Global Regularity of Three-Dimensional Ricci Limit Spaces

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March 1, 2018

Abstract

In their recent work [ST17], Miles Simon and the second author established a local bi-Hölder correspondence between weakly noncollapsed Ricci limit spaces in three dimensions and smooth manifolds. In particular, any open ball of finite radius in such a limit space must be bi-Hölder homeomorphic to some open subset of a complete smooth Riemannian three-manifold. In this work we build on the technology from [ST16, ST17] to improve this local correspondence to a global-local correspondence. That is, we construct a smooth three-manifold M, and prove that the entire (weakly) noncollapsed three-dimensional Ricci limit space is homeomorphic to M via a globally-defined homeomorphism that is bi-Hölder once restricted to any compact subset. Here the bi-Hölder regularity is with respect to the distance d_g on M, where g is any smooth complete metric on M.

A key step in our proof is the construction of local *pyramid* Ricci flows, existing on uniform regions of spacetime, that are inspired by Hochard's *partial* Ricci flows [Hoc16]. Suppose (M, g_0, x_0) is a complete smooth pointed Riemannian three-manifold that is (weakly) noncollapsed and satisfies a lower Ricci bound. Then, given any $k \in \mathbb{N}$, we construct a smooth Ricci flow $g(t)$ living on a subset of spacetime that contains, for each $j \in \{1, \ldots, k\}$, a cylinder $\mathbb{B}_{g_0}(x_0, j) \times [0, T_j]$, where T_j is dependent only on the Ricci lower bound, the (weakly) noncollapsed volume lower bound and the radius j (in particular independent of k) and with the property that $g(0) = g_0$ throughout $\mathbb{B}_{g_0}(x_0, k)$.

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1. Introduction

Given a sequence of n-dimensional complete, smooth, pointed Riemannian manifolds $(\mathcal{M}_i, g_i, x_i)$, for which $\mathrm{Ric}_{g_i} \geq -\alpha_0$ for some given α_0, Gromov's compactness theorem, see [Che01] for instance, tells us that, after passing to a subsequence, there exists a locally compact complete pointed metric space (X, d_X, x_0) for which $(\mathcal{M}_i, d_{g_i}, x_i) \to (X, d_X, x_0)$ in the pointed Gromov-Hausdorff sense; the relevant definition of pointed Gromov-Hausdorff convergence may be found in, for example, either [BBI01] or [Che01]. It is a natural

question to ask about the regularity of the limit space (X, d_X), continuing a long tradition of such results that originates with the study of limit spaces of manifolds with uniform lower sectional curvature bounds (see e.g. [BBI01] as a starting point). In this paper we consider the weakly noncollapsed setting, that is with the added assumption that $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, 1) \geq v_0 > 0$. We refer to this setting as *weakly* noncollapsed since we only require a *single* unit ball $\mathbb{B}_{g_i}(x_i, 1)$ to have a specified uniform lower volume bound as opposed to the stronger globally noncollapsed condition in which we require *all* balls $\mathbb{B}_{g_i}(x, 1)$ to have the uniform lower volume bound. This stronger globally noncollapsed hypothesis can be handled using Ricci flow techniques that are far simpler than those required in this paper.

Pioneering regularity results were obtained for the limit spaces (X, d_X, x_0) of sequences of n-dimensional manifolds with uniform lower Ricci bounds by Cheeger-Colding, see [Che01], as we now describe. In the weakly noncollapsed setting the *'regular set'* $\mathcal{R}$ of X is the set of points in X at which all tangent cones are n-dimensional Euclidean space; see [Che01]. Cheeger-Colding [CC97] proved that while the Hausdorff dimension of X is n, the singular set $\mathcal{S} := X \setminus \mathcal{R}$ has Hausdorff dimension no larger than $n - 2$, and the regular set is contained within an open set that is locally bi-Hölder homeomorphic to a smooth manifold.

Recently, Miles Simon and the second author obtained improved regularity in dimension three; in [ST17] it is proved that weakly noncollapsed Ricci limit spaces in dimension three are topological manifolds throughout the entire limit space, irrespective of singularities. In fact, given any point $x \in X$, including any singular point, there is a neighbourhood of x that is *bi-Hölder* homeomorphic to a ball in $\mathbb{R}^3$. Moreover, the theory in that paper establishes that for any $r > 0$, the ball $\mathbb{B}_{d_X}(x_0, r)$ is bi-Hölder homeomorphic to an open subset in a complete smooth Riemannian manifold. See Theorem 1.4 and Corollary 1.5 in [ST17] for full details. In this paper we use all the technology from [ST17] and key results and ideas from [ST16, Hoc16] in order to prove directly the following result.

Theorem 1.1 (***Ricci limit spaces are globally smooth manifolds***)**.** *Suppose that $\left(\mathcal{M}_i^3, g_i, x_i\right)$ is a sequence of complete, smooth, pointed Riemannian three-manifolds such that for some $\alpha_0 > 0$ and $v_0 > 0$, and for all $i \in \mathbb{N}$, we have $\mathrm{Ric}_{g_i} \geq -\alpha_0$ throughout $\mathcal{M}_i$, and $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, 1) \geq v_0 > 0$.*

Then there exist a smooth manifold M, a point $x_0 \in M$, and a complete distance metric $d : M \times M \to [0, \infty)$ generating the same topology as M such that after passing to a subsequence in i we have

$$\left(\mathcal{M}_i^3, d_{g_i}, x_i\right) \to (M, d, x_0)\,,$$

in the pointed Gromov-Hausdorff sense, and if g is any smooth complete Riemannian metric on M then the identity map $(M, d) \to (M, d_g)$ is locally bi-Hölder.

A key part of [ST17] is the use of Ricci flow to *'mollify'* the Riemannian manifolds $(\mathcal{M}_i, g_i)$ in the spirit of early work of Simon e.g. [Sim02, Sim12]. However, it is not expected that there exists any traditional smooth Ricci flow that starts from a general limit space (X, d_X), or even from a general *smooth* three-manifold with Ricci curvature bounded below [Top14], so in [ST17] a notion of local Ricci flow is used, which automatically generates not just a Ricci flow, but also the underlying smooth manifold for the flow, see e.g. Theorem 1.1 in [ST17]. This Ricci flow is posed within a class of flows with good estimates, and it is not reasonable to ask for uniqueness of solutions. A consequence of this is that if one takes a second local Ricci flow on a larger local region of the limit space, then restricts to the original local region, there is no guarantee that the natural identification of the two resulting smooth underlying manifolds will be smooth. Consequently, Theorem 1.1 of this paper does not immediately follow.

These considerations encourage us to look again at the idea of trying to imagine a Ricci flow starting from the entire Ricci limit space (X, d_X). We have already pointed out that this should be impossible in the traditional manner, but it is instructive to imagine why we cannot construct such a Ricci flow as a limit of local Ricci flows that exist on larger and larger balls $\mathbb{B}_{g_i}(x_i, i)$. The problem is that the degree of noncollapsing of such balls typically degenerates as $i \to \infty$, and therefore the existence time of the corresponding local Ricci flows must necessarily degenerate to zero.

The solution to these problems, refining an approach of Hochard [Hoc16], is to consider Ricci flows that live on a subset of spacetime that is not simply a parabolic cylinder $\mathcal{M} \times [0, T]$. Given a smooth, complete Riemannian three-manifold $(\mathcal{M}, g_0, x_0)$ satisfying the above Ricci lower bound and weakly noncollapsed condition, then for any $k \in \mathbb{N}$, we prove the existence of a smooth Ricci flow $g_k(t)$ that is defined on a subset of

spacetime that contains, for each $m \in \{1, \dots, k\}$, the cylinder $\mathbb{B}_{g_0}(x_0, m) \times [0, T_m]$, where crucially $T_m > 0$ depends only on α_0, v_0 and m, and in particular *not* on k. Further, the flow enjoys local curvature bounds on the set $\mathbb{B}_{g_0}(x_0, m) \times (0, T_m]$, which again depend only on α_0, v_0 and m.

Theorem 1.2 (***Pyramid Ricci flow construction***)**.** *Suppose that (M^3, g_0) is a complete smooth Riemannian three-manifold and $x_0 \in M$. Assume that for given $\alpha_0, v_0 > 0$ we have both $\mathrm{Ric}_{g_0} \geq -\alpha_0$ throughout M, and $\mathrm{Vol}\mathbb{B}_{g_0}(x_0, 1) \geq v_0 > 0$.*

Then there exist increasing sequences $C_k \geq 1$ and $\alpha_k > 0$, and a decreasing sequence $T_k > 0$, all defined for $k \in \mathbb{N}$, and depending only on α_0 and v_0, such that the following is true.

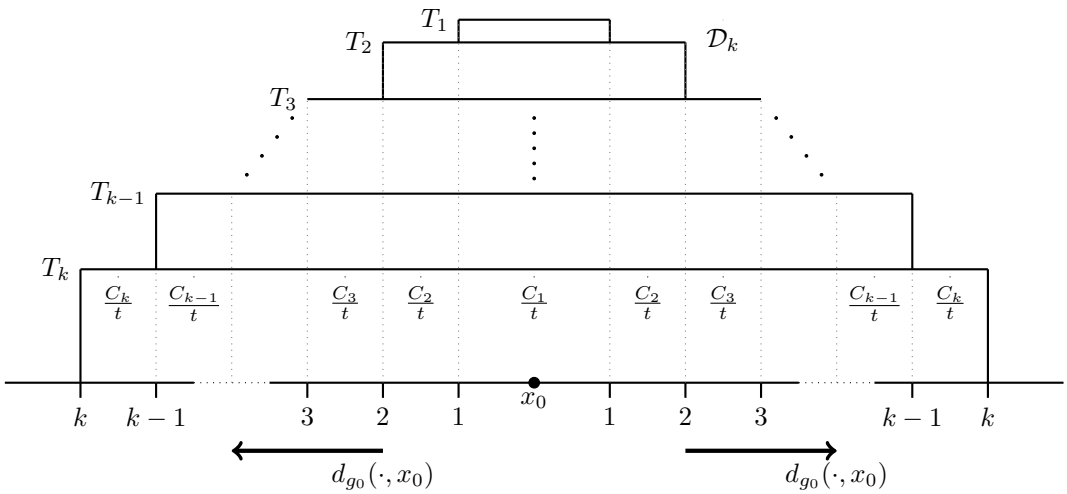
For any $k \in \mathbb{N}$ there exists a smooth Ricci flow solution $g_k(t)$, defined on a subset $\mathcal{D}_k$ of spacetime given by

$$\mathcal{D}_k := \bigcup_{m=1}^{k} \mathbb{B}_{g_0}(x_0, m) \times [0, T_m],$$

with $g_k(0) = g_0$ on $\mathbb{B}_{g_0}(x_0, k)$, and satisfying, for each $m \in \{1, \dots, k\}$,

$$\begin{cases} \mathrm{Ric}_{g_k(t)} \geq -\alpha_m & \text{on } \mathbb{B}_{g_0}(x_0, m) \times [0, T_m] \\ |\mathrm{Rm}|_{g_k(t)} \leq \frac{C_m}{t} & \text{on } \mathbb{B}_{g_0}(x_0, m) \times (0, T_m]. \end{cases} \tag{1.1}$$

The domain of definition $\mathcal{D}_k$ of the Ricci flow $g_k(t)$ has a pyramid structure, as illustrated in the following figure, and throughout this work we shall term such Ricci flows as *'Pyramid Ricci flows.'*.



