on a manifold M of any dimension n. Then there exists $\beta = \beta(n) \geq 1$ such that the following is true. Suppose $x_0 \in M$ and that $B_{g(0)}(x_0, r) \subset\subset M$ for some $r > 0$, and $|\mathrm{Rm}|_{g(t)} \leq c_0/t$, or more generally $\mathrm{Ric}_{g(t)} \leq (n-1)c_0/t$, on $B_{g(0)}(x_0, r) \cap B_{g(t)}(x_0, r - \beta\sqrt{c_0 t})$ for each $t \in (0, T]$ and some $c_0 > 0$. Then*

$$B_{g(0)}(x_0, r) \supset B_{g(t)}\left(x_0, r - \beta\sqrt{c_0 t}\right) \tag{2.5}$$

for all $t \in [0, T]$. More generally, for $0 \leq s \leq t \leq T$, we have

$$B_{g(s)}\left(x_0, r - \beta\sqrt{c_0 s}\right) \supset B_{g(t)}\left(x_0, r - \beta\sqrt{c_0 t}\right).$$

3 Distance estimates and bi-Hölder equivalence

In this section we explain how the local control on the curvature that we typically have in this paper leads to control on the distance function. The main novelty is a Hölder estimate (3.4).

Lemma 3.1 (Bi-Hölder distance estimate). *Suppose $(M^n, g(t))$ is a Ricci flow for $t \in (0, T]$, not necessarily complete, with the property that for some $x_0 \in M$ and $r > 0$, and all $t \in (0, T]$, we have $B_{g(t)}(x_0, 2r) \subset\subset M$. Suppose further that for some $c_0, \alpha > 0$, and for each $t \in (0, T]$, we have*

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha \\ \mathrm{Ric}_{g(t)} \leq \dfrac{(n-1)c_0}{t} \end{cases} \tag{3.1}$$

throughout $B_{g(t)}(x_0, 2r)$. By Remark 1.3, for all $x, y \in \Omega_T \subset M$, where

$$\Omega_T := \bigcap_{t \in (0,T]} B_{g(t)}(x_0, r),$$

the distance $d_{g(t)}(x, y)$ is unambiguous for all $t \in (0, T]$ and must be realised by a minimising geodesic lying within $B_{g(t)}(x_0, 2r)$. Then for any $0 < t_1 \leq t_2 \leq T$, we have

$$d_{g(t_1)}(x, y) - \beta\sqrt{c_0}(\sqrt{t_2} - \sqrt{t_1}) \leq d_{g(t_2)}(x, y) \leq e^{\alpha(t_2 - t_1)} d_{g(t_1)}(x, y), \tag{3.2}$$

where $\beta = \beta(n)$. In particular, the distance metrics $d_{g(t)}$ converge uniformly to a distance metric d_0 on Ω_T as $t \downarrow 0$, and

$$d_0(x, y) - \beta\sqrt{c_0 t} \leq d_{g(t)}(x, y) \leq e^{\alpha t} d_0(x, y), \tag{3.3}$$

for all $t \in (0, T]$. Furthermore, there exists $\gamma > 0$ depending on n, c_0 and upper bounds for r and T such that

$$d_{g(t)}(x, y) \geq \gamma\,[d_0(x, y)]^{1 + 2(n-1)c_0}, \tag{3.4}$$

for all $t \in (0, T]$. Finally, for all $t \in (0, T]$ and $R < R_0 := re^{-\alpha T} - \beta\sqrt{c_0 T} < r$, we have

$$B_{g(t)}(x_0, R_0) \subset \Omega_T \qquad \textit{and} \qquad B_{d_0}(x_0, R) \subset\subset \mathcal{M}, \tag{3.5}$$

where $\mathcal{M}$ is the component of $\mathrm{Interior}(\Omega_T)$ *containing x_0.*

To clarify, the first inclusion of (3.5) will be vacuous if T is sufficiently large that R_0 is nonpositive.

Remark 3.2. A consequence of this lemma is that the identity map $(\mathcal{M}, d_{g(t)}) \to (\mathcal{M}, d_0)$ is Hölder continuous with Hölder exponent $[1 + 2(n-1)c_0]^{-1}$, and the identity map $(\mathcal{M}, d_0) \to (\mathcal{M}, d_{g(t)})$ in the other direction is Lipschitz. The Hölder exponent can be seen to be sharp on solitons coming out of cones in two dimensions.

Remark 3.3. We have stated the lemma for Ricci flows that exist for $t \in (0, T]$, since that is the situation for limit flows considered in this paper. However, we can apply the lemma to Ricci flows that exist for $t \in [0, T]$, in which case we automatically have $d_0 = d_{g(0)}$, and the conclusions of the lemma hold on the whole interval $[0, T]$.

Remark 3.4. Estimate (3.4) only uses the *upper* curvature bound of the lemma, i.e. the Ricci lower bound is not required. In contrast to earlier distance estimates, we prove it by splitting the time interval $[0, t]$ into two subintervals $[0, t_3]$ and $[t_3, t]$, and controlling the distance on each using different techniques.

Proof of Lemma 3.1. The function $t \mapsto d_{g(t)}(x, y)$ is locally Lipschitz on $(0, T]$. At times at which this function is differentiable, we have

$$\frac{d}{dt} d_{g(t)}(x, y) = \frac{d}{dt} L_{g(t)}(\gamma),$$

where γ is a unit-speed minimising geodesic from x to y at the given time. On one hand, we have

$$\frac{d}{dt} d_{g(t)}(x, y) = \frac{d}{dt} L_{g(t)}(\gamma) = -\int_\gamma \mathrm{Ric}(\dot\gamma, \dot\gamma) \tag{3.6}$$

$$\leq \alpha\, d_{g(t)}(x, y), \tag{3.7}$$

and integrating in time gives the second inequality of (3.2). On the other hand, we know from Hamilton-Perelman [19, (3.6)] that

$$\frac{d}{dt} d_{g(t)}(x, y) = \frac{d}{dt} L_{g(t)}(\gamma) \geq -\frac{\beta}{2}\sqrt{c_0/t},$$

where $\beta = \beta(n) \geq 1$ is ultimately the β that appears in Lemma 2.2. Integrating in time now gives the first inequality of (3.2). Having established (3.2), the existence of d_0, and the estimate (3.3), is obvious. Note that the second inequality of (3.3) guarantees that $d_0(x, y) = 0$ implies $x = y$, as required for d_0 to be a metric.

The first step to proving (3.4) is to observe that for

$$0 < t \leq t_3 := \frac{1}{c_0}\left[\frac{1}{2\beta} d_0(x, y)\right]^2,$$

the first inequality of (3.3) implies that

$$d_{g(t)}(x, y) \geq \frac{1}{2} d_0(x, y). \tag{3.8}$$

In particular, for $t \leq t_3$ we have established a stronger conclusion than our desired (3.4). On the other hand, to prove (3.4) for times after t_3, we first use (3.8) at the last possible time t_3, giving

$$d_{g(t_3)}(x, y) \geq \frac{1}{2} d_0(x, y), \tag{3.9}$$

and then use the consequence of (3.6) that

$$\frac{d}{dt} d_{g(t)}(x, y) \geq -(n-1)\frac{c_0}{t} d_{g(t)}(x, y),$$

at times at which $t \mapsto d_{g(t)}(x, y)$ is differentiable, which implies, when integrated from time t_3 to any later time $t > t_3$, that

$$d_{g(t)}(x, y) \geq d_{g(t_3)}(x, y) \left[\frac{t}{t_3}\right]^{-(n-1)c_0}.$$

Inserting the formula for t_3, and using (3.9) we obtain

$$d_{g(t)}(x, y) \geq \gamma(n, c_0) \left[d_0(x, y)\right]^{1+2(n-1)c_0} t^{-(n-1)c_0}, \tag{3.10}$$

which is a little stronger than required.

It remains to prove the inclusions (3.5). On the one hand, for $s \in (0, t)$ we have $R_0 + \beta\sqrt{c_0(t-s)} < R_0 + \beta\sqrt{c_0 T} = re^{-\alpha T} < r$, and so by Lemma 2.2 applied from time s onwards, we have

$$\begin{aligned} B_{g(t)}(x_0, R_0) &\subset B_{g(s)}(x_0, R_0 + \beta\sqrt{c_0(t-s)}) \\ &\subset B_{g(s)}(x_0, R_0 + \beta\sqrt{c_0 T}) = B_{g(s)}(x_0, re^{-\alpha T}) \\ &\subset B_{g(s)}(x_0, r). \end{aligned} \tag{3.11}$$

On the other hand, for $s \in (t, T]$, we have $R_0 e^{\alpha(s-t)} < R_0 e^{\alpha T} < r$, and so by Lemma 2.1 we have

$$B_{g(t)}(x_0, R_0) \subset B_{g(s)}(x_0, R_0 e^{\alpha(s-t)}) \subset B_{g(s)}(x_0, r).$$

Combining, these two cases $s \in (0, t)$ and $s \in (t, T]$, we find that $B_{g(t)}(x_0, R_0) \subset \Omega_T$ as claimed in the first inclusion of (3.5).

An immediate consequence of this inclusion is that $B_{g(t)}(x_0, \frac{R+R_0}{2}) \subset\subset \text{Interior}(\Omega_T)$, because $\frac{R+R_0}{2} < R_0$. Therefore the second inclusion of (3.5) will follow immediately if we can show that $B_{d_0}(x_0, R) \subset B_{g(t)}(x_0, \frac{R+R_0}{2})$ for some $t \in (0, T]$. However, this will be true for sufficiently small $t > 0$ by the uniform convergence of $d_{g(t)}$ to d_0 as $t \downarrow 0$. □

An optimised version of estimate (3.10) would tell us that we can take γ as close as we like to 1 by taking c_0 sufficiently small.

4 Proof of the Local Existence Theorem 1.6

We shall need the following generalisations and consequences of the Local Lemma, [19, Lemma 2.1], the Double Bootstrap Lemma, [19, Lemma 9.1] and Hochard's Lemma, [15, Lemma 6.2], in order to prove Theorem 1.6. In all of the three lemmata appearing below, the Riemannian manifolds (N, g) appearing are not necessarily complete. However, it still makes sense to define the injectivity radius at $p \in N$ as the supremum of the radii r for which the exponential map at p is well-defined on the ball of radius r in T_pN, and is a diffeomorphism from that ball to its image.

Lemma 4.1 (cf. [19, Lemma 2.1])**.** *Let $(N^3, g(t))_{t\in[0,T]}$ be a smooth Ricci flow such that for some fixed $x \in N$ we have $B_{g(t)}(x, 1) \subset\subset N$ for all $t \in [0, T]$, and so that*

(i) $\text{VolB}_{g(0)}(x, 1) \geq v_0 > 0$, *and*
(ii) $\text{Ric}_{g(t)} \geq -1$ *on* $B_{g(t)}(x, 1)$ *for all* $t \in [0, T]$.