As the distance from the central point x_0 increases, not only does the existence time of the flow decrease, but the C/t curvature decay estimate worsens. This is in contrast to the *partial* Ricci flow construction of Hochard, and is essential to obtain the uniform estimates on the domain of existence. Another distinction to partial Ricci flows is that by virtue of the theory of Miles Simon and the second author in [ST16, ST17], in particular the so-called Double Bootstrap lemma, our flows have lower Ricci bounds that do not degenerate as $t \downarrow 0$. These uniform lower Ricci bounds will be crucial for obtaining our bi-Hölder estimates in Theorem 1.1, and to make the application to Ricci limit spaces, thanks to the bi-Hölder regularity from Lemma 3.1 in [ST17] (see Lemma A.7 of this paper).

Although we do not need it to prove Theorem 1.1, we record now that our pyramid Ricci flows constructed in Theorem 1.2 allow us to prove the following hybrid of the local and global existence results from [ST17].

Theorem 1.3 (***Global-Local Ricci flows***)**.** *Suppose that (M, g_0, x_0) is a complete, smooth, pointed, Riemannian three-manifold and, for given $\alpha_0, v_0 > 0$, we have both $\mathrm{Ric}_{g_0} \geq -\alpha_0$ throughout M, and $\mathrm{Vol}\mathbb{B}_{g_0}(x_0, 1) \geq v_0 > 0$. Then there exist increasing sequences $C_j \geq 1$ and $\alpha_j > 0$ and a decreasing sequence $T_j > 0$, all defined for $j \in \mathbb{N}$, and depending only on α_0 and v_0, for which the following is true.*

There exists a smooth Ricci flow $g(t)$, defined on a subset of spacetime that contains, for each $j \in \mathbb{N}$, the cylinder $\mathbb{B}_{g_0}(x_0, j) \times [0, T_j]$, satisfying that $g(0) = g_0$ throughout M, and further that, again for each $j \in \mathbb{N}$,

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha_j & \text{on } \mathbb{B}_{g_0}(x_0, j) \times [0, T_j] \\ |\mathrm{Rm}|_{g(t)} \leq \frac{C_j}{t} & \text{on } \mathbb{B}_{g_0}(x_0, j) \times (0, T_j]\,. \end{cases} \tag{1.2}$$

To reiterate, in this result we only assume weak noncollapsing, and thus we must not expect global existence for positive times.

Analogously to Theorem 1.8 from [ST17], we can obtain this sort of global-local existence starting also from a *weakly* noncollapsed Ricci limit space, and in doing so we establish most of Theorem 1.1.

Theorem 1.4. (***Ricci flow from a weakly noncollapsed 3D Ricci limit space***) *Suppose that $\left(\mathcal{M}_i^3, g_i, x_i\right)$ is a sequence of complete, smooth, pointed Riemannian three-manifolds such that for given $\alpha_0, v_0 > 0$ we have $\mathrm{Ric}_{g_i} \geq -\alpha_0$ throughout $\mathcal{M}_i$, and $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, 1) \geq v_0 > 0$, for each $i \in \mathbb{N}$.*

Then there exist increasing sequences $C_k \geq 1$ and $\alpha_k > 0$ and a decreasing sequence $T_k > 0$, all defined for $k \in \mathbb{N}$, and depending only on α_0 and v_0, for which the following holds.

There exist a smooth three-manifold M, a point $x_0 \in M$, a complete distance metric $d : M \times M \to [0, \infty)$ generating the same topology as we already have on M, and a smooth Ricci flow $g(t)$ defined on a subset of spacetime $M \times (0, \infty)$ that contains $\mathbb{B}_d(x_0, k) \times (0, T_k]$ for each $k \in \mathbb{N}$, with $d_{g(t)} \to d$ locally uniformly on M as $t \downarrow 0$, and after passing to a subsequence in i we have that $(\mathcal{M}_i, d_{g_i}, x_i)$ converges in the pointed Gromov-Hausdorff sense to (M, d, x_0). Moreover, for any $k \in \mathbb{N}$,

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha_k & \text{on } \mathbb{B}_d(x_0, k) \times (0, T_k] \\ |\mathrm{Rm}|_{g(t)} \leq \frac{C_k}{t} & \text{on } \mathbb{B}_d(x_0, k) \times (0, T_k]\,. \end{cases} \tag{1.3}$$

This theorem will be a special case of the more elaborate Theorem 5.1 that will explicitly arrive at $g(t)$ as a limit of pyramid Ricci flows via pull-back by diffeomorphisms generated by a local form of Hamilton-Cheeger-Gromov compactness that we give in Lemma B.3. A further special case of Theorem 5.1 will be Theorem 1.1, and the following stronger assertion.

Theorem 1.5 (***Regular GH approximations***)**.** *In the setting of Theorem 1.1, we may assume the following additional conclusions:*

There exists a sequence of smooth maps $\varphi_i : \mathbb{B}_d(x_0, i) \to \mathcal{M}_i$, diffeomorphic onto their images, and mapping x_0 to x_i such that for any $R > 0$ we have $d_{g_i}(\varphi_i(x), \varphi_i(y)) \to d(x, y)$ uniformly for $x, y \in \mathbb{B}_d(x_0, R)$ as $i \to \infty$.

Moreover, for sufficiently large i, $\varphi_i|_{\mathbb{B}_d(x_0,R)}$ is bi-Hölder with Hölder exponent depending only on α_0, v_0 and R.

Finally, for any $r \in (0, R)$, and for sufficiently large i, $\varphi_i|_{\mathbb{B}_d(x_0,R)}$ maps onto $\mathbb{B}_{g_i}(x_i, r)$.

Thus, not only do we have the pointed Gromov-Hausdorff convergence of Theorem 1.1, we can also find Gromov-Hausdorff approximations that are smooth and bi-Hölder (neglecting a thin boundary layer) cf. Theorem 1.4 from [ST17].

In this paper we utilise numerous results, and mild variants of results, from [ST16] and [ST17]. For convenience we collect all such material in Appendix A. The remainder of the paper is structured as follows. In Section 2 we prove the *pyramid extension lemma* 2.1 that is related to the extension lemma 4.4 from [ST17]. We use this in Section 3 to prove Theorem 1.2, constructing the pyramid Ricci flows. In Section 4 we prove Theorem 1.3. Finally, in Section 5 we prove Theorem 5.1, which in turn establishes Theorems 1.1, 1.5 and 1.4.

Acknowledgements: This work was supported by EPSRC grant number EP/K00865X/1 and an EPSRC doctoral fellowship EP/M506679/1. The second author would like to thank Miles Simon for many discussions on this topic.

2. The Pyramid Extension Lemma

The following result interpolates between the local existence theorem (Theorem 1.6) and the extension lemma (Lemma 4.4) of Simon-Topping [ST17].

Lemma 2.1 (***Pyramid Extension Lemma***)**.** *Suppose (M, g_0, x_0) is a pointed complete Riemannian 3-manifold satisfying $\mathrm{Ric}_{g_0} \geq -\alpha_0 < 0$ throughout M, and $\mathrm{Vol}\mathbb{B}_{g_0}(x_0, 1) \geq v_0 > 0$. Then there exist increasing sequences $C_k \geq 1$ and $\alpha_k > 0$, and a decreasing sequence $S_k > 0$, all defined for $k \in \mathbb{N}$ and depending only on α_0 and v_0, with the following properties.*

First, for each $k \in \mathbb{N}$ there exists a Ricci flow $g(t)$ on $\mathbb{B}_{g_0}(x_0, k)$ for $t \in [0, S_k]$ such that $g(0) = g_0$ where defined and so that $|\mathrm{Rm}|_{g(t)} \leq C_k/t$ for all $t \in (0, S_k]$ and $\mathrm{Ric}_{g(t)} \geq -\alpha_k$ for all $t \in [0, S_k]$.

Moreover, given any Ricci flow $\tilde{g}(t)$ on $\mathbb{B}_{g_0}(x_0, k+1)$ over a time interval $t \in [0, S]$ with $\tilde{g}(0) = g_0$ where defined, and with $|\mathrm{Rm}|_{\tilde{g}(t)} \leq c_0/t$ for some $c_0 > 0$ and all $t \in (0, S]$, there exists $\tilde{S}_k > 0$ depending on k, α_0, v_0 and c_0 only, such that we may choose the Ricci flow $g(t)$ above to agree with the restriction of $\tilde{g}(t)$ to $\mathbb{B}_{g_0}(x_0, k)$ for times $t \in [0, \min\{S, \tilde{S}_k, S_k\}]$.

To understand this lemma, it is crucial to understand the dependencies of the constants. Notably, the existence time S_k does not depend on the constant c_0.

Proof of Lemma 2.1. By making a uniform parabolic rescaling (scaling distances by a factor of 14), it suffices to prove the lemma under the apparently stronger hypothesis that $\tilde{g}(t)$ is assumed to be defined not just on $\mathbb{B}_{g_0}(x_0, k+1)$ but on the larger ball $\mathbb{B}_{g_0}(x_0, k+14)$, still satisfying the curvature decay $|\mathrm{Rm}|_{\tilde{g}(t)} \leq c_0/t$.

We also observe that it is not important that the sequences C_k, α_k and S_k are monotonic; this can always be arranged with hindsight by, for example, iteratively replacing C_k by $\max\{C_k, C_{k-1}\}$ for $k = 2, 3, \ldots$.

By Bishop-Gromov, for all $k \in \mathbb{N}$, there exists $v_k > 0$ depending only on k, α_0 and v_0 such that if $x \in \mathbb{B}_{g_0}(x_0, k+14)$ and $r \in (0, 1]$ then $\mathrm{Vol}\mathbb{B}_{g_0}(x, r) \geq v_k r^3$.

Let us fix, for the remainder of the proof, the $k \in \mathbb{N}$ for which we want to prove the lemma. The first part of the lemma, giving the initial existence statement for $g(t)$, follows immediately by the local existence theorem A.3 for some $C_k \geq 1$, $\alpha_k > 0$ and $S_k > 0$ depending only on α_0 and v_k, i.e. on α_0, k and v_0. We will allow ourselves to increase C_k and α_k, and decrease S_k, in order to establish the remaining claims of the lemma.

We increase C_k to be at least as large as the constant C_0 retrieved from Lemma A.1 with v_0 there equal to v_k here. Thus C_k still depends only on k, α_0 and v_0, and in particular, not on c_0, and can be fixed for the remainder of the proof.

Before applying Lemma A.1, we appeal to the double bootstrap lemma A.2 centred at each $x \in \mathbb{B}_{g_0}(x_0, k+12)$, which tells us that there exists $\hat{S} > 0$ depending only on c_0 and α_0 so that for all $t \in [0, \min\{S, \hat{S}\}]$ we have $\mathrm{Ric}_{\tilde{g}(t)} \geq -100\alpha_0 c_0$ throughout $\mathbb{B}_{g_0}(x_0, k+12)$. (In due course, we will require a lower Ricci bound that does not depend on c_0.) In addition, the shrinking balls lemma A.5 tells us that for all $x \in \mathbb{B}_{g_0}(x_0, k+10)$ we have $\mathbb{B}_{\tilde{g}(t)}(x, 1) \subset \mathbb{B}_{g_0}(x, 2) \subset \mathbb{B}_{g_0}(x_0, k+12)$ where the Ricci curvature is controlled, for all $t \in [0, \min\{S, \tilde{S}_k\}]$, for some $\tilde{S}_k \in (0, \hat{S}]$ also depending only on c_0 and α_0.

Thus, for such x we can apply Lemma A.1 to deduce that $|\mathrm{Rm}|_{\tilde{g}(t)}(x) \leq C_k/t$ for all $t \in (0, \min\{S, \tilde{S}_k\}]$, where we have reduced $\tilde{S}_k > 0$ depending additionally on v_k (i.e. on k, α_0, v_0 and c_0).

Now we have a curvature decay estimate that does not depend on c_0 (albeit for a time depending on c_0) we can return to the double bootstrap lemma A.2, which then tells us that on the smaller ball $\mathbb{B}_{g_0}(x_0, k+8)$ we have $\mathrm{Ric}_{\tilde{g}(t)} \geq -\alpha_k$ for $t \in [0, \min\{S, \tilde{S}_k\}]$, where $\alpha_k := 100\alpha_0 C_k$ will be increased once more below (but only ever depending on k, α_0 and v_0) and where we have reduced $\tilde{S}_k > 0$ without adding any additional dependencies.

We can also exploit these new estimates to get better volume bounds via Lemma A.4. We apply that result with $R = k+8$ to obtain that for every $t \in [0, \min\{S, \tilde{S}_k\}]$, where we have reduced $\tilde{S}_k > 0$ again without adding any additional dependencies, we have $\mathbb{B}_{\tilde{g}(t)}(x_0, k+7) \subset \mathbb{B}_{g_0}(x_0, k+8)$, and for every $x \in \mathbb{B}_{\tilde{g}(t)}(x_0, k+6)$, we have $\mathrm{Vol}\mathbb{B}_{\tilde{g}(t)}(x, 1) \geq \varepsilon_k > 0$, where ε_k depends only on v_0, k, and α_0.

We need one final reduction of $\tilde{S}_k > 0$ in order to ensure appropriate nesting of balls defined at different times. By the expanding balls lemma A.6, exploiting our lower Ricci bounds (even the weaker bound suffices

here) we deduce that

$$\mathbb{B}_{g_0}(x_0, k+4) \subset \mathbb{B}_{\tilde{g}(t)}(x_0, k+5) \qquad \text{for all } t \in [0, \min\{S, \tilde{S}_k\}], \tag{2.1}$$

with $\tilde{S}_k > 0$ reduced appropriately, without additional dependencies.

At this point we can fix $\tilde{S}_k$ and try to find our desired extension $g(t)$ of $\tilde{g}(t)$ by considering $\tilde{g}(\tau)$ for $\tau := \min\{S, \tilde{S}_k\}$ and restarting the flow from there. We cannot restart the flow using any variant of Shi's existence theorem (as was done in the extension lemma from [ST17], for example) since we would not have appropriate control on the existence time. Instead, we appeal to the local existence theorem A.3. In order to do so, note that $\tilde{g}(\tau)$ satisfies the estimates $\mathrm{Ric}_{\tilde{g}(\tau)} \geq -\alpha_k$ on $\mathbb{B}_{g(\tau)}(x_0, k+7)$, and $\mathrm{Vol}\mathbb{B}_{\tilde{g}(\tau)}(x, 1) \geq \varepsilon_k > 0$ for each $x \in \mathbb{B}_{\tilde{g}(\tau)}(x_0, k+6)$.

The output of the local existence theorem A.3, applied with g_0 there equal to $\tilde{g}(\tau)$ here, and with $s_0 = k+7$, is that after reducing the $S_k > 0$ that we happened to find at the start of the proof, still depending only on α_0, k and v_0, there exists a Ricci flow $h(t)$ on $\mathbb{B}_{\tilde{g}(\tau)}(x_0, k+5)$ for $t \in [0, S_k]$, with $h(0) = \tilde{g}(\tau)$ where defined, and such that $\mathrm{Ric}_{h(t)} \geq -\alpha_k$ (after possibly increasing α_k, still depending only on α_0, k and v_0). That theorem also gives us decay of the curvature tensor, although we do not need that. By the inclusion of (2.1), this flow is defined throughout $\mathbb{B}_{g_0}(x_0, k+4)$.

Define a concatenated Ricci flow on $\mathbb{B}_{g_0}(x_0, k+4)$ for $t \in [0, \tau + S_k]$ by

$$g(t) := \begin{cases} \tilde{g}(t) & 0 \leq t \leq \tau \\ h(t-\tau) & \tau < t \leq \tau + S_k. \end{cases} \tag{2.2}$$

This already satisfies the required lower Ricci bound $\mathrm{Ric}_{g(t)} \geq -\alpha_k$.

At this point we truncate the flow $g(t)$ to live only on the time interval $[0, S_k]$ (i.e. we chop off some of the end, not the beginning). The flow now lives on a time interval of length independent of c_0 and S.

The main final step is to apply Lemma A.1 once more. For every $x \in \mathbb{B}_{g_0}(x_0, k+2)$, by the shrinking balls lemma A.5, after possibly reducing $S_k > 0$, without adding dependencies, we may assume that $\mathbb{B}_{g(t)}(x, 1) \subset \mathbb{B}_{g_0}(x, 2) \subset \mathbb{B}_{g_0}(x_0, k+4)$. Thus, after a possible further reduction of $S_k > 0$, and with C_k as fixed earlier, the local lemma A.1 tells us that $|\mathrm{Rm}|_{g(t)} \leq C_k/t$ for all $t \in (0, S_k]$, and throughout $\mathbb{B}_{g_0}(x_0, k+2)$. By restricting $g(t)$ to $\mathbb{B}_{g_0}(x_0, k)$ we are done. ■

3. Pyramid Ricci Flow Construction - Proof of Theorem 1.2

Proof of Theorem 1.2. For our given constants α_0 and v_0, we appeal to Lemma 2.1 for increasing sequences $C_k \geq 1$ and $\alpha_k > 0$, and a decreasing sequence $S_k > 0$, all defined for $k \in \mathbb{N}$ and depending only on α_0 and v_0. Moreover, we construct a sequence $\tilde{S}_k$ as follows. For each $k \in \mathbb{N}$, we appeal to Lemma 2.1 with our given constants α_0 and v_0 and with $c_0 = C_{k+1}$. The sequences $C_k \geq 1$ and $\alpha_k > 0$ are suitable for the sequences required by the theorem.

An induction argument is required to get the constants T_k. We begin by setting T_1 to be S_1. The inductive step is as follows: Suppose we have picked $T_1, \ldots T_{k-1}$ for any integer $k \geq 2$. Then we set T_k to be the minimum of S_k, $\tilde{S}_{k-1}$ and T_{k-1}.

Note that when we pick T_k, it depends on S_k, i.e. on k, α_0 and v_0, and it also depends on $\tilde{S}_{k-1}$, i.e. additionally on C_k, but that itself only depends on k, α_0 and v_0.

Fix $l \in \mathbb{N}$. To construct $g_l(t)$, we appeal to Lemma 2.1 l times.

First we use the first part of that lemma with $k = l$. This initial flow lives on $\mathbb{B}_{g_0}(x_0, l)$ for a time S_l, and thus certainly for T_l.

Since $T_l \leq \tilde{S}_{l-1}$, we can appeal a second time to the lemma, this time with $k = l-1$, in order to extend the flow $g_l(t)$ to the longer time interval $[0, T_{l-1}]$, albeit on the smaller ball $\mathbb{B}_{g_0}(x_0, l-1)$.

We repeat this process inductively for the remaining values of k down until it is finally repeated for $k = 1$. The resulting smooth Ricci flow $g_l(t)$ is now defined, for each $m \in \{1, \ldots, l\}$, on $\mathbb{B}_{g_0}(x_0, m)$ over the time interval $t \in [0, T_m]$, still satisfying that $g_l(0) = g_0$ where defined. Moreover, our repeated applications of

Lemma 2.1 provide the estimates

$$\begin{cases} \mathrm{Ric}_{g_l(t)} \geq -\alpha_m & \text{on } \mathbb{B}_{g_0}(x_0, m) \times [0, T_m] \\ |\mathrm{Rm}|_{g_l(t)} \leq \frac{C_m}{t} & \text{on } \mathbb{B}_{g_0}(x_0, m) \times (0, T_m] \end{cases} \tag{3.1}$$

for each $m \in \{1, \ldots, l\}$, which completes the proof. ■

4. Global-Local Mollification

Proof of Theorem 1.3. For the α_0 and v_0 of the theorem, we begin by retrieving sequences C_j, α_j and T_j from Theorem 1.2. Our first step is to modify them by throwing away the first two terms of each, i.e. replacing the three sequences by C_{j+2}, α_{j+2} and T_{j+2}.