Then there exist $C_0 = C_0(v_0) \geq 1$ and $\hat{T} = \hat{T}(v_0) > 0$ such that $|\text{Rm}|_{g(t)}(x) \leq C_0/t$, and $\text{inj}_{g(t)}(x) \geq \sqrt{t/C_0}$ for all $0 < t \leq \min(\hat{T}, T)$.

Proof. Lemma 2.1 of [19] tells us that there exist $C_0 = C_0(v_0) \geq 1$, $\hat{T} = \hat{T}(v_0) > 0$ and $\eta_0 = \eta_0(v_0) > 0$ such that $|\mathrm{Rm}|_{g(t)} \leq \frac{C_0}{t}$ on $B_{g(t)}(x, 1/2)$ and $\mathrm{VolB}_{g(t)}(x, 1/2) \geq \eta_0$ for all $t \in (0, \min(\hat{T}, T)]$. (The lemma there gave the volume of the unit ball, but we are assuming a Ricci lower bound throughout the unit ball, so Bishop-Gromov applies.) The injectivity radius estimate of Cheeger-Gromov-Taylor [8] and the Bishop-Gromov comparison principle then tell us (after scaling $g(t)$ to $\tilde{g} = \frac{1}{t}g(t)$ for each t, and then scaling back), that there exists $i_0 = i_0(\eta_0, C_0) = i_0(v_0) > 0$ such that $\mathrm{inj}_{g(t)}(x) \geq i_0\sqrt{t}$. By increasing C_0 if necessary, we may assume without loss of generality that $i_0 \geq \frac{1}{\sqrt{C_0}}$. □

Lemma 4.2 (cf. Double Bootstrap Lemma [19, Lemma 9.1])**.** *Let $(N^3, g(t))_{t\in[0,T]}$ be a smooth Ricci flow, and $x \in N$, such that $B_{g(0)}(x, 2)$ is compactly contained in N and so that throughout $B_{g(0)}(x, 2)$ we have*

(i) $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ *for some $c_0 \geq 1$ and all $t \in (0, T]$, and*
(ii) $\mathrm{Ric}_{g(0)} \geq -\delta_0$ *for some $\delta_0 > 0$.*

Then there exists $\hat{S} = \hat{S}(c_0, \delta_0) > 0$ such that $\mathrm{Ric}_{g(t)}(x) \geq -100\delta_0 c_0$ for all $0 \leq t \leq \min(\hat{S}, T)$.

Proof. Using Lemma 2.2, we see that $B_{g(t)}(x, 2-\beta\sqrt{c_0 t}) \subset B_{g(0)}(x, 2)$ for some universal constant $\beta \geq 1$, for all $t \in [0, T]$. Choosing $\hat{S} = \frac{1}{\beta^2 c_0}$, we see that $B_{g(t)}(x, 1) \subset B_{g(0)}(x, 2)$ for all $0 \leq t \leq \min(\hat{S}, T)$. The conclusions of the lemma now follow from direct application of [19, Lemma 9.1], after decreasing $\hat{S}$ again if necessary. □

The proof of Theorem 1.6 will involve defining a local smooth $(B_{g_0}(x_0, r_1), g(t))$ solution for $t \in [0, t_1]$ for some small (uncontrolled) $t_1 > 0$, and $r_1 = s_0 - 1$, and then inductively defining smooth local extensions $(B_{g_0}(x_0, r_i), g(t))$ for $t \in [0, t_i]$ where t_i is a geometrically increasing sequence and r_i is a decreasing sequence that nevertheless enjoys a good lower bound. The general strategy of construction of a flow on dyadic time intervals was first used by Hochard [15]. However, in our approach the solutions will each satisfy both $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ and $\mathrm{Ric}_{g(t)} \geq -\alpha(v_0, \alpha_0)$ on $B_{g_0}(x_0, r_i)$ for all $t \in [0, t_i]$. The main inductive step is achieved through the Extension Lemma 4.4 below, whose proof involves conformally modifying the metric, as in [15, Lemma 6.2], and then using the two lemmata from above. We rewrite Hochard's Lemma here in a scaled form for ease of application in the proofs that follow.

Lemma 4.3 (Variant of Hochard, [15, Lemma 6.2])**.** *Let (N^n, g) be a smooth (not necessarily complete) Riemannian manifold and let $U \subset N$ be an open set. Assume that for some $\rho \in (0, 1]$, we have $\sup_U |\mathrm{Rm}|_g \leq \rho^{-2}$, $B_g(x, \rho) \subset\subset N$ and $\mathrm{inj}_g(x) \geq \rho$ for all $x \in U$. Then there exist a constant $\gamma = \gamma(n) \geq 1$, an open set $\tilde{U} \subset U$ and a smooth metric $\tilde{g}$ defined on $\tilde{U}$ such that each connected component of $(\tilde{U}, \tilde{g})$ is a complete Riemmanian manifold satisfying*

(1) $\sup_{\tilde{U}} |\mathrm{Rm}|_{\tilde{g}} \leq \gamma\rho^{-2}$
(2) $U_\rho \subset \tilde{U} \subset U$
(3) $\tilde{g} = g$ on $\tilde{U}_\rho \supset U_{2\rho}$,

where $U_s = \{x \in U \mid B_g(x, s) \subset\subset U\}$.

The strategy of Hochard to prove this lemma is to conformally blow up the metric in a neighbourhood of the boundary of U so that it looks essentially hyperbolic. Once we have a complete metric locally, we can run the Ricci flow with standard existence theory. The idea of cutting off a metric locally and replacing it with a complete hyperbolic metric in order to start the flow was introduced in [20] in a much simpler situation. A global conformal deformation of metric that is related to Hochard's construction was carried out in Section 8, in particular Theorem 8.4, of [17].

Proof. Scale the metric g by $h = \rho^{-2}g$. The new metric h satisfies $\sup_U |\mathrm{Rm}|_h \le 1$, $B_h(x,1) \subset\subset N$ and $\mathrm{inj}_h(x) \ge 1$ for all $x \in U$. Let $C(n)$ be the constant from Hochard's Lemma 6.2 in [15]. Without loss of generality this constant satisfies $C(n) > 1$ since otherwise we can set $C(n)$ to be the maximum of the old $C(n)$ and 2, and note that the conclusions of that lemma will still be correct. We use Hochard's Lemma 6.2 [15] applied to the Riemmanian manifold (N,h) and the set U appearing in the statement of this lemma, with the choice of $k = C^2(n)$. We conclude that there exists an open set $\tilde{U} \subset U$ and a metric $\tilde{h}$ defined on $\tilde{U}$ such that each connected component of $(\tilde{U},\tilde{h})$ is smooth and complete and satisfies

$$(1)\quad \sup_{\tilde{U}} |\mathrm{Rm}|_{\tilde{h}} \le \gamma \tag{4.1}$$

$$(2)\quad U_1 \subset \tilde{U} \subset U \tag{4.2}$$

$$(3)\quad \tilde{h} = h \text{ on } \tilde{U}_1, \tag{4.3}$$

where $U_s = \{x \in U \mid B_h(x,s) \subset\subset U\}$, and $\gamma := C^2(n)$. Let x be any point in U_2. Then $B_h(x,2) \subset\subset U$ by definition, and hence, by the triangle inequality, $B_h(x,1) \subset\subset U_1 \subset \tilde{U}$, and thus $x \in \tilde{U}_1$. This shows that $U_2 \subset \tilde{U}_1$, in view of the fact that $x \in U_2$ was arbitrary. Hence, the conclusion (3) above may be replaced by $\tilde{h} = h$ on $\tilde{U}_1 \supset U_2$. Scaling back, that is, defining $\tilde{g} = \rho^2\tilde{h}$, completes the proof. □

The Extension Lemma 4.4 shows us how we can extend a smooth solution for a short, but well-defined time, still maintaining bounds like $|\mathrm{Rm}|_{g(t)} \le \frac{c_0}{t}$, if the initial Ricci curvature and volume are bounded from below. The cost is that we have to decrease the size of the region on which the smooth solution is defined. The extension will only be possible if the size of the radius r_1 of the ball (with respect to $g(0)$) where the solution is defined is not too small, $r_1 \ge 2$ will suffice, and the time for which the solution is defined is not too large.

In the following lemma, no metrics are assumed to be complete.

Lemma 4.4 (Extension Lemma). *For $v_0 > 0$ given, there exist $c_0 \ge 1$ and $\tau > 0$ such that the following is true. Let $r_1 \ge 2$, and (M, g_0) be a smooth three-dimensional Riemannian manifold such that $B_{g_0}(x_0, r_1) \subset\subset M$, and*

(i) $\mathrm{Ric}_{g_0} \ge -\alpha_0$ *for some $\alpha_0 \ge 1$ on $B_{g_0}(x_0, r_1)$, and*
(ii) $\mathrm{VolB}_{g_0}(x,r) \ge v_0 r^3$ *for all $r \le 1$ and all $x \in B_{g_0}(x_0, r_1 - r)$.*

Assume further that we are given a smooth Ricci flow $(B_{g_0}(x_0, r_1), g(t))$, $t \in [0, \ell_1]$, where $\ell_1 \le \frac{\tau}{200\alpha_0 c_0}$, with $g(0)$ equal to the restriction of g_0, for which

(a) $|\mathrm{Rm}|_{g(t)} \le \frac{c_0}{t}$ *and*
(b) $\mathrm{Ric}_{g(t)} \ge -\frac{\tau}{\ell_1}$

on $B_{g_0}(x_0, r_1)$ for all $t \in (0, \ell_1]$. Then, setting $\ell_2 = \ell_1(1 + \frac{1}{4c_0})$ and $r_2 = r_1 - 6\sqrt{\frac{\ell_2}{\tau}} \ge 1$, the Ricci flow $g(t)$ can be extended smoothly to a Ricci flow on the smaller ball $B_{g_0}(x_0, r_2)$, for the longer time interval $t \in [0, \ell_2]$, with

(a′) $|\mathrm{Rm}|_{g(t)} \le \frac{c_0}{t}$ *and*
(b′) $\mathrm{Ric}_{g(t)} \ge -\frac{\tau}{\ell_2}$

throughout $B_{g_0}(x_0, r_2)$ for all $t \in (0, \ell_2]$.

A version of the Extension Lemma can be given without assumption (b) since we can always obtain a lower Ricci bound by application of the Double Bootstrap Lemma 4.2. However, in practice we will always have this lower bound already, and its inclusion simplifies the proof and emphasises the natural symmetry between the hypothesis and conclusion.

Proof. For the given v_0, let $C_0 \geq 1$ and $\hat{T} > 0$ be the constants given by Lemma 4.1. With this choice of C_0 we choose $c_0 = 4\gamma C_0 > C_0$, where $\gamma \geq 1$ is the constant coming from Hochard's Lemma 4.3 above, and set $\delta_0 = \frac{1}{100c_0}$, and we let $\hat{S}$ be the constant we obtain from Lemma 4.2 for these choices. We define $\tau := \min\{\hat{T}, \hat{S}\}$ so that we are free to apply Lemma 4.1 and Lemma 4.2 with $\hat{T}$, respectively $\hat{S}$, replaced by τ. We also reduce τ if necessary so that

$$\beta^2 c_0 \tau \leq 1, \qquad \tau \leq 1, \qquad \text{and } \tau \leq C_0/4, \tag{4.4}$$

where $\beta \geq 1$ is the constant from Lemma 2.2. The constants are now fixed. Note that

$$\ell_1 \leq \ell_2 \leq 2\ell_1 \leq \tau \leq 1. \tag{4.5}$$

Claim 1: For all $x \in U := B_{g_0}(x_0, r_1 - 2\sqrt{\frac{\ell_1}{\tau}})$, we have $B_{g(t)}(x, \sqrt{t/C_0}) \subset\subset B_{g_0}(x_0, r_1)$, $\mathrm{inj}_{g(t)}(x) \geq \sqrt{t/C_0}$ and $|\mathrm{Rm}|_{g(t)}(x) \leq C_0/t$, for all $t \in (0, \ell_1]$.

Note that by assumption, we have c_0/t curvature decay; the claim improves this to C_0/t curvature decay, albeit on a smaller ball, as well as obtaining an injectivity radius bound. The original c_0/t decay will nevertheless be required to control the nesting of balls.

Proof of Claim 1: For $x \in B_{g_0}(x_0, r_1 - 2\sqrt{\frac{\ell_1}{\tau}})$, the triangle inequality implies that $B_{g_0}(x, 2\sqrt{\frac{\ell_1}{\tau}}) \subset\subset B_{g_0}(x_0, r_1)$ and hence by hypothesis, $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ and $\mathrm{Ric}_{g(t)} \geq -\frac{\tau}{\ell_1}$ on $B_{g_0}(x, 2\sqrt{\frac{\ell_1}{\tau}})$ for all $t \in (0, \ell_1]$. Scaling the solution to $\hat{g}(t) := \frac{\tau}{\ell_1} g(t\frac{\ell_1}{\tau})$ we see that we have a solution $\hat{g}(t)$ on $B_{g_0}(x_0, r_1) \supset\supset B_{\hat{g}(0)}(x, 2)$, $t \in [0, \tau]$ with $|\mathrm{Rm}|_{\hat{g}(t)} \leq \frac{c_0}{t}$ and $\mathrm{Ric}_{\hat{g}(t)} \geq -1$ on $B_{\hat{g}(0)}(x, 2)$ for all $t \in (0, \tau]$.

Applying Lemma 2.2 to $\hat{g}(t)$, we find that $B_{\hat{g}(t)}(x, 2 - \beta\sqrt{c_0 t}) \subset B_{\hat{g}(0)}(x, 2)$ for all $t \in [0, \tau]$, and in particular, $B_{\hat{g}(t)}(x, 1) \subset B_{\hat{g}(0)}(x, 2)$ because of (4.4).

This puts us in a position to apply Lemma 4.1 to $\hat{g}(t)$, giving that $\mathrm{inj}_{\hat{g}(t)}(x) \geq \sqrt{t/C_0}$ and $|\mathrm{Rm}|_{\hat{g}(t)}(x) \leq C_0/t$ for all $0 < t \leq \tau$. Scaling back, we see that $B_{g(t)}(x, \sqrt{\ell_1/\tau}) \subset\subset B_{g_0}(x_0, r_1)$, $\mathrm{inj}_{g(t)}(x) \geq \sqrt{t/C_0}$ and and $|\mathrm{Rm}|_{g(t)}(x) \leq C_0/t$ for all $t \in (0, \ell_1]$, which is a little stronger than Claim 1 because $t \leq \ell_1$ and $C_0 \geq \tau$ by (4.4), so $\frac{t}{C_0} \leq \frac{\ell_1}{\tau}$. //

Claim 1, specialised to $t = \ell_1$, puts us in exactly the situation we require in order to apply Lemma 4.3 with $U = B_{g_0}(x_0, r_1 - 2\sqrt{\frac{\ell_1}{\tau}})$, $N = B_{g_0}(x_0, r_1)$, $g = g(\ell_1)$ and $\rho^2 := \frac{\ell_1}{C_0} \leq 1$ (recall (4.5)). The output is a new, possibly disconnected, smooth manifold $(\tilde{U}, \tilde{g})$, each component of which is complete, such that

$$\begin{aligned} &(1)\ |\mathrm{Rm}|_{\tilde{g}} \leq \frac{\gamma C_0}{\ell_1} = \frac{c_0}{4\ell_1} \text{ on } \tilde{U} \\ &(2)\ \ U_{\sqrt{\frac{\ell_1}{C_0}}} \subset \tilde{U} \subset U \\ &(3)\ \ \tilde{g} = g(\ell_1) \text{ on } \tilde{U}_{\sqrt{\frac{\ell_1}{C_0}}} \supset U_{2\sqrt{\frac{\ell_1}{C_0}}}, \end{aligned} \tag{4.6}$$

where $U_r = \{x \in U \mid B_{g(\ell_1)}(x, r) \subset\subset U\}$.

Before restarting the flow with $\tilde{g}$ on one component of $\tilde{U}$, we take a closer look at where $g(\ell_1)$ equals $\tilde{g}$:

Claim 2: We have $B_{g_0}(x_0, r_1 - 4\sqrt{\frac{\ell_1}{\tau}}) \subset U_{2\sqrt{\frac{\ell_1}{C_0}}}$, where the metrics $g(\ell_1)$ and $\tilde{g}$ agree.

Proof of Claim 2: By definition of U, for every $x \in B_{g_0}(x_0, r_1 - 4\sqrt{\frac{\ell_1}{\tau}})$, we have $B_{g_0}(x, 2\sqrt{\frac{\ell_1}{\tau}}) \subset\subset U$. By assumption, we have $|\mathrm{Rm}|_{g(t)} \leq c_0/t$ on $B_{g_0}(x_0, r_1)$, and hence on U and on $B_{g_0}(x, 2\sqrt{\frac{\ell_1}{\tau}})$ for all $t \in (0, \ell_1]$, and so Lemma 2.2 tells us that $B_{g_0}(x, 2\sqrt{\frac{\ell_1}{\tau}}) \supset B_{g(t)}(x, 2\sqrt{\frac{\ell_1}{\tau}} - \beta\sqrt{c_0 t})$ for all $t \in [0, \ell_1]$. Specialising to $t = \ell_1$, and recalling that $\beta\sqrt{c_0\ell_1} \leq \sqrt{\ell_1/\tau}$ by (4.4), we see that $B_{g(\ell_1)}(x, \sqrt{\frac{\ell_1}{\tau}}) \subset\subset U$, and by (4.4) this gives $B_{g(\ell_1)}(x, 2\sqrt{\frac{\ell_1}{C_0}}) \subset\subset U$ as required to establish that $x \in U_{2\sqrt{\frac{\ell_1}{C_0}}}$. //

We now restart the flow at time ℓ_1, using Shi's complete bounded-curvature Ricci flow, starting at the connected component of $(\tilde{U}, \tilde{g})$ that contains x_0. By Claim 2 (and the inequality $\ell_1 < \ell_2$) the new Ricci flow will live on a superset of $B_{g_0}(x_0, r_1 - 4\sqrt{\frac{\ell_1}{\tau}}) \supset B_{g_0}(x_0, r_1 - 4\sqrt{\frac{\ell_2}{\tau}})$, and we call it still $g(t)$ for t beyond ℓ_1. By the standard *doubling time estimate* [10, Lemma 6.1], any smooth complete bounded-curvature Ricci flow $h(t)$ such that $|\mathrm{Rm}|_{h(0)} \leq K$ must satisfy $|\mathrm{Rm}|_{h(t)} \leq 2K$ for $t \leq \frac{1}{16K}$. In our situation, where $K = \frac{c_0}{4\ell_1}$, this tells us that $g(t)$ will exist for $t \in [\ell_1, \ell_1(1 + \frac{1}{4c_0})] = [\ell_1, \ell_2]$ and satisfy $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{2\ell_1} \leq \frac{c_0}{\ell_2} \leq \frac{c_0}{t}$ throughout its domain of definition by (4.5). Thus, we have constructed a smooth extension to our Ricci flow on $B_{g_0}(x_0, r_1 - 4\sqrt{\frac{\ell_2}{\tau}})$, now existing for $t \in [0, \ell_2]$, and this satisfies $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$.

It remains to show that by reducing the radius of our g_0 ball where the extension is defined to $r_2 = r_1 - 6\sqrt{\frac{\ell_2}{\tau}}$, we can be sure of the lower Ricci bound as claimed in the lemma. Pick an arbitrary $x \in B_{g_0}(x_0, r_2)$ at which we would like to establish the lower bound $\mathrm{Ric}_{g(t)} \geq -\frac{\tau}{\ell_2}$. Observe that the extended Ricci flow is defined throughout $B_{g_0}(x, 2\sqrt{\frac{\ell_2}{\tau}}) \subset\subset B_{g_0}(x_0, r_1 - 4\sqrt{\frac{\ell_2}{\tau}})$ for all $t \in [0, \ell_2]$.

We scale up so that ℓ_2 goes to τ. That is we define $\hat{g}(s) := \frac{\tau}{\ell_2} g(\frac{s \cdot \ell_2}{\tau})$. Then we have, for this scaled solution, $B_{\hat{g}(0)}(x, 2)$ compactly contained within the domain of definition of the flow, for all $t \in [0, \tau]$, and $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ on $B_{\hat{g}(0)}(x, 2)$ for all $t \in (0, \tau]$.

Moreover, we have $\mathrm{Ric}_{\hat{g}(0)} \geq -\delta_0$ on $B_{\hat{g}(0)}(x, 2)$ for the choice of δ_0 made above, because keeping in mind that $\ell_1 \leq \frac{\tau}{200\alpha_0 c_0}$, we have

$$\mathrm{Ric}_{\hat{g}(0)} \geq -\alpha_0 \frac{\ell_2}{\tau} \geq -2\alpha_0 \frac{\ell_1}{\tau} \geq -\frac{1}{100c_0} = -\delta_0.$$

Using Lemma 4.2 with the c_0 and δ_0 we have chosen, we see that $\mathrm{Ric}_{\hat{g}(t)}(x) \geq -1$ for all $t \in [0, \tau]$. Rescaling back, we see that $\mathrm{Ric}_{g(t)}(x) \geq -\frac{\tau}{\ell_2}$ for all $t \in [0, \ell_2]$ as required to complete the proof. □

Remark 4.5. With hindsight, we see that $(B_{g_0}(x_0, r_2), g(\ell_2))$ is the restriction to a local region of the final time of a complete Ricci flow $g(t)$ for $t \in [\ell_1, \ell_2]$ with the property that $|\mathrm{Rm}|_{g(t)} \leq c_0/\ell_1$. Global derivative bounds for such flows, applied over the time interval $[\ell_1, \ell_2]$, give us a bound $|\nabla^k \mathrm{Rm}|_{g(\ell_2)} \leq C/\ell_2^{1+k/2}$ at the end time. Moreover, this flow $g(t)$ has initially, at $t = \ell_1$, a lower injectivity radius bound, and such a bound will remain at time ℓ_2 because of the curvature bound for the flow, giving $\mathrm{inj}_{g(\ell_2)} \geq \sqrt{\ell_2/C}$. Thus at the *end* time of the extension, the extended metric is isometrically embedded within a complete Riemannian manifold with good bounds on its curvature and all its derivatives, as well as its injectivity radius.