With a view to later applying the shrinking balls lemma A.5 for each $j \in \mathbb{N}$ we reduce T_j, without additional dependencies; with hindsight, it will suffice to ensure that

$$T_j < \frac{1}{4\beta^2 C_j} \tag{4.1}$$

where $\beta \geq 1$ is the universal constant arising in the shrinking balls lemma A.5.

For each $i \in \mathbb{N}$ let $g_i(t)$ denote the *pyramid Ricci flow* obtained in Theorem 1.2 defined on a subset $\mathcal{D}_i \subset M \times [0, \infty)$ that now contains $\mathbb{B}_{g_0}(x_0, l+2) \times [0, T_l]$ for each $l \in \{1, \ldots, i\}$, having deleted the first two terms of the sequences C_j, α_j and T_j. If we fix $j \in \mathbb{N}$, then for $i \geq j$ the estimates of (1.1) hold for $g_i(t)$ on the g_0 ball of radius $j+2$. Consider an arbitrary point $y \in \mathbb{B}_{g_0}(x_0, j+1)$. We have the curvature estimate $|\mathrm{Rm}|_{g_i(t)} \leq \frac{C_j}{t}$ throughout $\mathbb{B}_{g_0}(y, 1) \times (0, T_j]$. The shrinking balls lemma A.5 tells us that $\mathbb{B}_{g_i(t)}\left(y, \frac{1}{2}\right) \subset \mathbb{B}_{g_0}\left(y, \frac{1}{2} + \beta\sqrt{C_j T_j}\right) \subset\subset \mathbb{B}_{g_0}(y, 1)$ for any $t \in [0, T_j]$, provided $\frac{1}{2} + \beta\sqrt{C_j T_j} < 1$. The restriction (4.1) ensures this is the case, and hence we establish the curvature bound $|\mathrm{Rm}|_{g_i(t)} \leq \frac{C_j}{t}$ throughout $\mathbb{B}_{g_i(t)}\left(y, \frac{1}{2}\right)$ for any $t \in (0, T_j]$.

These estimates allow us to repeat the argument of Miles Simon and the second author in Theorem 1.7 in [ST17] and deduce that after passing to a subsequence in i, we have smooth convergence $g_i(t) \to g(t)$, for some smooth Ricci flow $g(t)$ on $\mathbb{B}_{g_0}(x_0, j)$, defined for $t \in [0, T_j]$, with $g(0) = g_0$ on $\mathbb{B}_{g_0}(x_0, j)$, and satisfying the curvature estimates

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha_j & \text{on } \mathbb{B}_{g_0}(x_0, j) \times [0, T_j] \\ |\mathrm{Rm}|_{g(t)} \leq \frac{C_j}{t} & \text{on } \mathbb{B}_{g_0}(x_0, j) \times (0, T_j]. \end{cases} \tag{4.2}$$

We can now repeat this process for each $j = 1, 2, \ldots$ and take a diagonal subsequence to obtain a smooth limit Ricci flow $g(t)$ on a subset of spacetime that contains $\mathbb{B}_{g_0}(x_0, j) \times [0, T_j]$ for each $j \in \mathbb{N}$, with $g(0) = g_0$ throughout M, and satisfying (4.2) for every $j \in \mathbb{N}$. ■

Remark 4.1. By dropping further terms from the sequences C_j, α_j and T_j we could invoke Lemma 3.1 from [ST17], i.e. Lemma A.7 from this paper, to obtain local bi-Hölder distance estimates relating $d_{g(t)}$ for $t > 0$ and d_{g_0}, invoke Lemma A.4, for $R = 2$ and $\varepsilon = 1$, to deduce $\mathrm{Vol}\mathbb{B}_{g(t)}(x_0, 1) \geq \mathcal{V}(\alpha_0, v_0) > 0$ for all $t \in [0, T_1]$, and finally, by using the Shi estimates from Lemma B.1 together with the expanding balls lemma A.6, we could control all derivatives of $\mathrm{Rm}_{g(t)}$ on each $\mathbb{B}_{g_0}(x_0, j)$ over the time interval $t \in (0, T_j]$.

5. Pyramid Ricci Flow Compactness Theorem

The following overarching theorem effectively includes Theorems 1.1, 1.4 and 1.5. The Ricci flows $g_k(t)$ arising here are pyramid Ricci flows coming from Theorem 1.2.

Theorem 5.1. *In the setting of Theorem 1.4, in addition to the conclusions of that theorem, including the existence of M, x_0, d, $g(t)$ and the sequences C_k, α_k and T_k, we may assume also that the following holds.*

For each $k \in \mathbb{N}$, there exist Ricci flows $g_k(t)$ defined on the subset of $\mathcal{M}_k \times [0,\infty)$ defined by

$$\mathcal{D}_k := \bigcup_{m=1}^{k} \mathbb{B}_{g_k}(x_k, m+2) \times [0, T_m]\,,$$

with the properties that $g_k(0) = g_k$ on $\mathbb{B}_{g_k}(x_k, k+2)$ and

$$\begin{cases} \mathrm{Ric}_{g_k(t)} \geq -\alpha_m & \text{on } \mathbb{B}_{g_k}(x_k, m+2) \times [0, T_m] \\ |\mathrm{Rm}|_{g_k(t)} \leq \frac{C_m}{t} & \text{on } \mathbb{B}_{g_k}(x_k, m+2) \times (0, T_m]\,, \end{cases} \tag{5.1}$$

for each $m \in \{1, \ldots, k\}$.

Moreover, for each $m \in \mathbb{N}$, the flows $g_k(t)$ converge to $(B_d(x_0, m), g(t))$ for $t \in (0, T_m]$, in the following sense: There exists a sequence of smooth maps $f_k^m : B_d(x_0, m) \to \mathbb{B}_{g_k}(x_k, m+1) \subset \mathcal{M}_k$, mapping x_0 to x_k, such that for each $\delta \in (0, T_m)$ we have $(f_k^m)^* g_k(t) \to g(t)$ smoothly uniformly on $B_d(x_0, m) \times [\delta, T_m]$.

Moreover, there exists a sequence of smooth maps $\varphi_k : \mathbb{B}_d(x_0, k) \to \mathbb{B}_{g_k}(x_k, k+1) \subset \mathcal{M}_k$, diffeomorphic onto their images, mapping x_0 to x_k, such that, for any $R > 0$, as $k \to \infty$ we have the convergence

$$d_{g_k}(\varphi_k(x), \varphi_k(y)) \to d(x, y)$$

uniformly as x, y vary over $\mathbb{B}_d(x_0, R)$, and for sufficiently large k, $\varphi_k|_{\mathbb{B}_d(x_0,R)}$ is bi-Hölder with Hölder exponent depending only on α_0, v_0 and R. Moreover, for any $r \in (0, R)$, and for sufficiently large k, $\varphi_k|_{\mathbb{B}_d(x_0,R)}$ maps onto $\mathbb{B}_{g_k}(x_k, r)$.

Finally, if g is any smooth complete Riemannian metric on M then the identity map $(M, d) \to (M, d_g)$ is locally bi-Hölder.

To clarify, by smooth uniform convergence, we mean uniform C^l convergence for arbitrary $l \in \mathbb{N}$. We remark that the bi-Hölder assertion for the maps φ_k in this theorem can be taken with respect to the distance metrics d and d_{g_k}, although one could replace g_k by any complete smooth metric.

Proof of Theorem 5.1. For the α_0 and v_0 of the theorem (as in Theorem 1.4) we begin by retrieving sequences C_j, α_j and T_j from Theorem 1.2.

Throughout the proof $\eta := \frac{1}{10}$ will be fixed. With a view to later applying Lemma A.7 and both the expanding and shrinking balls lemmas, for each $j \in \mathbb{N}$ we reduce T_j, without additional dependencies, and with hindsight it will suffice to ensure that

$$\begin{cases} (i) \quad (4j+8)(1 - e^{-\alpha_j T_j}) < 1 - 8\beta\sqrt{C_j T_j} \qquad \text{(in particular } \beta\sqrt{C_j T_j} < \tfrac{1}{8}\text{) and} \\ (ii) \quad (j+1)(e^{\alpha_j T_j} - 1) \leq \eta \end{cases} \tag{5.2}$$

where $\beta \geq 1$ is the universal constant arising in the shrinking balls lemma A.5. For $j = 2, 3, \ldots$, if necessary we inductively replace T_j by $\min\left\{T_j, T_{j-1}, \frac{1}{j}\right\}$ to ensure the monotonicity of the sequence T_j remains, and to force $T_j \downarrow 0$ as $j \to \infty$.

We modify these sequences further by dropping the first two terms, i.e. by replacing each C_j, α_j and T_j by C_{j+2}, α_{j+2} and T_{j+2} respectively. This does not affect the monotonicity or dependencies. We may fix the values $C_j \geq 1$ and $\alpha_j > 0$ for each $j \in \mathbb{N}$ for the remainder of the proof. Before fixing T_j, we (potentially) reduce the value further.

With a view to appealing to Cheeger-Gromov-Hamilton compactness via Lemma B.3, we reduce T_j, without additional dependencies, so that the conclusions of Lemma B.3 for hypotheses $R = j+1$, $\eta = \frac{1}{10}$, $n = 3$, $v = v_0$, $\alpha = \alpha_j$ and $c_0 = C_j$ are valid for all times $t \in (0, T_j]$. As above, we may assume that T_j remains monotonically decreasing. After these reductions, we can now fix the value of T_j for each $j \in \mathbb{N}$ for the remainder of the proof.