Proof of Theorem 1.6. We may assume, without loss of generality, that $\alpha_0 \geq 1$: if $\alpha_0 < 1$ then $\mathrm{Ric}_{g_0} \geq -\alpha_0$ implies $\mathrm{Ric}_{g_0} \geq -1$ and so we replace α_0 in this case by 1. From the Bishop-Gromov comparison principle, by reducing v_0 to a smaller positive number depending on α_0 and the original v_0, we may assume without loss of generality that $\mathrm{VolB}_{g_0}(x,r) \geq v_0 r^3$ for all $x \in B_{g_0}(x_0, s_0 - 1)$ and for all $r \in (0,1]$. Let c_0 and τ be the constants given by Lemma 4.4 for this new v_0.

Since $B_{g_0}(x_0, s_0)$ is compactly contained in M, we can be sure that $\sup_{B_{g_0}(x_0,s_0)} |\mathrm{Rm}|_{g_0} \leq \rho^{-2} < \infty$, and $B_{g_0}(x, \rho) \subset\subset M$ and $\mathrm{inj}_{g_0}(x) \geq \rho$ for all $x \in B_{g_0}(x_0, s_0)$, for some $\rho \in (0, \frac{1}{2}]$ depending on (M, g_0), x_0 and s_0. By using Hochard's Lemma, Lemma 4.3 of this paper, with $U := B_{g_0}(x_0, s_0)$, we can find a connected subset $\tilde{M} \subset U \subset M$ containing $B_{g_0}(x_0, s_0 - \frac{1}{2})$, and a smooth, complete metric $\tilde{g}_0$ on $\tilde{M}$ with $\sup_{\tilde{M}} |\mathrm{Rm}|_{\tilde{g}_0} < \infty$ such that on $B_{g_0}(x_0, s_1)$, where $s_1 := s_0 - 1 \geq 3$, the metric remains unchanged, i.e. $\tilde{g}_0 = g_0$ there. By renaming $(\tilde{M}, \tilde{g}_0)$ as (M, g_0) we have reduced the theorem to the following:

Claim: Suppose that (M^3, g_0) is a *complete* Riemannian manifold with *bounded curvature*, $x_0 \in M$, $s_1 \geq 3$, and

$$\begin{cases} \mathrm{Ric}_{g_0} \geq -\alpha_0 \leq -1 & \text{on } B_{g_0}(x_0, s_1) \\ \mathrm{VolB}_{g_0}(x,r) \geq v_0 r^3 > 0 & \text{for all } r \in (0,1] \text{ and } x \in B_{g_0}(x_0, s_1 - r). \end{cases} \tag{4.7}$$

Then there exist $T, \alpha, c_0 > 0$ depending only on α_0 and v_0, and a Ricci flow $g(t)$ defined for $t \in [0,T]$ on $B_{g_0}(x_0, s_1 - 1)$, with $g(0) = g_0$ where defined, such that

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha & \text{on } B_{g_0}(x_0, s_1 - 1) \\ |\mathrm{Rm}|_{g(t)} \leq c_0/t & \text{on } B_{g_0}(x_0, s_1 - 1) \end{cases} \tag{4.8}$$

for all $t \in (0, T]$.

Now that we have reduced to the case that (M, g_0) is complete, with bounded curvature, we can take Shi's Ricci flow: There exists a smooth, complete, bounded-curvature Ricci flow $g(t)$ on M for some nontrivial time interval $[0, t_1]$ with $g(0) = g_0$. In view of the boundedness of the curvature, after possibly reducing t_1 to a smaller positive value, we may trivially assume that $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ for all $t \in (0, t_1]$ and $\mathrm{Ric}_{g(t)} \geq -\frac{\tau}{t_1}$ for all $t \in [0, t_1]$.

Of course, what is lacking from our flow is uniform control on its existence time. If $t_1 \geq \frac{\tau}{200\alpha_0 c_0}$, then we *do* have such control, but otherwise we will be able to iteratively apply the Extension Lemma 4.4 in a manner analogous to that employed by Hochard [15], to get successive extensions until we have a flow defined for a uniform time.

Having found t_1 and s_1, for $i \in \mathbb{N}$ we define t_i and s_i iteratively by setting $t_{i+1} = \nu^{-2} t_i$, where $\nu := (1 + \frac{1}{4c_0})^{-1/2} \in (0,1)$, and $s_{i+1} = s_i - \mu\sqrt{t_{i+1}}$, where $\mu := 6\tau^{-1/2}$. Thus

$$\begin{aligned} s_i &= s_{i-1} - \mu\sqrt{t_i} \\ &= s_{i-2} - \mu\sqrt{t_i} - \mu\sqrt{t_{i-1}} \\ &= s_{i-2} - \mu\sqrt{t_i}[1 + \nu] \\ &= s_1 - \mu\sqrt{t_i}[1 + \nu + \cdots + \nu^{i-2}] \\ &> s_1 - \frac{\mu}{1-\nu}\sqrt{t_i}. \end{aligned} \tag{4.9}$$

Our iterative assertion, that we have established above for $i = 1$ is:

$$I(i) \begin{cases} \text{We have constructed a smooth Ricci flow } g(t) \text{ on } B_{g_0}(x_0, s_i) \text{ for } t \in [0, t_i], \\ \text{with } g(0) = g_0 \text{ on this ball, such that on } B_{g_0}(x_0, s_i), \text{ and for } t \in (0, t_i], \text{ we have} \\ |\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t} \text{ and } \mathrm{Ric}_{g(t)} \geq -\frac{\tau}{t_i}. \end{cases}$$

The Extension Lemma 4.4, applied with $r_1 = s_i$ and $\ell_1 = t_i$, tells us that $I(i)$ implies $I(i+1)$, provided that t_i remains below $\frac{\tau}{200\alpha_0 c_0}$, and s_i does not get too small. More precisely we iteratively apply that lemma until either $t_i > \frac{\tau}{200\alpha_0 c_0}$ or $t_{i+1} > \frac{(1-\nu)^2}{\mu^2}$. The latter requirement ensures that $\frac{\mu}{1-\nu}\sqrt{t_i} \leq 1$ for $i \geq 2$ up to when we stop iterating, which in turn ensures that $s_i \geq s_1 - 1 \geq 2$ as required for the radius r_1 in the extension lemma. Either way, the final assertion $I(i)$ tells us that we have constructed the desired Ricci flow $g(t)$ for the time interval $[0, t_i]$, which has positive length bounded below depending only on α_0 and v_0, defined on the domain $B_{g_0}(x_0, s_i)$ where $s_i > s_1 - \frac{\mu}{1-\nu}\sqrt{t_i} \geq s_1 - 1$. This completes the proof of the claim, and hence of Theorem 1.6. □

5 Proof of the Mollification theorem 1.1

The existence assertion of Theorem 1.1 will follow rapidly from the Local Existence Theorem 1.6. Before we can apply that theorem, we must observe that a standard volume comparison argument tells us that there exists some smaller $v_0 > 0$ depending only on the original v_0, ε and α_0, such that for all $x \in B_{g_0}(x_0, 1 - \frac{\varepsilon}{4})$, we have $\mathrm{VolB}_{g_0}(x, \frac{\varepsilon}{4}) \geq v_0$.

We can then rescale the initial metric g_0 to $\tilde{g}_0 := \frac{16}{\varepsilon^2} g_0$, i.e. expand distances by a factor of $4/\varepsilon$, to put ourselves in exactly the situation of Theorem 1.6, with $s_0 = 4/\varepsilon \geq 40$, for some new $\alpha_0 > 0$ depending only on the old α_0 and ε. The output of Theorem 1.6 is a Ricci flow on $B_{\tilde{g}_0}(x_0, s_0 - 2)$ with estimates, and after returning to the original scaling, we have a Ricci flow $g(t)$ on $B_{g_0}(x_0, 1 - \varepsilon/2)$, for $t \in [0, T]$, where $T > 0$ depends only on α_0, v_0 and ε, with $g(0) = g_0$ where defined, and so that

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha \\ |\mathrm{Rm}|_{g(t)} \leq c_0/t \end{cases} \tag{5.1}$$

on $B_{g_0}(x_0, 1 - \varepsilon/2)$, for all $t \in (0, T]$, for some $\alpha, c_0 > 0$ depending only on α_0, v_0 and ε.

It remains to establish the desired properties of $g(t)$, and in order to do this we may have to reduce T to a smaller positive number, with the same dependencies. Looking first at the desired curvature control of (1.3), we observe that the first assertion was already obtained in the second inequality of (5.1) above on an even larger domain. This larger domain is useful, however, since it allows us to invoke Shi's local derivative estimates to give the bounds for the higher derivatives claimed in the second inequality of (1.3), as we now explain. First we apply the Shrinking Balls Lemma 2.2 to deduce that after possibly reducing $T > 0$ depending on c_0 and ε, we have $B_{g(t/2)}(x_0, 1 - \frac{2}{3}\varepsilon) \subset B_{g_0}(x_0, 1 - \frac{\varepsilon}{2})$, for all $t \in [0, T]$. Then, we reduce $T > 0$ further if necessary, depending on α and ε, to be sure that $B_{g(t/2)}(x_0, 1 - \frac{5}{6}\varepsilon) \supset B_{g_0}(x_0, 1 - \varepsilon)$ for all $t \in [0, T]$, this time by Lemma 2.1. To obtain the required bounds for the higher derivatives at time t, we can then apply Shi's estimates over the time interval $[t/2, t]$ on the ball $B_{g(t/2)}(x_0, 1 - \frac{2}{3}\varepsilon)$. A reference for Shi's estimates in almost the required form is [11, Theorem 14.14], although one needs to observe that the constant $C(\alpha, K, r, m, n)$ in that theorem can be given as $C(\alpha, r, m, n)K$ by a simple rescaling argument, and indeed to rescale in order to obtain the estimates on the whole of $B_{g(t/2)}(x_0, 1 - \frac{5}{6}\varepsilon) \supset B_{g_0}(x_0, 1 - \varepsilon)$ at time t.

At this point we can restrict the Ricci flow to $B_{g_0}(x_0, 1 - \varepsilon)$.

Next, we apply the Shrinking Balls Lemma 2.2 again to deduce that after possibly reducing $T > 0$ depending on c_0 and ε, we have $B_{g(t)}(x_0, 1-2\varepsilon) \subset B_{g_0}(x_0, 1-\varepsilon)$, for all $t \in [0, T]$. In turn, this allows us to apply our Lower Volume Control Lemma, [19, Lemma 2.3] with $\gamma = 1-2\varepsilon$. In order to do so, we first observe that by volume comparison, there is a positive lower bound for $\mathrm{VolB}_{g_0}(x_0, 1-2\varepsilon)$ depending only on v_0 and α_0. (There is not even a dependence on ε because we are assuming $\varepsilon \leq 1/10$.) The output of that lemma is that after possibly reducing $T > 0$ a little further, without adding any dependencies, there exists $v > 0$ as claimed so that $\mathrm{VolB}_{g(t)}(x_0, 1-2\varepsilon) \geq v$ for all $t \in [0, T]$, which is the remaining part of (1.2).

Prior to addressing the claims on the distance function, we must verify that for $s, t \in [0, T]$, we have $B_{g(s)}(x_0, \frac{1}{2} - 2\varepsilon) \subset B_{g(t)}(x_0, \frac{1}{2} - \varepsilon)$, provided that we restricted $T > 0$ sufficiently, depending only on α_0, v_0 and ε. To see this, observe that either we have $s \in [t, T]$, in which case it follows from the Shrinking Balls Lemma 2.2 (provided we restrict $T > 0$ depending only on c_0 and ε) or we have $s \in (0, t)$, in which case it follows from the Expanding Balls Lemma 2.1 (and T must be reduced also depending on α and ε).

The final parts of the theorem concerning the distance function then follow immediately from Lemma 3.1 by setting $r = \frac{1}{2} - \varepsilon$, cf. Remark 3.3, completing the proof of the theorem. □

Remark 5.1. We are not claiming that the Ricci flow $(\mathfrak{B}, g(t))$ we construct in this theorem is (isometric to) a restriction to a local region of a complete Ricci flow. However, by developing a little the statement and proof of Theorem 1.6, one could add the conclusion in Theorem 1.1 that for $\nu := (1 + \frac{1}{4c_0})^{-1/2} \in (0, 1)$ as used in the proof of Theorem 1.6, if we define $\tau_j := \nu^{2j} \downarrow 0$ then for sufficiently large j, $(\mathfrak{B}, g(\tau_j))$ can be isometrically embedded within a complete Riemannian manifold $(\mathcal{M}_j^3, g_j)$ such that

$$\begin{cases} |\mathrm{Rm}|_{g_j} \leq C(\alpha_0, v_0, \varepsilon)/\tau_j \\ \left|\nabla^k \mathrm{Rm}\right|_{g_j} \leq C(k, \alpha_0, v_0, \varepsilon)/\tau_j^{1+\frac{k}{2}} \end{cases} \tag{5.2}$$

globally throughout $\mathcal{M}_j$, and so that

$$\mathrm{inj}_{g_j}(\mathcal{M}_j) \geq \eta\sqrt{\tau_j}$$

for some $\eta = \eta(\alpha_0, v_0, \varepsilon) > 0$. See Remark 4.5, and note that in the proof of Theorem 1.6, we may as well assume that $t_1 = \nu^{2m}$ for some $m \in \mathbb{N}$, by reducing t_1 a little if necessary, and so $t_i = \nu^{2(m+1-i)}$.

6 Ricci limit spaces in 3D are bi-Hölder to smooth manifolds

In this section we prove Theorem 1.4. The result considers a sequence of coarsely controlled manifolds and obtains compactness; a subsequence converges to an optimally-regular limit. In particular, the limit is much more regular than we learn from Gromov compactness. The strategy is to regularise the coarsely controlled manifolds using Ricci flow, and in particular using the Mollification Theorem 1.1. Once we have regularity, then we have compactness, and a subsequence of the Ricci flows will converge essentially to a Ricci flow whose initial data represents the desired optimally-regular limit of the original sequence of manifolds.

A number of subtleties arise in the course of the proof, even once Theorem 1.1 has been proved. Because we must work locally, and we only have uniform regularity for positive times, we have to take care over which region we try to extract a limit. We have no real choice other than to work in a time $t > 0$ ball, but then we have to ensure that we end up with a limit that is defined on a time 0 ball of positive radius. For this, we need the Ricci lower bound estimates of [19], as well as the c_0/t decay of the full curvature tensor.

We also have to ensure that the limit Ricci flow has initial data (a metric space) that agrees with the limit of the original initial metrics. For this, we need strong uniform control on the evolution of the Riemannian distance, and again this requires our lower Ricci bounds and upper c_0/t curvature decay.

It may be helpful to record a result ensuring that appropriate smooth *local* convergence of Riemannian manifolds will imply convergence of the distance functions on a sufficiently smaller region.

Lemma 6.1 (Convergence of distance functions under local convergence)**.** *Suppose (M_i, g_i) is a sequence of smooth n-dimensional Riemannian manifolds, not necessarily complete, and that $x_i \in M_i$ for each i. Suppose that there exist a smooth, possibly incomplete n-dimensional Riemannian manifold $(\mathcal{N}, \hat{g})$ and a point $x_0 \in \mathcal{N}$ with $B_{\hat{g}}(x_0, 2r) \subset\subset \mathcal{N}$ for some $r > 0$, and a sequence of smooth maps $\varphi_i : \mathcal{N} \to M_i$, diffeomorphic onto their images and mapping x_0 to x_i, such that $\varphi_i^* g_i \to \hat{g}$ smoothly on $\overline{B_{\hat{g}}(x_0, 2r)}$. Then*

1. *If $0 < a \leq 2r$, and $a < b$, then $\varphi_i(B_{\hat{g}}(x_0, a)) \subset B_{g_i}(x_i, b)$ for sufficiently large i.*
2. *If $0 < a < b \leq 2r$, then $B_{g_i}(x_i, a) \subset\subset \varphi_i(B_{\hat{g}}(x_0, b))$ for sufficiently large i.*
3. *For every $s \in (0, r)$, we have convergence of the distance functions*

$$d_{g_i}(\varphi_i(x), \varphi_i(y)) \to d_{\hat{g}}(x, y)$$

as $i \to \infty$, uniformly as x and y vary within $B_{\hat{g}}(x_0, s)$.

Proof of Lemma 6.1. For Part 1, if $x \in B_{\hat{g}}(x_0, a)$, then we can take a minimising $\hat{g}$-geodesic σ from x_0 to x, so that $\varphi_i \circ \sigma$ is a path of length less than b for sufficiently large i, independent of x.

For Part 2, first note that since we are free to adjust a and b a little, it suffices to prove an inclusion rather than a compact inclusion. If the inclusion failed for every i after taking a subsequence, then we could take a sequence of points $z_i \in B_{g_i}(x_i, a)$ not in $\varphi_i(B_{\hat{g}}(x_0, b))$. We could then take a smooth path $\sigma_i : [0, 1] \to M_i$ connecting x_i to z_i with g_i-length no more than a, and thus lying within $B_{g_i}(x_i, a)$. (We do not know at this stage whether σ_i can be taken to be a minimising geodesic.) By truncating σ_i, and modifying z_i accordingly, we may assume that $\sigma_i([0, 1)) \subset \varphi_i(B_{\hat{g}}(x_0, b))$. The path $\varphi_i^{-1} \circ \sigma_i$ must then have $\hat{g}$-length only a little more than the g_i-length of σ_i, and in particular less than b, for sufficiently large i, which is a contradiction.

For Part 3, let $\delta > 0$ be arbitrary; we would like to show that for sufficiently large i, we have

$$|d_{g_i}(\varphi_i(x), \varphi_i(y)) - d_{\hat{g}}(x, y)| \leq \delta,$$

for every $x, y \in B_{\hat{g}}(x_0, s)$. For any such x, y, let σ be a minimising geodesic with respect to $\hat{g}$ that connects these points, and note that σ must remain within $B_{\hat{g}}(x_0, 2s) \subset\subset B_{\hat{g}}(x_0, 2r)$, cf. Remark 1.3. For each i we map σ forwards to $\varphi_i \circ \sigma$. By the convergence $\varphi_i^* g_i \to \hat{g}$, for sufficiently large i (independent of the particular x and y we chose but depending on r) the length cannot increase by more than δ, i.e.

$$d_{g_i}(\varphi_i(x), \varphi_i(y)) - d_{\hat{g}}(x, y) \leq \delta.$$

To prove the reverse direction, set $s_1 = (s + r)/2$, so $s < s_1 < r$, and throw away finitely many terms in i so that the image under φ_i of $B_{\hat{g}}(x_0, s)$ is contained in $B_{g_i}(x_i, s_1)$ (by Part 1). Therefore, for any two points $x, y \in B_{\hat{g}}(x_0, s)$, any minimising geodesic (with respect to g_i) connecting $\varphi_i(x)$ and $\varphi_i(y)$ must lie within the ball $B_{g_i}(x_i, 2s_1)$, and at least one such geodesic must exist becaused $B_{g_i}(x_i, 2s_1) \subset\subset \varphi_i(B_{\hat{g}}(x_0, 2r))$ by Part 2. Thus by considering the length of preimages under φ_i of minimising geodesics between $\varphi_i(x)$ and $\varphi_i(y)$, we see that

$$d_{\hat{g}}(x, y) - d_{g_i}(\varphi_i(x), \varphi_i(y)) \leq \delta,$$

for sufficiently large i, independent of x and y (but depending on r). □

Proof of Theorem 1.4. We begin by applying the Mollification Theorem 1.1 for each i, with $\varepsilon = 1/100$ fixed. The result is a collection of positive constants T, v, α, c_0 as in that theorem, and a sequence of Ricci flows $g_i(t)$, $t \in [0, T]$, defined on the balls $B_{g_i}(x_i, 1-\varepsilon)$, with $g_i(0) = g_i$ where defined, such that for all $t \in [0, T]$ we have $B_{g_i(t)}(x_i, 1 - 2\varepsilon) \subset\subset B_{g_i}(x_i, 1 - \varepsilon)$ where the Ricci flows are defined, and

$$\begin{cases} \mathrm{Ric}_{g_i(t)} \geq -\alpha & \text{on } B_{g_i}(x_i, 1-\varepsilon) \\ \mathrm{VolB}_{g_i(t)}(x_i, 1 - 2\varepsilon) \geq v > 0, \end{cases} \tag{6.1}$$

while for all $t \in (0, T]$ we have

$$\begin{cases} |\mathrm{Rm}|_{g_i(t)} \leq c_0/t \\ \left|\nabla^k \mathrm{Rm}\right|_{g_i(t)} \leq C/t^{1+\frac{k}{2}} \end{cases} \tag{6.2}$$

on $B_{g_i}(x_i, 1-\varepsilon)$ for any $k \in \mathbb{N}$, where C depends on k, α_0 and v_0 (since ε has been fixed). Moreover, Theorem 1.1 also tells us that for each $i \in \mathbb{N}$, $s \in [0, T]$ and $X, Y \in B_{g_i(s)}(x_i, \frac{1}{2} - 2\varepsilon)$, we have $X, Y \in B_{g_i(t)}(x_i, \frac{1}{2} - \varepsilon)$ for all $t \in [0, T]$ so the infimum length of curves within $(B_{g_i}(x_i, 1-\varepsilon), g_i(t))$ connecting X and Y is realised by a geodesic within $B_{g_i(t)}(x_i, 1 - 2\varepsilon) \subset B_{g_i}(x_i, 1 - \varepsilon)$ where $g_i(t)$ is defined. Moreover, for any $t \in [0, T]$ we have

$$d_{g_i}(X, Y) - \beta\sqrt{c_0 t} \leq d_{g_i(t)}(X, Y) \leq e^{\alpha t} d_{g_i}(X, Y). \tag{6.3}$$

During the proof it will be necessary to take these flows, and other flows with the same estimates, and argue that balls of a given radius $r \in (0, \frac{1}{2})$ at a given time lie within balls of slightly larger radius $r + \varepsilon$ at any different time. For this to work, we will apply Lemmata 2.1 and 2.2, which will do the job provided that T is small compared with ε (and ε^2) depending also on α and c_0 (we have $n = 3$). We will also need to be able to bound R_0 in an application of Lemma 3.1. With hindsight, it will be enough to reduce $T > 0$ if necessary so that

$$T \leq \frac{\varepsilon^2}{100 c_0 \beta^2} \qquad \text{and} \qquad T \leq \frac{\varepsilon}{100\alpha}, \tag{6.4}$$

where β is from Lemma 2.2.