For each $k \in \mathbb{N}$ let $g_k(t)$ denote the smooth *pyramid Ricci flow*, defined on the subset $\mathcal{D}_k \subset \mathcal{M}_k \times [0,\infty)$ obtained via Theorem 1.2. That is

$$\mathcal{D}_k = \bigcup_{m=1}^{k} \mathbb{B}_{g_k}(x_k, m+2) \times [0, T_m]\,. \tag{5.3}$$

(Recall that we have dropped the first two terms of the sequences, so we can work on a radius $m+2$ rather than m.) In particular, we have $g_k(0) = g_k$ where defined and for each $m \in \{1, \dots, k\}$ we have

$$\begin{cases} \mathrm{Ric}_{g_k(t)} \geq -\alpha_m & \text{on } \mathbb{B}_{g_k}(x_k, m+2) \times [0, T_m] \\ |\mathrm{Rm}|_{g_k(t)} \leq \frac{C_m}{t} & \text{on } \mathbb{B}_{g_k}(x_k, m+2) \times (0, T_m]\,. \end{cases} \tag{5.4}$$

Fix $m \in \mathbb{N}$. For every $k \geq m$ the flow $g_k(t)$ is defined throughout $\mathbb{B}_{g_k}(x_k, m+2) \times [0, T_m]$. Combining (5.4) with $\mathrm{Vol}\mathbb{B}_{g_k}(x_k, m+1) \geq v_0 > 0$ allows us to appeal to Lemma B.3 with $R = m+1$, $\eta = \frac{1}{10}$, $n = 3$, $v = v_0$, $\alpha = \alpha_m$ and $c_0 = C_m$ to deduce that, after passing to a subsequence in k, we obtain a smooth three-manifold $\mathcal{N}_m$, a point $x_\infty^m \in \mathcal{N}_m$ and a smooth Ricci flow $\hat{g}_m(t)$ on $\mathcal{N}_m \times (0, T_m]$ with the following properties. First, for any $t \in (0, T_m]$ we have the inclusion

$$\mathbb{B}_{\hat{g}_m(t)}\left(x_\infty^m, m+1-\eta\right) \subset\subset \mathcal{N}_m. \tag{5.5}$$

Second, we have

$$\mathbb{B}_{\hat{g}_m(t)}\left(x_\infty^m, m+1-2\eta\right) \subset M_m, \tag{5.6}$$

for all $t \in (0, T_m]$, where M_m is the connected component of the interior of

$$\bigcap_{s \in (0, T_m]} \mathbb{B}_{\hat{g}_m(s)}\left(x_\infty^m, m+1-\eta\right) \subset \mathcal{N}_m \tag{5.7}$$

that contains x_∞^m. Combining (5.5) and (5.7) allows us to conclude that

$$M_m \subset\subset \mathcal{N}_m. \tag{5.8}$$

Moreover, Lemma B.3 gives us a sequence of smooth maps $F_k^m : M_m \to \mathbb{B}_{g_k}(x_k, m+1) \subset \mathcal{M}_k$, for $k \geq m$, mapping x_∞^m to x_k, diffeomorphic onto their images and such that $(F_k^m)^* g_k(t) \to \hat{g}_m(t)$ smoothly uniformly on $M_m \times [\delta, T_m]$ as $k \to \infty$, for every $\delta \in (0, T_m)$. Finally, we have

$$\begin{cases} \mathrm{Ric}_{\hat{g}_m(t)} \geq -\alpha_m & \text{on } M_m \times (0, T_m] \\ |\mathrm{Rm}|_{\hat{g}_m(t)} \leq \frac{C_m}{t} & \text{on } M_m \times (0, T_m]\,. \end{cases} \tag{5.9}$$

By taking an appropriate diagonal subsequence in k, we can be sure that these limits exist for *every* $m \in \mathbb{N}$.

We now wish to relate the limit flows $\hat{g}_m(t)$ that we have constructed, for different m. Let us fix m. Then $\hat{g}_m(T_{m+1})$ is a smooth limit of the metrics $g_k(T_{m+1})$ (modulo the diffeomorphisms F_k^m) defined on M_m. On the other hand, $\hat{g}_{m+1}(T_{m+1})$ is a smooth limit of the metrics $g_k(T_{m+1})$ (modulo the diffeomorphisms F_k^{m+1}) defined on M_{m+1}. Intuitively, M_{m+1} should be "bigger" than M_m since it arises from the compactness of the metrics on larger radius balls. This intuition is made precise in the following claim.

<u>Claim:</u> For sufficiently large k we have

$$F_k^m(M_m) \subset F_k^{m+1}(M_{m+1}). \tag{5.10}$$

Indeed, we have the stronger inclusion that for any $t \in (0, T_{m+1}]$ and sufficiently large k, depending on t,

$$F_k^m(M_m) \subset F_k^{m+1}\left(\mathbb{B}_{\hat{g}_{m+1}(t)}\left(x_\infty^{m+1}, m+2-2\eta\right)\right) \tag{5.11}$$

which immediately yields (5.10) via (5.6) by fixing $t = T_{m+1}$.

Proof: Recall that by definition of F_k^m, for all $k \geq m \in \mathbb{N}$ we have

$$F_k^m(M_m) \subset \mathbb{B}_{g_k}(x_k, m+1).$$

For each $t \in (0, T_{m+1}]$, and sufficiently large k, depending on t, we may appeal to Part 2 of Lemma A.8, with $2r = m+2-2\eta$, $b = 2r$, $a = m+2-3\eta$, $x_0 = x_\infty^{m+1}$, $(\mathcal{N}, \hat{g}) = (M_{m+1}, \hat{g}_{m+1}(t))$ and the sequence $\{\varphi_i\}$ being the sequence $\{F_k^{m+1}\}_{k \geq m+1}$, to deduce that $F_k^{m+1}(\mathbb{B}_{\hat{g}_{m+1}(t)}(x_\infty^{m+1}, m+2-2\eta)) \supset \mathbb{B}_{g_k(t)}(x_k, m+2-3\eta)$. Thus, in order to prove (5.11), it suffices to prove that

$$\mathbb{B}_{g_k}(x_k, m+1) \subset \mathbb{B}_{g_k(t)}(x_k, m+2-3\eta). \tag{5.12}$$

We prove this through a combination of the shrinking and expanding balls lemmas.

Recall from (5.4) we know that $\mathrm{Ric}_{g_k(t)} \geq -\alpha_m$ throughout $\mathbb{B}_{g_k}(x_k, m+2) \times [0, T_m]$ and $|\mathrm{Rm}|_{g_k(t)} \leq \frac{C_m}{t}$ throughout $\mathbb{B}_{g_k}(x_k, m+2) \times (0, T_m]$. Therefore we can appeal to the shrinking balls lemma A.5 to deduce that $\mathbb{B}_{g_k(t)}(x_k, m+2-3\eta) \subset \mathbb{B}_{g_k}(x_k, m+2)$ provided $m+2-3\eta \leq m+2-\beta\sqrt{C_m t}$, which will be the case if $\beta\sqrt{C_m T_{m+1}} \leq 3\eta$, since $t \leq T_{m+1}$. But (i) in (5.2) tells us that $\beta\sqrt{C_m T_m} < \frac{1}{8}$, which is slightly stronger than required (recalling the monotonicity of the sequence T_j).

Thus we may conclude that $\mathrm{Ric}_{g_k(t)} \geq -\alpha_m$ throughout $\mathbb{B}_{g_k(t)}(x_k, m+2-3\eta) \times [0, T_{m+1}]$. The expanding balls lemma A.6 then tells us that $\mathbb{B}_{g_k(t)}(x_k, m+1+\eta) \supset \mathbb{B}_{g_k}(x_k, m+1)$, provided $(m+1+\eta)e^{-\alpha_m t} \geq m+1$, which will itself be true if $(m+1)(e^{\alpha_m T_{m+1}} - 1) \leq \eta$. Since $T_m \geq T_{m+1}$ we observe that (ii) of (5.2) ensures this is the case. But this inclusion is stronger than the inclusion (5.12) that we need. ††

By the uniqueness of smooth limits the metrics must agree in the sense that there is a smooth map $\psi_m : M_m \to M_{m+1}$ that is an isometry when domain and target are given the metrics $\hat{g}_m(T_{m+1})$ and $\hat{g}_{m+1}(T_{m+1})$ respectively, and which sends x_∞^m to x_∞^{m+1}.

Indeed, after passing to another subsequence, we could see ψ_m as a smooth limit, as $k \to \infty$, of maps $(F_k^{m+1})^{-1} \circ F_k^m$, which are well-defined because of the claim, and which are independent of time, and it is apparent that in fact ψ_m is an isometry also when domain and target are given the metrics $\hat{g}_m(t)$ and $\hat{g}_{m+1}(t)$ respectively, for any $t \in (0, T_{m+1}]$.

Moreover seeing ψ_m as such a limit and appealing to (5.11) allows us to conclude that

$$\psi_m(M_m) \subset \mathbb{B}_{\hat{g}_{m+1}(t)}\left(x_\infty^{m+1}, m+2-2\eta\right) \tag{5.13}$$

for any $t \in (0, T_{m+1}]$.

At this point we can already define a smooth extension of $\hat{g}_{m+1}(t)$ to the longer time interval $t \in (0, T_m]$, albeit on the smaller region $\psi_m(M_m)$, by taking $(\psi_m^{-1})^*(\hat{g}_m(t))$. However we would like to make such an extension for each m, and we must pause to construct the manifold on which this final flow will live.

The maps $\psi_m : M_m \to M_{m+1}$ allow us to apply Theorem C.1 to the collection $\{M_m\}_{m \in \mathbb{N}}$. Doing so gives a smooth three-manifold M, a point $x_0 \in M$, and smooth maps $\theta_m : M_m \to M$, mapping x_∞^m to x_0, diffeomorphic onto their images, satisfying $\theta_m(M_m) \subset \theta_{m+1}(M_{m+1})$ and $\theta_{m+1}^{-1} \circ \theta_m = \psi_m$, and such that

$$M = \bigcup_{m \in \mathbb{N}} \theta_m(M_m). \tag{5.14}$$

In a moment, we will strengthen the inclusion $\theta_m(M_m) \subset \theta_{m+1}(M_{m+1})$ to assert that the images of M_m are contained within bounded subsets of M.

We can thus consider pull-back Ricci flows $(\theta_m^{-1})^*\hat{g}_m(t)$ on $\theta_m(M_m) \subset M$ for each m, and because ψ_m is an isometry, these pull-backs agree where they overlap. The union of the pull-backs we call $g(t)$. Moreover, the curvature estimates of (5.9) immediately give that for each $m \in \mathbb{N}$ we have

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha_m & \text{on } \theta_m(M_m) \times (0, T_m] \\ |\mathrm{Rm}|_{g(t)} \leq \frac{C_m}{t} & \text{on } \theta_m(M_m) \times (0, T_m]\,. \end{cases} \tag{5.15}$$

Furthermore, from (5.6) and (5.8) we have that

$$\mathbb{B}_{g(s)}(x_0, m+1-3\eta) = \theta_m(\mathbb{B}_{\hat{g}_m(s)}(x_\infty^m, m+1-3\eta)) \subset\subset \theta_m(M_m) \tag{5.16}$$

for any $0 < s \leq T_m$.

Since $\theta_{m+1}^{-1} \circ \theta_m \equiv \psi_m$, (5.13) implies $\theta_{m+1}^{-1}(\theta_m(M_m)) \subset \mathbb{B}_{\hat{g}_{m+1}(t)}\left(x_\infty^{m+1}, m+2-2\eta\right) \subset M_{m+1}$ for any $t \in (0, T_{m+1}]$. Therefore we can strengthen the inclusion $\theta_m(M_m) \subset \theta_{m+1}(M_{m+1})$ to

$$\theta_m(M_m) \subset \mathbb{B}_{g(t)}(x_0, m+2-2\eta) \subset \theta_{m+1}(M_{m+1}) \tag{5.17}$$

for any $t \in (0, T_{m+1}]$.