The curvature bounds coming from the Mollification Theorem 1.1 allow us to obtain compactness using the argument of Hamilton-Cheeger-Gromov. Our situation is a bit easier than theirs in that we have already argued that all the derivatives of the curvature are bounded, and as is standard, by differentiating the Ricci flow equation we also have uniform control on all space-*time* derivatives of the curvature. However, there is an extra subtlety arising from working only locally in that we have to carefully choose the region on which we work in order for the standard argument to go through verbatim. In particular, we do not work on a time $t = 0$ ball since its geometry with respect to a time $t > 0$ metric is uncontrolled, and the standard covering arguments used in the compactness theory would fail.

Instead we work on a time $t > 0$ ball, and we choose to work at the final time $t = T$. We have established above that the Ricci flows $g_i(t)$ are each defined on the ball

$B_{g_i(T)}(x_i, 1-2\varepsilon)$, for $t \in [0,T]$. By Remark 1.3, if we halve the radius, and consider the ball $B_{g_i(T)}(x_i, \frac{1}{2}-\varepsilon)$, then the distance function is unambiguous and is realised by a minimising geodesic lying within the region $B_{g_i(T)}(x_i, 1-2\varepsilon)$ where the metric $g_i(T)$ is defined. By our volume and curvature bounds, we can obtain compactness on, say, $\overline{B_{g_i(T)}(x_i, \frac{1}{2}-2\varepsilon)}$. More precisely, after passing to a subsequence there exist a (typically incomplete) manifold $(\mathcal{N}, g_\infty)$ and a point $x_0 \in \mathcal{N}$ such that $B_{g_\infty}(x_0, \frac{1}{2}-2\varepsilon) \subset\subset \mathcal{N}$, and a sequence of smooth maps $\varphi_i : \mathcal{N} \to B_{g_i(T)}(x_i, 1-2\varepsilon) \subset M_i$, diffeomorphic onto their image, and mapping x_0 to x_i, such that $\varphi_i^* g_i(T) \to g_\infty$ smoothly on $\overline{B_{g_\infty}(x_0, \frac{1}{2}-2\varepsilon)}$.

We observe that with our set-up, this compactness assertion follows from the usual proofs in the smooth case. As an alternative to requiring detailed knowledge of the proofs, we note that by Remark 5.1, after reducing T to make it an integral power of $\nu^2 = (1+\frac{1}{4c_0})^{-1}$, we could extend Theorem 1.1 and obtain that in fact $(B_{g_i}(x_i, 1-\varepsilon), g_i(T))$ can be isometrically embedded in a complete Riemannian manifold with uniform (i-independent) bounds on all derivatives of the curvature and a positive uniform lower bound on the injectivity radius. Thus compactness can be obtained from the standard theorems, after which we can restrict to $\overline{B_{g_\infty}(x_0, \frac{1}{2}-2\varepsilon)}$ in the limit.

Once we have compactness at time T, Hamilton's original argument, using the uniform curvature bounds for positive time, allows us to pass to a further sequence in i to obtain a smooth Ricci flow $g(t)$ living on $\overline{B_{g_\infty}(x_0, \frac{1}{2}-2\varepsilon)}$ for $t \in (0,T]$ such that $\varphi_i^* g_i(t) \to g(t)$ smoothly locally on $\overline{B_{g_\infty}(x_0, \frac{1}{2}-2\varepsilon)} \times (0,T]$. (Here $g(T) = g_\infty$ where $g(T)$ is defined.)

Because of the smooth convergence, the curvature estimates in (6.1) and (6.2) pass to the limit, and we have

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha \\ |\mathrm{Rm}|_{g(t)} \leq c_0/t \end{cases} \tag{6.5}$$

on the whole domain $\overline{B_{g_\infty}(x_0, \frac{1}{2}-2\varepsilon)}$ for all $t \in (0,T]$.

We have constructed a Ricci flow on a reasonably-sized ball with respect to $g(T)$, but we have to be concerned that this region might be much smaller with respect to earlier metrics, and might even be contained within a $g(t)$ ball of radius $r(t)$ that converges to zero as $t \downarrow 0$. We will be rescued from this possibility by the refinement of the Expanding Balls Lemma, given in Lemma 2.1; because we reduced T in (6.4), that lemma turns our control on the Ricci tensor in (6.5) into the inclusion

$$B_{g(t)}(x_0, \frac{1}{2}-3\varepsilon) \subset\subset B_{g(T)}(x_0, \frac{1}{2}-2\varepsilon), \tag{6.6}$$

for any $t \in (0,T]$. This tells us, via Remark 1.3, that if $x, y \in B_{g(t)}(x_0, \frac{1}{4}-\frac{3}{2}\varepsilon)$, the corresponding ball of half the radius, then the $g(t)$-distance between x and y is realised by a minimising geodesic lying within the domain of definition of the Ricci flow $g(t)$ (see Remark 1.3). Also, we can apply Lemma 3.1 with $r = \frac{1}{4}-\frac{3}{2}\varepsilon$ to give us an extension of $d_{g(t)}$ to a metric d_0 on Ω_T, as defined in that lemma, together with the distance estimates

$$d_0(x,y) - \beta\sqrt{c_0 t} \leq d_{g(t)}(x,y) \leq e^{\alpha t} d_0(x,y), \tag{6.7}$$

and

$$d_{g(t)}(x,y) \geq \gamma \left[d_0(x,y)\right]^{1+4c_0}, \tag{6.8}$$

for all $t \in (0,T]$, $x, y \in \Omega_T$, and some $\gamma < \infty$ depending only on α_0 and v_0. (Note that these estimates imply that d_0 generates the same topology as we have already on Ω_T.) Moreover, by our restrictions on T from (6.4), we can be sure that $R_0 > \frac{1}{4}-2\varepsilon$, and so we can set $R = \frac{1}{4}-2\varepsilon$, in which case (3.5) tells us that

$$B_{g(t)}(x_0, \frac{1}{4}-2\varepsilon) \subset\subset \mathcal{M} \qquad \text{and} \qquad B_{d_0}(x_0, \frac{1}{4}-2\varepsilon) \subset\subset \mathcal{M}, \tag{6.9}$$

for all $t \in (0, T]$, where $\mathcal{M}$ is the connected component of Interior(Ω_T) containing x_0.

In summary, we have so far constructed a smooth limit Ricci flow $g(t)$ for $t \in (0, T]$ on $\overline{B_{g_\infty}(x_i, \frac{1}{2} - 2\varepsilon)}$, and extended its distance function uniformly to $t = 0$ on an open subdomain $\mathcal{M}$ that compactly contains both $B_{g(t)}(x_0, \frac{1}{4} - 2\varepsilon)$, for each $t \in (0, T]$, and also $B_{d_0}(x_0, \frac{1}{4} - 2\varepsilon)$.

In what follows, we will need to consider the distance with respect to $g_i(t)$ between image points $\varphi_i(x)$ and $\varphi_i(y)$, and we pause to verify that such distances are realised by minimising geodesics for arbitrary x, y in the domain of definition of the Ricci flow $g(t)$. By Part 1 of Lemma 6.1, after omitting finitely many terms in i, the image of the entire flow domain $\overline{B_{g(T)}(x_0, \frac{1}{2} - 2\varepsilon)}$ under φ_i must lie within $B_{g_i(T)}(x_0, \frac{1}{2} - \frac{3}{2}\varepsilon)$, say, which in turn, by the Shrinking Balls Lemma 2.2, and (6.4), must lie within $B_{g_i(t)}(x_0, \frac{1}{2} - \varepsilon)$ for all $t \in [0, T]$. Since the Mollification Theorem told us that $B_{g_i(t)}(x_i, 1-2\varepsilon) \subset\subset B_{g_i}(x_i, 1-\varepsilon)$ where $g_i(t)$ is defined, we see that for all $x, y \in \overline{B_{g(T)}(x_0, \frac{1}{2} - 2\varepsilon)}$, the distance between $\varphi_i(x)$ and $\varphi_i(y)$ with respect to any $g_i(t)$ is realised by a minimising geodesic lying within $B_{g_i(t)}(x_i, 1 - 2\varepsilon)$, cf. Remark 1.3.

Our essential task now is to compare distances $d_0(x, y)$ and $d_{g_i}(\varphi_i(x), \varphi_i(y))$ for x and y in $\mathcal{M}$. The rough strategy is as follows. First, by the distance estimates (6.7) that came from Lemma 3.1, we know that $d_0(x, y)$ is close to $d_{g(t)}(x, y)$ for $t \in (0, T]$ small. Second, by the convergence of $g_i(t)$ to $g(t)$, we expect that $d_{g(t)}(x, y)$ should be close to $d_{g_i(t)}(\varphi_i(x), \varphi_i(y))$. Third, by the distance estimates (6.3) coming from the Mollification Theorem, $d_{g_i(t)}(\varphi_i(x), \varphi_i(y))$ should be close to $d_{g_i}(\varphi_i(x), \varphi_i(y))$.

Claim: As $i \to \infty$, we have convergence

$$d_{g_i}(\varphi_i(x), \varphi_i(y)) \to d_0(x, y)$$

uniformly as x, y vary over $\mathcal{M}$.