For each $m \in \mathbb{N}$ we have a sequence

$$f_k^m : \theta_m(M_m) \to \mathbb{B}_{g_k}(x_k, m+1) \subset \mathcal{M}_k$$

of smooth maps, for $k \geq m$, defined by $f_k^m := F_k^m \circ \theta_m^{-1}$, that map x_0 to x_k and are diffeomorphic onto their images. Moreover, from the choice of our diagonal subsequence, for any $\delta \in (0, T_m)$ we have

$$(f_k^m)^* g_k(t) \to g(t) \tag{5.18}$$

smoothly uniformly on $\theta_m(M_m) \times [\delta, T_m]$ as $k \to \infty$.

The obvious idea for constructing a distance metric d on M is to define $d := \lim_{t\downarrow 0} d_{g(t)}$, if we can be sure that this limit exists. The existence is a consequence of Lemma A.7, which may be applied with $r = \frac{m}{2} + \frac{1}{4}$, $\alpha = \alpha_m$, $c_0 = C_m$ and $T = T_m$, which is possible due to the curvature estimates of (5.15), and the fact that from (5.16) we have $\mathbb{B}_{g(s)}(x_0, m+1-3\eta) \subset\subset \theta_m(M_m)$ for any $0 < s \leq T_m$.

The result is a distance metric d on $\Sigma_m := \bigcap_{t\in(0,T_m]} \mathbb{B}_{g(t)}\left(x_0, \frac{m}{2} + \frac{1}{4}\right)$ arising as the uniform limit of $d_{g(t)}$ as $t \downarrow 0$. Moreover, for any $x, y \in \Sigma_m$ and any $0 < s \leq T_m$ we have

$$d(x,y) - \beta\sqrt{C_m s} \leq d_{g(s)}(x,y) \leq e^{\alpha_m s} d(x,y) \tag{5.19}$$

and

$$\kappa_m(m, \alpha_0, v_0)\,[d(x,y)]^{1+4C_m} \leq d_{g(s)}(x,y), \tag{5.20}$$

where $\kappa_m > 0$. As stated in Lemma A.7, these estimates ensure d generates the same topology as we already have on Σ_m.

If we can estimate the R_0 from (A.14) by $R_0 > \frac{m}{2} + \frac{1}{8}$, then (A.14) gives that for any $t \in (0, T_m]$ we have

$$\mathbb{B}_d\left(x_0, \frac{m}{2} + \frac{1}{8}\right) \subset\subset \mathcal{O}_m \quad \text{and} \quad \mathbb{B}_{g(t)}\left(x_0, \frac{m}{2} + \frac{1}{8}\right) \subset\subset \Sigma_m \tag{5.21}$$

where $\mathcal{O}_m$ is the connected component of the interior of Σ_m that contains x_0. This lower bound for R_0 is true provided $\left(\frac{m}{2} + \frac{1}{4}\right) e^{-\alpha_m T_m} - \beta\sqrt{C_m T_m} > \frac{m}{2} + \frac{1}{8}$, i.e. if $1 - 8\beta\sqrt{C_m T_m} > (4m+2)(1 - e^{-\alpha_m T_m})$. Restriction (i) in (5.2) implies this inequality and hence the inclusions of (5.21) are valid.

A particular consequence of the first of these inclusions, via (5.19) and (5.20), is that for any $x, y \in \mathbb{B}_d(x_0, \frac{m}{2})$ and any $0 < s \leq T_m$ we have

$$d(x,y) - \beta\sqrt{C_m s} \leq d_{g(s)}(x,y) \leq e^{\alpha_m s} d(x,y) \tag{5.22}$$

and

$$\kappa_m(m, \alpha_0, v_0)\,[d(x,y)]^{1+4C_m} \leq d_{g(s)}(x,y). \tag{5.23}$$

The natural idea for extending d to the entirety of M is to repeat this procedure for all $m \in \mathbb{N}$. Of course this will require the sets $\{\Sigma_m\}_{m\in\mathbb{N}}$ to exhaust M. That this is indeed the case is a consequence of the following claim.

<u>Claim:</u> For every $m \in \mathbb{N}$ we have $\theta_m(M_m) \subset\subset \Sigma_{2m+4}$.

Proof: Recall from (5.17) we know that $\theta_m(M_m) \subset \mathbb{B}_{g(t)}(x_0, m+2-2\eta)$ for any $t \in (0, T_{m+1}]$. Moreover (5.21) gives that for any $t \in (0, T_{2m+4}]$ we have $\mathbb{B}_{g(t)}(x_0, m+2) \subset\subset \Sigma_{2m+4}$. Working with $t = T_{2m+4}$ in both of these inclusions gives the desired inclusion. ††

Knowing that the collection $\{\Sigma_m\}_{m\in\mathbb{N}}$ exhausts M allows us to repeat for all $m \in \mathbb{N}$ and extend d to the entirety of M whilst ensuring d generates the same topology as we already have on M.

Moreover, it is clear that (M, d) is a complete metric space. To elaborate, consider a Cauchy sequence in M with respect to d. This sequence must be bounded and so contained within $\mathbb{B}_d\left(x_0, \frac{m}{2}\right)$ for some $m \in \mathbb{N}$. The first inclusion of (5.21) tells us that the closure of this ball is compact, so we may pass to a convergent subsequence. By virtue of the sequence being Cauchy, this establishes the sequence itself is convergent.

The estimates (5.22) and (5.23) give the local bi-Hölder regularity of the identity map on M that is claimed at the end of Theorem 5.1, as we now explain. Let $m \in \mathbb{N}$ and consider $\mathbb{B}_d\left(x_0, \frac{m}{2}\right) \subset\subset M$. For our arbitrary complete metric g on M, the distance metric d_g is bi-Lipschitz equivalent to $d_{g(T_m)}$ once restricted to $\mathbb{B}_d\left(x_0, \frac{m}{2}\right)$. The estimates (5.22) and (5.23) tell us that the identity map $\left(\mathbb{B}_d\left(x_0, \frac{m}{2}\right), d\right) \to \left(\mathbb{B}_d\left(x_0, \frac{m}{2}\right), d_{g(T_m)}\right)$ is Lipschitz continuous, whilst the identity map $\left(\mathbb{B}_d\left(x_0, \frac{m}{2}\right), d_{g(T_m)}\right) \to \left(\mathbb{B}_d\left(x_0, \frac{m}{2}\right), d\right)$ is Hölder continuous, with Lipschitz constant and Hölder exponent depending only on α_0, v_0 and m. The arbitrariness of $m \in \mathbb{N}$ gives the desired local bi-Hölder regularity of the identity map $(M, d) \to (M, d_g)$.

Having d defined globally on M allows us to simplify several of the techniques utilised in [ST17]. For example, given $m \in \mathbb{N}$ the local uniform convergence of $d_{g(t)}$ to d as $t \downarrow 0$ tells us that for some $t_0 > 0$ we have $\mathbb{B}_d(x_0, m) \subset \mathbb{B}_{g(t)}\left(x_0, m+\frac{1}{2}\right)$ for every $t \in (0, \min\{t_0, T_m\}]$. Hence from (5.16) (recalling the definition of η)

$$\mathbb{B}_d(x_0, m) \subset\subset \theta_m(M_m) \tag{5.24}$$

and so the estimates of (5.15) are valid on $\mathbb{B}_d(x_0, m) \times (0, T_m]$. In fact, this establishes that the flow $g(t)$ lives where specified by the theorem.

We now turn our attention to defining the smooth maps φ_i. For each $m \in \mathbb{N}$, by (5.16) and (5.18) we have $(f_k^m)^* g_k(T_m) \to g(T_m)$ smoothly on $\overline{\mathbb{B}_{g(T_m)}(x_0, m+1-4\eta)}$ and so, by appealing to Lemma A.8, we may choose $K(m)$ such that for all $k \geq K(m)$ we have

$$\left|d_{g_k(T_m)}\left(f_k^m(x), f_k^m(y)\right) - d_{g(T_m)}(x, y)\right| \leq \tfrac{1}{m}, \tag{5.25}$$

$$\left(1+\tfrac{1}{m}\right)^{-1} d_{g(T_m)}(x, y) \leq d_{g_k(T_m)}\left(f_k^m(x), f_k^m(y)\right) \leq \left(1+\tfrac{1}{m}\right) d_{g(T_m)}(x, y) \tag{5.26}$$

for all $x, y \in \mathbb{B}_{g(T_m)}\left(x_0, \frac{m}{2}+\frac{1}{4}\right)$, and

$$f_k^m\left(\mathbb{B}_{g(T_m)}\left(x_0, \tfrac{m}{2}-\tfrac{1}{2}\right)\right) \supset\supset \mathbb{B}_{g_k(T_m)}\left(x_k, \tfrac{m}{2}-\tfrac{3}{4}\right), \tag{5.27}$$

where (5.27) will be required later to ensure the image of the (not yet defined) map φ_i is large enough. We may assume that $K(m)$ is strictly increasing in m, otherwise we can fix $K(1)$, and then inductively replace $K(m)$ for $m = 2, 3, \ldots$ by the maximum of $K(m)$ and $K(m-1)+1$. Pass to a further subsequence in k by selecting the entries $K(1), K(2), K(3), \ldots$, so estimates (5.25), (5.26) and (5.27) now hold for all $k \geq m$.

For each $i \in \mathbb{N}$ we define a map $\varphi_i : \theta_i(M_i) \to \mathbb{B}_{g_i}(x_i, i+1) \subset \mathcal{M}_i$ by $\varphi_i := f_i^i$. In particular, each φ_i is defined throughout $\mathbb{B}_d(x_0, i)$ thanks to (5.24). These are diffeomorphisms onto their images, map x_0 to x_i and satisfy versions of the above estimates. Namely

$$\left|d_{g_i(T_i)}\left(\varphi_i(x), \varphi_i(y)\right) - d_{g(T_i)}(x, y)\right| \leq \tfrac{1}{i}, \tag{5.28}$$

$$\left(1+\tfrac{1}{i}\right)^{-1} d_{g(T_i)}(x, y) \leq d_{g_i(T_i)}\left(\varphi_i(x), \varphi_i(y)\right) \leq \left(1+\tfrac{1}{i}\right) d_{g(T_i)}(x, y) \tag{5.29}$$

for all $x, y \in \mathbb{B}_{g(T_i)}\left(x_0, \frac{i}{2}+\frac{1}{4}\right)$, and

$$\varphi_i\left(\mathbb{B}_{g(T_i)}\left(x_0, \tfrac{i}{2}-\tfrac{1}{2}\right)\right) \supset\supset \mathbb{B}_{g_i(T_i)}\left(x_i, \tfrac{i}{2}-\tfrac{3}{4}\right). \tag{5.30}$$

In what follows we will fix some $i_0 \in \mathbb{N}$ and consider the maps φ_i for $i \geq i_0$ restricted to the ball $\mathbb{B}_d(x_0, i_0)$. With this in mind we record the following observations.