Proof of Claim: Let $\delta > 0$. We must make sure that for sufficiently large i, depending on δ, we have

$$|d_{g_i}(\varphi_i(x), \varphi_i(y)) - d_0(x, y)| \leq \delta \qquad \text{for every } x, y \in \mathcal{M}. \tag{6.10}$$

By the distance estimates (6.7) that came from our application of Lemma 3.1, there exists $t_1 \in (0, T]$ such that for all $t \in [0, t_1]$, we have

$$|d_0(x, y) - d_{g(t)}(x, y)| < \delta/3 \qquad \text{for all } x, y \in \mathcal{M}. \tag{6.11}$$

We need a similar estimate for $g_i(t)$. By the distance estimates (6.3) coming from the Mollification Theorem, we know that there exists $t_2 \in (0, T]$ such that for all $t \in [0, t_2]$, we have

$$|d_{g_i}(X, Y) - d_{g_i(t)}(X, Y)| < \delta/3, \quad \text{whenever } \exists\, s \in [0, T] \text{ s.t. } X, Y \in B_{g_i(s)}(x_i, \frac{1}{2} - 2\varepsilon). \tag{6.12}$$

We fix $t_0 = \min\{t_1, t_2\}$ so that both (6.11) and (6.12) hold for $t = t_0$. (In fact, we could have naturally picked t_1 and t_2 the same from the outset.)

By definition of Ω_T, and hence $\mathcal{M}$, we have

$$\mathcal{M} \subset B_{g(t_0)}(x_0, \tfrac{1}{4} - \tfrac{3}{2}\varepsilon), \tag{6.13}$$

and by (6.6) we can pick $r > \frac{1}{4} - \frac{3}{2}\varepsilon$ so that

$$B_{g(t_0)}(x_0, 2r) \subset\subset B_{g(T)}(x_0, \tfrac{1}{2} - 2\varepsilon),$$

where $g(t_0)$ and the maps φ_i are defined and we have the convergence $\varphi_i^* g_i(t_0) \to g(t_0)$. Therefore we can apply Lemma 6.1, with $s = \frac{1}{4} - \frac{3}{2}\varepsilon$ and $\hat{g}$ and g_i there equal to $g(t_0)$ and $g_i(t_0)$ here, respectively, to conclude that

$$d_{g_i(t_0)}(\varphi_i(x), \varphi_i(y)) \to d_{g(t_0)}(x, y) \tag{6.14}$$

as $i \to \infty$, uniformly in $x, y \in \mathcal{M}$.

By combining (6.11) and (6.12) for $t = t_0$, and (6.14), we will have proved (6.10) and hence the claim, provided that (6.12) holds for $X = \varphi_i(x)$ and $Y = \varphi_i(y)$. But by (6.14) applied for first $(x, y) = (x, x_0)$ and then $(x, y) = (x_0, y)$, we see by (6.13) that $\varphi_i(x), \varphi_i(y) \in B_{g_i(t_0)}(x_i, \frac{1}{2} - 2\varepsilon)$ for sufficiently large i as required. //

What the claim tells us is that for arbitrarily small $\delta > 0$, the maps φ_i are δ-Gromov Hausdorff approximations from $\mathcal{M}$ to $\varphi_i(\mathcal{M})$ for sufficiently large i. Unusually for such approximations, the maps are smooth.

Another immediate consequence of the claim is that for all $\eta > 0$, for sufficiently large i we have $\varphi_i(\mathcal{B}) \subset B_{g_i}(x_i, 1/10 + \eta)$, where $\mathcal{B} := \overline{B_{d_0}(x_0, 1/10)} \subset \mathcal{M}$ (recall (6.9)) which is the second inclusion of (1.6). To obtain the first inclusion, we first need to clarify how large the image $\varphi_i(\mathcal{M})$ is. By the first part of (6.9), with $t = T$, we have $B_{g(T)}(x_0, \frac{1}{4} - 2\varepsilon) \subset\subset \mathcal{M}$. Therefore, by Part 2 of Lemma 6.1 we have

$$\varphi_i(\mathcal{M}) \supset \varphi_i(B_{g(T)}(x_0, \tfrac{1}{4} - 2\varepsilon)) \supset B_{g_i(T)}(x_i, \tfrac{1}{4} - 3\varepsilon),$$

after deleting finitely many terms in i. Using once more the Expanding Balls Lemma 2.1, and (6.4), we can conclude that

$$\varphi_i(\mathcal{M}) \supset B_{g_i(T)}(x_i, \tfrac{1}{4} - 3\varepsilon) \supset B_{g_i(0)}(x_i, \tfrac{1}{4} - 4\varepsilon) \supset B_{g_i}(x_i, 1/10 - \eta).$$

Now that we are sure the image of the maps φ_i is large enough, we immediately obtain from the Claim the restricted statement

$$\varphi_i(\mathcal{B}) \supset B_{g_i}(x_i, 1/10 - \eta),$$

after dropping finitely many terms in i, which is the first inclusion of (1.6).

Now we have proved the existence and claimed properties of the maps φ_i, it is easy to perturb them to Gromov-Hausdorff approximations $(\mathcal{B}, d_0) \to (\overline{B_{g_i}(x_i, 1/10)}, d_{g_i})$, and we obtain the claimed Gromov-Hausdorff convergence $(\overline{B_{g_i}(x_i, 1/10)}, d_{g_i}) \to (\mathcal{B}, d_0)$.

Finally we turn to the Lipschitz and Hölder claims of the theorem. Whichever metric g we take on $\mathcal{M}$, the distance d_g will be bi-Lipschitz equivalent to $d_{g(T)}$ once we restrict to $\mathcal{B}$, so we need only prove the claims with g replaced by the (incomplete) metric $g(T)$ from the Ricci flow we have constructed. The second inequality of (6.7) tells us that $d_{g(T)}(x, y) \leq e^{\alpha T} d_0(x, y)$, which ensures that the identity map $(\mathcal{B}, d_0) \to (\mathcal{B}, d_{g(T)})$ is Lipschitz continuous. Meanwhile, (6.8) tells us that $d_{g(T)}(x, y) \geq \gamma \, [d_0(x, y)]^{1+4c_0}$, which implies that the identity map $(\mathcal{B}, d_{g(T)}) \to (\mathcal{B}, d_0)$ is Hölder continuous with Hölder exponent $\frac{1}{1+4c_0}$. □

7 Proof of the Anderson, Cheeger, Colding, Tian conjecture

Proof of Corollary 1.5. First note that it is expected that it is impossible to flow from (M_i, g_i) since we are not assuming uniform global noncollapsing. As a result, this time it is more convenient to start by appealing to Gromov compactness to get a complete limit length space

$$(M_i, d_{g_i}, y_i) \to (X, d_X, y_0)$$

in the pointed Gromov-Hausdorff sense, for some subsequence in i.

To show that the topological space M induced by (X, d_X) is in fact a manifold with bi-Hölder charts as claimed in the corollary, we must show that given an arbitrary $x \in X$, there is a neighbourhood of x that is bi-Hölder homeomorphic to a ball in $\mathbb{R}^3$, or indeed to some open subset of a complete Riemannian three-manifold.

By definition of the pointed convergence above, and the fact that the limit is a length space, for $r := d_X(x, y_0)$ there exists a sequence f_i of $\varepsilon(i)$-Gromov-Hausdorff approximations $B_{d_X}(y_0, r+1) \to B_{d_{g_i}}(y_i, r+1)$, where $\varepsilon(i) \downarrow 0$ as $i \to \infty$, with $f_i(y_0) = y_i$ for each i. Defining $x_i := f_i(x)$, we obtain the pointed Gromov-Hausdorff convergence

$$(\overline{B_{g_i}(x_i, 1/10)}, d_{g_i}, x_i) \to (\overline{B_{d_X}(x, 1/10)}, d_X, x),$$

with the limit being compact.

By the hypotheses (1.7) and the fact that $d_{g_i}(y_i, x_i)$ converges to $d_X(y_0, x) = r$, and so is bounded, we must have a uniform lower bound $\mathrm{VolB}_{g_i}(x_i, 1) \geq \tilde{v}_0$, for some $\tilde{v}_0 > 0$ independent of i, by Bishop-Gromov. Therefore we can apply Theorem 1.4 with v_0 there equal to $\tilde{v}_0$ here, to show that there exist a smooth three-dimensional manifold without boundary $\mathcal{M}$, containing a point x_0, and a metric $d_0 : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ generating the same topology as $\mathcal{M}$ such that $\overline{B_{d_0}(x_0, 1/10)} \subset\subset \mathcal{M}$ and so that after passing to a subsequence the compact metric spaces $(\overline{B_{g_i}(x_i, 1/10)}, d_{g_i})$ Gromov-Hausdorff converge to $(\mathcal{B}, d_0)$, where $\mathcal{B} = \overline{B_{d_0}(x_0, 1/10)}$. In fact, that theorem implies the pointed Gromov-Hausdorff convergence

$$(\overline{B_{g_i}(x_i, 1/10)}, d_{g_i}, x_i) \to (\mathcal{B}, d_0, x_0).$$

Moreover, that theorem tells us that the identity map from $(\mathcal{B}, d_0)$ to $(\mathcal{B}, d_g)$, for any smooth complete metric g on $\mathcal{M}$, is bi-Hölder.

By uniqueness of compact limits under pointed Gromov-Hausdorff convergence [4, Theorem 8.1.7], $(\mathcal{B}, d_0)$ must be isometric to the restriction of (X, d_X) to the closed ball centred at x of radius $1/10$, via an isometry that identifies x_0 and x.

Consequently, we find that $(B_{d_0}(x_0, 1/10), d_0)$ is isometric to $(B_{d_X}(x, 1/10), d_X)$, and we have found a bi-Hölder homeomorphism from this neighbourhood to $(B_{d_0}(x_0, 1/10), d_g)$. □

8 Proof of the Global Existence Theorem 1.7

Before beginning the proof, we recall the following special case of a result of B.L. Chen [9, Theorem 3.1]. Similar results were independently proved by the first author [18, Theorem 1.3].

Lemma 8.1. *Suppose $(M^n, g(t))$ is a Ricci flow for $t \in [0, T]$, not necessarily complete, with the property that for some $y_0 \in M$ and $r > 0$, and all $t \in [0, T]$, we have $B_{g(t)}(y_0, r) \subset\subset M$ and*

$$|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t} \qquad \textit{throughout } B_{g(t)}(y_0, r), \textit{ for all } t \in (0, T], \tag{8.1}$$

and for some $c_0 \geq 1$. Then if $|\mathrm{Rm}|_{g(0)} \leq r^{-2}$ on $B_{g(0)}(y_0, r)$, we must have

$$|\mathrm{Rm}|_{g(t)}(y_0) \leq e^{Cc_0} r^{-2}$$

where C depends only on n.

We will also need a slightly nonstandard version of Shi's derivative estimates, as phrased in [20, Lemma A.4]. For a slightly stronger result, and proof, see [11, Theorem 14.16].