Given a fixed $i_0 \in \mathbb{N}$, restriction (ii) in (5.2) (recalling the definition of η) ensures that $i_0 e^{\alpha_{i_0} T_{i_0}} < i_0 + \frac{1}{2}$. Hence (5.22) and the monotonicity of the sequence T_i imply that for all $i \geq i_0$ we have the inclusion

$$\mathbb{B}_d\left(x_0, \frac{i_0}{2}\right) \subset \mathbb{B}_{g(T_i)}\left(x_0, \frac{i_0}{2} + \frac{1}{4}\right). \tag{5.31}$$

This inclusion implies that for $i \geq i_0$ both (5.28) and (5.29) are valid for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$. Moreover, restriction (i) in (5.2) ensures that $\beta\sqrt{C_i T_i} < \frac{1}{8}$, and so (5.19) and (5.21) (with i here being used as the m there) yields that $\mathbb{B}_d\left(x_0, \frac{i}{2}\right) \supset \mathbb{B}_{g(T_i)}\left(x_0, \frac{i}{2} - \frac{1}{2}\right)$. Hence (5.30) implies $\varphi_i\left(\mathbb{B}_d\left(x_0, \frac{i}{2}\right)\right) \supset \mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i}{2} - \frac{3}{4}\right)$.

Now we restrict φ_i to the ball $\mathbb{B}_d(x_0, i)$. Above we have shown that for any $i \in \mathbb{N}$ we have

$$\varphi_i\left(\mathbb{B}_d\left(x_0, \tfrac{i}{2}\right)\right) \supset\supset \mathbb{B}_{g_i(T_i)}\left(x_i, \tfrac{i}{2} - \tfrac{3}{4}\right). \tag{5.32}$$

Moreover, given $i_0 \in \mathbb{N}$ we have shown that for all $i \geq i_0$ we have

$$\left|d_{g_i(T_i)}\left(\varphi_i(x), \varphi_i(y)\right) - d_{g(T_i)}(x, y)\right| \leq \tfrac{1}{i}, \tag{5.33}$$

$$\left(1 + \tfrac{1}{i}\right)^{-1} d_{g(T_i)}(x, y) \leq d_{g_i(T_i)}\left(\varphi_i(x), \varphi_i(y)\right) \leq \left(1 + \tfrac{1}{i}\right) d_{g(T_i)}(x, y) \tag{5.34}$$

for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$.

We now turn our attention to the properties of these maps restricted to balls of the form $\mathbb{B}_d(x_0, R)$. We first establish the uniform convergence and bi-Hölder regularity claims. For this purpose we take i_0 to be $i_0 := 2\left(\lfloor R \rfloor + 1\right) \in \mathbb{N}$.

For $i \geq i_0$ the *pyramid Ricci flow* $g_i(t)$ is defined on $\mathcal{D}_i$ (recall (5.3)), and in particular (5.4) gives that $\mathrm{Ric}_{g_i(t)} \geq -\alpha_{i_0}$ throughout $\mathbb{B}_{g_i}(x_i, i_0+2) \times [0, T_{i_0}]$ and $|\mathrm{Rm}|_{g_i(t)} \leq \frac{C_{i_0}}{t}$ throughout $\mathbb{B}_{g_i}(x_i, i_0+2) \times (0, T_{i_0}]$. But restriction (i) of (5.2) tells us that $\beta\sqrt{C_{i_0} T_{i_0}} < \frac{1}{8}$, so the shrinking balls lemma A.5 gives that

$$\mathbb{B}_{g_i(s)}\left(x_i, i_0 + 2 - \tfrac{1}{8}\right) \subset \mathbb{B}_{g_i}\left(x_i, i_0 + 2 - \tfrac{1}{8} + \beta\sqrt{C_{i_0} T_{i_0}}\right) \subset\subset \mathbb{B}_{g_i}(x_i, i_0 + 2)$$

for any $s \in [0, T_{i_0}]$. These estimates allow us to apply Lemma A.7 to the flow $g_i(t)$ with $r = \frac{i_0}{2} + 1 - \frac{1}{16}$, $n = 3$, $\alpha = \alpha_{i_0}$, $c_0 = C_{i_0}$ and $T = T_{i_0}$ to quantify the uniform convergence of $d_{g_i(s)}$ to d_{g_i} as $s \downarrow 0$ on $\Omega_i^{i_0} := \bigcap_{0<t\leq T_{i_0}} \mathbb{B}_{g_i(t)}\left(x_i, \frac{i_0}{2} + 1 - \frac{1}{16}\right)$. For any $z, w \in \Omega_i^{i_0}$ and any $0 < s \leq T_{i_0}$ we have

$$d_{g_i}(z, w) - \beta\sqrt{C_{i_0} s} \leq d_{g_i(s)}(z, w) \leq e^{\alpha_{i_0} s} d_{g_i}(z, w) \tag{5.35}$$

and

$$\gamma(i_0, \alpha_0, v_0)\left[d_{g_i}(z, w)\right]^{1+4C_{i_0}} \leq d_{g_i(s)}(z, w), \tag{5.36}$$

where $\gamma > 0$.

If we can estimate the R_0 from (A.14) by $R_0 > \frac{i_0}{2} + \frac{1}{2}$, then (A.14) gives that

$$\mathbb{B}_{g_i(s)}\left(x_i, \frac{i_0}{2} + \frac{1}{2}\right) \subset\subset \Omega_i^{i_0} \tag{5.37}$$

for any $0 \leq s \leq T_{i_0}$, recalling that $g_i(0) = g_i$ on $\mathbb{B}_{g_i}(x_i, i+2)$. This lower bound for R_0 will be true provided $\left(\frac{i_0}{2} + 1 - \frac{1}{16}\right) e^{-\alpha_{i_0} T_{i_0}} - \beta\sqrt{C_{i_0} T_{i_0}} > \frac{i_0}{2} + \frac{1}{2}$. This inequality is itself true if $\frac{7}{2} - 8\beta\sqrt{C_{i_0} T_{i_0}} > \left(4i_0 + 8 - \frac{1}{2}\right)\left(1 - e^{-\alpha_{i_0} T_{i_0}}\right)$. Restriction (i) in (5.2) implies this latter inequality, and so the inclusions of (5.37) are valid.

We are now ready to establish the claimed uniform convergence. To do so we closely follow the argument of Miles Simon and the second author utilised in the proof of Theorem 1.4 in [ST17].

<u>Claim:</u> As $i \to \infty$, we have convergence

$$d_{g_i}(\varphi_i(x), \varphi_i(y)) \to d(x, y) \tag{5.38}$$

uniformly as x, y vary over $\mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$.

Proof: Let $\varepsilon > 0$. We must make sure that for sufficiently large i, depending on ε, we have

$$|d_{g_i}(\varphi_i(x), \varphi_i(y)) - d(x, y)| < \varepsilon \tag{5.39}$$

for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$. By the distance estimates (5.35) and the inclusions of (5.37) there exists a $\tau_1 > 0$, depending only on $\varepsilon, i_0, \alpha_0$ and v_0, such that for all $i \geq i_0$ and any $s \in (0, \min\{\tau_1, T_{i_0}\}]$ we have

$$|d_{g_i}(z, w) - d_{g_i(s)}(z, w)| < \frac{\varepsilon}{3} \tag{5.40}$$

whenever there exists $t \in [0, T_{i_0}]$ such that $z, w \in \mathbb{B}_{g_i(t)}\left(x_i, \frac{i_0}{2} + \frac{1}{2}\right)$.

By the distance estimates (5.22) (for $m = i_0$) there exists a $\tau_2 > 0$, depending only on $\varepsilon, i_0, \alpha_0$ and v_0, such that for any $s \in (0, \min\{\tau_2, T_{i_0}\}]$ we have

$$|d(x, y) - d_{g(s)}(x, y)| < \frac{\varepsilon}{3} \tag{5.41}$$

for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$.

Let $\tau := \min\{\tau_1, \tau_2\} > 0$ (though we could have naturally picked the same τ_1 and τ_2 to begin with) and choose $i_1 \in \mathbb{N}$ such that for all $i \geq i_1$ we have $T_i < \tau$; this is possible since $T_i \downarrow 0$ as $i \to \infty$. Therefore for $i \geq \max\{i_0, i_1\}$ both (5.40) and (5.41) hold for $s = T_i$.

From (5.33), for all $i \geq \max\left\{i_0, \frac{3}{\varepsilon}\right\}$ we have

$$\left|d_{g_i(T_i)}(\varphi_i(x), \varphi_i(y)) - d_{g(T_i)}(x, y)\right| < \frac{1}{i} < \frac{\varepsilon}{3} \tag{5.42}$$

for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$.

Let $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$ and let $i \geq \max\left\{i_0, i_1, \frac{3}{\varepsilon}, 5\right\}$. Appealing to (5.31) gives $x, y \in \mathbb{B}_{g(T_i)}\left(x_0, \frac{i_0}{2} + \frac{1}{4}\right)$, thus (5.42) tells us that $\varphi_i(x), \varphi_i(y) \in \mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i_0}{2} + \frac{1}{4} + \frac{1}{i}\right)$. Since $i \geq 5$ this tells us that (5.40) is valid for $z = \varphi_i(x)$ and $w = \varphi_i(y)$. Combining (5.40), (5.41) and (5.42) establishes (5.39) and completes the proof of the claim. ††

The uniform convergence on $\mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$ immediately gives uniform convergence on $\mathbb{B}_d(x_0, R)$ since $\frac{i_0}{2} \geq R$.

The bi-Hölder estimates for $\varphi_i|_{\mathbb{B}_d(x_0,R)}$ are an easy consequence of those we have already obtained. If $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$ then for $i \geq i_0$ (5.31) yields that $x, y \in \mathbb{B}_{g(T_i)}\left(x_0, \frac{i_0}{2} + \frac{1}{4}\right)$. Then (5.33) gives $\varphi_i(x), \varphi_i(y) \in \mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i_0}{2} + \frac{1}{4} + \frac{1}{i}\right)$. Thus for $i \geq \max\{i_0, 5\}$ we have $\varphi_i(x), \varphi_i(y) \in \mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i_0}{2} + \frac{1}{2}\right)$. Therefore by (5.37) both the estimates (5.35) and (5.36) are valid for $z = \varphi_i(x)$ and $w = \varphi_i(y)$.