Lemma 8.2. *Suppose $(M^n, g(t))$ is a Ricci flow for $t \in [0,T]$, not necessarily complete, with the property that for some $y_0 \in M$ and $r > 0$, we have $B_{g(0)}(y_0, r) \subset\subset M$, and $|\mathrm{Rm}|_{g(t)} \le r^{-2}$ throughout $B_{g(0)}(y_0, r)$ for all $t \in [0,T]$, and so that for some $l_0 \in \mathbb{N}$ we initially have $|\nabla^l \mathrm{Rm}|_{g(0)} \le r^{-2-l}$ throughout $B_{g(0)}(y_0, r)$ for all $l \in \{1, \ldots, l_0\}$. Then there exists $C < \infty$ depending only on l_0, n and an upper bound for T/r^2 such that*

$$|\nabla^l \mathrm{Rm}|_{g(t)}(y_0) \le C r^{-2-l}$$

for every $l \in \{1, \ldots, l_0\}$ and all $t \in [0,T]$.

Proof of Theorem 1.7. Pick any point $x_0 \in M$. For each integer $k \ge 2$, apply Theorem 1.6 to the manifold (M, g_0), with $s_0 = k+2$. The conclusion is that there exist $T, v, \alpha, c_0 > 0$, depending only on α_0 and v_0, so that there exists a Ricci flow $g_k(t)$ defined on $B_{g_0}(x_0, k)$ for $t \in [0,T]$ with $g_k(0) = g_0$ on $B_{g_0}(x_0, k)$ with the properties that

$$\begin{cases} \mathrm{Ric}_{g_k(t)} \ge -\alpha \\ |\mathrm{Rm}|_{g_k(t)} \le c_0/t \end{cases} \tag{8.2}$$

on $B_{g_0}(x_0, k)$ for all $t \in (0,T]$.

For $r_0 > 0$ and $k \ge r_0 + 2$, Lemma 8.1 can now be applied to $g_k(t)$ centred at arbitrary points $y_0 \in B_{g_0}(x_0, r_0 + 1)$. Note that the curvature estimate $|\mathrm{Rm}|_{g(t)} \le c_0/t$ holds on $B_{g_0}(y_0, 1)$, and by the Shrinking Balls Lemma 2.2, this contains $B_{g_k(t)}(y_0, 1/2)$ for all $t \in [0,T]$ after possibly reducing T to a smaller positive value depending only on c_0, which in turn depends only on α_0 and v_0. The output of Lemma 8.1 is that there exists $K_0 < \infty$, depending only on $\sup_{B_{g_0}(x_0, r_0+2)} |\mathrm{Rm}|_{g_0}$ and c_0 and in particular independent of k, such that $|\mathrm{Rm}|_{g_k(t)} \le K_0$ throughout $B_{g_0}(x_0, r_0 + 1)$, for all $t \in [0,T]$.

We may then apply Lemma 8.2 to $g_k(t)$ centred at arbitrary points $y_0 \in B_{g_0}(x_0, r_0)$. This time we deduce that for each $l \in \mathbb{N}$ there exists $K_1 < \infty$, depending only on l, g_0 and r_0, and in particular independent of k, such that

$$|\nabla^l \mathrm{Rm}|_{g(t)} \le K_1$$

throughout $B_{g_0}(x_0, r_0)$, for all $t \in [0,T]$. Working in coordinate charts, by Ascoli-Arzelà we can pass to a subsequence in k and obtain a smooth limit Ricci flow $g(t)$ on $B_{g_0}(x_0, r_0)$ for $t \in [0,T]$ with $g(0) = g_0$ (it is not necessary to take a Cheeger-Hamilton limit here since $g_k(0) = g_0$ on $B_{g_0}(x_0, r_0)$ for sufficiently large k). Moreover, the limit will inherit the curvature bounds

$$\begin{cases} \mathrm{Ric}_{g(t)} \ge -\alpha \\ |\mathrm{Rm}|_{g(t)} \le c_0/t \end{cases} \tag{8.3}$$

on $B_{g_0}(x_0, r_0)$ for all $t \in (0,T]$.

We can now repeat this process for larger and larger $r_0 \to \infty$, and take a diagonal subsequence to obtain a smooth limit Ricci flow $g(t)$ on the whole of M for $t \in [0,T]$ with $g(0) = g_0$. This limit flow must be complete because for arbitrary $t \in (0,T]$ and arbitrarily large $r > 0$, the Shrinking Balls Lemma 2.2 tells us that $B_{g(t)}(x_0, r) \subset B_{g_0}(x_0, r + \beta\sqrt{c_0 t}) \subset\subset M$.

Given our curvature control (8.3) and the completeness of $g(t)$, the distance estimates are standard, although they also follow from our more elaborate Lemma 3.1. □

9 Starting a Ricci flow with a Ricci limit space

In this section we prove Theorem 1.8.

Proof. Because we are making a stronger assumption in Theorem 1.8 than in Corollary 1.5, we can apply Theorem 1.7 to each (M_i, g_i) to give a sequence of Ricci flows $g_i(t)$ on M_i with $g_i(0) = g_i$, defined over a uniform time interval $[0, T]$, and enjoying the uniform estimates

$$\begin{cases} \mathrm{Ric}_{g_i(t)} \geq -\alpha & \\ \mathrm{VolB}_{g_i(t)}(x, 1) \geq v > 0 & \text{for all } x \in M_i \\ |\mathrm{Rm}|_{g_i(t)} \leq c_0/t & \text{throughout } M_i \end{cases} \tag{9.1}$$

for all $t \in (0, T]$, and

$$d_{g_i(t_1)}(x, y) - \beta\sqrt{c_0}(\sqrt{t_2} - \sqrt{t_1}) \leq d_{g_i(t_2)}(x, y) \leq e^{\alpha(t_2 - t_1)} d_{g_i(t_1)}(x, y), \tag{9.2}$$

for any $0 \leq t_1 \leq t_2 \leq T$, and any $x, y \in M_i$. Because of the uniform curvature bounds for positive times, Hamilton's compactness theorem tells us that we can pass to a subsequence in i so that $(M_i, g_i(t), x_i) \to (M, g(t), x_\infty)$ in the smooth Cheeger-Gromov sense, where M is a smooth manifold, $g(t)$ is a complete Ricci flow on M for $t \in (0, T]$, $x_\infty \in M$ and we pass the estimates (9.1) and (9.2) to the limit to give

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha & \\ \mathrm{VolB}_{g(t)}(x, 1) \geq v > 0 & \text{for all } x \in M \\ |\mathrm{Rm}|_{g(t)} \leq c_0/t & \text{throughout } M \end{cases} \tag{9.3}$$

for all $t \in (0, T]$, and

$$d_{g(t_1)}(x, y) - \beta\sqrt{c_0}(\sqrt{t_2} - \sqrt{t_1}) \leq d_{g(t_2)}(x, y) \leq e^{\alpha(t_2 - t_1)} d_{g(t_1)}(x, y), \tag{9.4}$$

for any $0 < t_1 \leq t_2 \leq T$, and any $x, y \in M$. This final estimate tells us that there exists a metric d_0 on M to which $d_{g(t)}$ converges locally uniformly as $t \downarrow 0$. It also tells us, when combined with the smooth convergence of $(M_i, g_i(t), x_i)$ to $(M, g(t), x_\infty)$ for positive times and (9.2) that

$$(M_i, d_{g_i}, x_i) \to (M, d_0, x_\infty)$$

in the pointed Gromov-Hausdorff sense, which can be considered an easier version of the argument in Section 6. □

References

[1] M. Anderson *Degenerations of metrics with bounded curvature and applications to critical metrics of Riemannian functionals.* Proc. Sympos. Pure Math. Amer. Math. Soc. **54** (1993) 53–79.

[2] C. Böhm and B. Wilking, *Manifolds with positive curvature operators are space forms.* Annals of Math. **167** (2008) 1079–1097.

[3] S. Brendle and R. Schoen, *Manifolds with 1/4-pinched curvature are space forms.* J. Amer. Math. Soc. **22** (2009) 287–307.

[4] D. Burago, Y. Burago and S. Ivanov, 'A course in metric geometry.' Graduate studies in math. **33** AMS 2001.

[5] J. Cheeger, 'Degeneration of Riemannian Metrics under Ricci Curvature Bounds.' Edizioni della Normale (2001).

[6] J. Cheeger and T.H. Colding *On the structure of spaces with Ricci curvature bounded below I.* J. Differential Geometry, **45** (1997) 406–480.

[7] J. Cheeger, T.H. Colding and G. Tian *On the sigularities of spaces with bounded Ricci curvature.* Geom. Funct. Anal. (2002) 12: 873. doi:10.1007/PL00012649

[8] J. Cheeger, M. Gromov and M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds.* J. Differential Geometry, **17** (1982) 15–53.

[9] B.-L. Chen, *Strong uniqueness of the Ricci flow.* J. Differential Geometry **82** (2009) 363–382.

[10] B. Chow, P. Lu and L. Ni, 'Hamilton's Ricci Flow,' Graduate Studies in Mathematics **77**. AMS (2006).

[11] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo and L. Ni. Ricci flow: Techniques and Applications: Part II: Analytic aspects. 'Mathematical Surveys and Monographs,' **144** A.M.S. 2008.

[12] P. Gianniotis, *The Ricci flow on manifolds with boundary.* J. Differential Geometry **104** (2016) 291–324.

[13] R. S. Hamilton, *Three-manifolds with positive Ricci curvature.* J. Differential Geometry **17** (1982) 255–306.

[14] R. S. Hamilton, *The formation of singularities in the Ricci flow.* Surveys in differential geometry, Vol. II (Cambridge, MA, 1993) 7–136, Internat. Press, Cambridge, MA, 1995.

[15] R. Hochard, *Short-time existence of the Ricci flow on complete, non-collapsed* 3*-manifolds with Ricci curvature bounded from below.* http://arxiv.org/abs/1603.08726v1

[16] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications.* http://arXiv.org/abs/math/0211159v1 (2002).

[17] M. Simon, *Ricci flow of non-collapsed three manifolds whose Ricci curvature is bounded from below.* Journal für die reine und angewandte Mathematik. **662** (2012) 59–94.

[18] M. Simon, *Local results for flows whose speed or height is bounded by c/t.* Int. Math. Res. Notices (2008) 2008: rnn097.

[19] M. Simon and P. M. Topping, *Local control on the geometry in 3D Ricci flow.* http://arxiv.org/abs/1611.06137

[20] P. M. Topping, *Ricci flow compactness via pseudolocality, and flows with incomplete initial metrics.* J. Eur. Math. Soc. **12**, 1429–1451.

[21] D. Yang, *Convergence of Riemannian manifolds with integral bounds on curvature. I.* Ann. Sci.École Norm. Sup. **25** (1992) 77–105.

MS: institut für analysis und numerik (IAN), universität magdeburg, universitätsplatz 2, 39106 magdeburg, germany

PT: mathematics institute, university of warwick, coventry, CV4 7AL, uk
http://www.warwick.ac.uk/~maseq