As a first consequence, we can deduce that for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$ and all $i \geq \max\{i_0, 5\}$ we have

$$\begin{aligned} d(x, y) &\overset{(5.23)}{\leq} \left[\frac{1}{\kappa_{i_0}(i_0, \alpha_0, v_0)} d_{g(T_i)}(x, y)\right]^{\frac{1}{1+4C_{i_0}}} \\ &\overset{(5.34)}{\leq} \left[\frac{\left(1 + \frac{1}{i}\right)}{\kappa_{i_0}(i_0, \alpha_0, v_0)} d_{g_i(T_i)}(\varphi_i(x), \varphi_i(y))\right]^{\frac{1}{1+4C_{i_0}}} \\ &\overset{(5.35)}{\leq} \left[\frac{\left(1 + \frac{1}{i}\right) e^{\alpha_{i_0} T_i}}{\kappa_{i_0}(i_0, \alpha_0, v_0)} d_{g_i}(\varphi_i(x), \varphi_i(y))\right]^{\frac{1}{1+4C_{i_0}}}. \end{aligned}$$

The monotonicity of the sequence T_i allows us to define $B(i_0, \alpha_0, v_0) := \left[\frac{2e^{\alpha_{i_0} T_{i_0}}}{\kappa_{i_0}(i_0, \alpha_0, v_0)}\right]^{\frac{1}{1+4C_{i_0}}} > 0$ and conclude that for all $i \geq \max\{i_0, 5\}$ we have

$$d(x, y) \leq B(i_0, \alpha_0, v_0) \left[d_{g_i}(\varphi_i(x), \varphi_i(y))\right]^{\frac{1}{1+4C_{i_0}}}. \tag{5.43}$$

Similarly, a second consequence is that for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$ and all $i \geq \max\{i_0, 5\}$ we have

$$\begin{aligned} d_{g_i}(\varphi_i(x), \varphi_i(y)) &\overset{(5.36)}{\leq} \left[\frac{1}{\gamma(i_0, \alpha_0, v_0)} d_{g_i(T_i)}(\varphi_i(x), \varphi_i(y))\right]^{\frac{1}{1+4C_{i_0}}} \\ &\overset{(5.34)}{\leq} \left[\frac{\left(1+\frac{1}{i}\right)}{\gamma(i_0, \alpha_0, v_0)} d_{g(T_i)}(x, y)\right]^{\frac{1}{1+4C_{i_0}}} \\ &\overset{(5.22)}{\leq} \left[\frac{\left(1+\frac{1}{i}\right) e^{\alpha_{i_0} T_i}}{\gamma(i_0, \alpha_0, v_0)} d(x, y)\right]^{\frac{1}{1+4C_{i_0}}}. \end{aligned}$$

The monotonicity of the sequence T_i allows us to define $A(i_0, \alpha_0, v_0) := \left[\frac{2e^{\alpha_{i_0} T_{i_0}}}{\gamma(i_0, \alpha_0, v_0)}\right] > 0$ and conclude that for all $i \geq \max\{i_0, 5\}$ we have

$$d_{g_i}(\varphi_i(x), \varphi_i(y)) \leq A(i_0, \alpha_0, v_0)^{\frac{1}{1+4C_{i_0}}} \left[d(x, y)\right]^{\frac{1}{1+4C_{i_0}}}. \tag{5.44}$$

Combining (5.43) and (5.44) yields that for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$ and all $i \geq \max\{i_0, 5\}$

$$\frac{1}{A(i_0, \alpha_0, v_0)} \left[d_{g_i}(\varphi_i(x), \varphi_i(y))\right]^{1+4C_{i_0}} \leq d(x, y) \leq B(i_0, \alpha_0, v_0) \left[d_{g_i}(\varphi_i(x), \varphi_i(y))\right]^{\frac{1}{1+4C_{i_0}}}. \tag{5.45}$$

This establishes that for all $i \geq \max\{i_0, 5\}$ the restriction of φ_i to $\mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$ is bi-Hölder with Hölder exponent depending only on i_0, α_0 and v_0. Since $\frac{i_0}{2} \geq R$ and i_0 is determined by R, we deduce from (5.45) that, for all $i \geq \max\{i_0, 5\}$, the restriction of φ_i to $\mathbb{B}_d(x_0, R)$ is bi-Hölder with Hölder exponent depending only on α_0, v_0 and R as desired.

Next we turn our attention to the claim that the image of $\mathbb{B}_d(x_0, R)$ under φ_i is eventually arbitrarily close to being the whole of $\mathbb{B}_{g_i}(x_i, R)$. We know $\varphi_i\left(\mathbb{B}_d\left(x_0, \frac{i}{2}\right)\right) \supset \mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i}{2} - \frac{3}{4}\right)$ from (5.32). We claim that $\mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i}{2} - \frac{3}{4}\right) \supset \mathbb{B}_{g_i}\left(x_i, \frac{i}{2} - 1\right)$. To begin with we can appeal to the shrinking balls lemma A.5 to deduce that

$$\mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i}{2} - \frac{3}{4}\right) \subset \mathbb{B}_{g_i}\left(x_i, \frac{i}{2} - \frac{3}{4} + \frac{1}{8}\right) \subset\subset \mathbb{B}_{g_i}(x_i, i+2)$$

since $1 - 8\beta\sqrt{C_i T_i} > 0$. This inclusion implies that $\mathrm{Ric}_{g_i(t)} \geq -\alpha_i$ throughout $\mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i}{2} - \frac{3}{4}\right) \times [0, T_i]$. The expanding balls lemma A.6 now gives our desired inclusion provided $\left(\frac{i}{2} - \frac{3}{4}\right) e^{-\alpha_i T_i} \geq \frac{i}{2} - 1$, that is if $\left(i - \frac{3}{2}\right)\left(1 - e^{-\alpha_i T_i}\right) \leq \frac{1}{2}$. However this is guaranteed to be true by (ii) in (5.2), which imposed the stronger condition $(i+1)(e^{\alpha_i T_i} - 1) \leq \eta$. Therefore for all $i \geq 2(R+1)$ we have that

$$\varphi_i\left(\mathbb{B}_d(x_0, i)\right) \supset \varphi_i\left(\mathbb{B}_d(x_0, \frac{i}{2})\right) \supset \mathbb{B}_{g_i}(x_i, R). \tag{5.46}$$

Now suppose $r \in (0, R)$ as in the theorem. By the uniform convergence claim (5.38), we know that for sufficiently large i, let's say for $i \geq i_2$, we have $|d_{g_i}(\varphi_i(x), \varphi_i(y)) - d(x, y)| < \frac{R-r}{2}$ for all $x, y \in \overline{\mathbb{B}_d(x_0, R)}$, and in particular,

$$d(x_0, y) < d_{g_i}(x_i, \varphi_i(y)) + \frac{R-r}{2} \qquad \text{for all } y \in \overline{\mathbb{B}_d(x_0, R)}. \tag{5.47}$$

We claim that this implies our desired inclusion

$$\mathbb{B}_{g_i}(x_i, r) \subset \varphi_i(\mathbb{B}_d(x_0, R)) \qquad \text{for } i \geq i_2. \tag{5.48}$$

If not, then, keeping in mind (5.46), there exists $z \in \mathbb{B}_{g_i}(x_i, r)$ such that $y := \varphi_i^{-1}(z) \notin \mathbb{B}_d(x_0, R)$. Because we have $d(x_0, y) > R$, we can move a point $\hat{z}$ along a minimising geodesic from x_i to z until the first time that $d(x_0, \varphi_i^{-1}(\hat{z})) = R$, then replace z by $\hat{z}$. This guarantees that additionally we have $d(x_0, y) = R$ and

$y \in \overline{\mathbb{B}_d(x_0, R)}$. But then by (5.47) we have

$$\begin{aligned} R = d(x_0, y) &< d_{g_i}(x_i, z) + \tfrac{R-r}{2} \\ &< r + \tfrac{R-r}{2} \\ &< R, \end{aligned} \tag{5.49}$$

a contradiction. Thus (5.48) holds as desired.

Finally we observe that, for sufficiently large $i \in \mathbb{N}$, slight modifications of the maps φ_i give ε-Gromov-Hausdorff approximations $\mathbb{B}_d(x_0, R)$ to $\mathbb{B}_{g_i}(x_i, R)$. Since $R > 0$ is arbitrary, we deduce that $(\mathcal{M}_i, d_{g_i}, x_i) \to (M, d, x_0)$ in the pointed Gromov-Hausdorff sense as $i \to \infty$. ■

A. Appendix - Results from Simon-Topping papers

Here we collect statements of the various results from [ST16] and [ST17] that we require. We first record a scaled variant of Lemma 4.1 in [ST17], where we have weakened the required Ricci lower bound to $-\gamma$ rather than -1. Lemma 4.1 in [ST17] corresponds to the $\gamma = 1$ case. The same statement is actually given as Lemma 2.1 in [ST16], but with less good dependencies given for the curvature estimates achieved. The following result makes explicit ideas that are implicit in [ST16] and [ST17].

Lemma A.1 (Variant of the local lemma 4.1 in [ST17])**.** *Let $\left(M^3, g(t)\right)$, for $0 \le t \le T$, be a smooth Ricci flow such that for some fixed $x \in M$ we have $\mathbb{B}_{g(t)}(x, 1) \subset\subset M$ for all $0 \le t \le T$, and so that for any $0 < r \le 1$, $\mathrm{Vol}\mathbb{B}_{g(0)}(x, r) \ge v_0 r^3 > 0$ and $\mathrm{Ric}_{g(t)} \ge -\gamma$ on $\mathbb{B}_{g(t)}(x, 1)$ for some $\gamma > 0$ and all $0 \le t \le T$. Then there exist $C_0 = C_0(v_0) \ge 1$ and $S = S(v_0, \gamma) > 0$ such that for all $0 < t \le \min\{T, S\}$ we have both*

$$|\mathrm{Rm}|_{g(t)}(x) \le \frac{C_0}{t} \quad \textit{and} \quad \mathrm{inj}_{g(t)}(x) \ge \sqrt{\frac{t}{C_0}}. \tag{A.1}$$

Proof. Without loss of generality we assume that $\gamma \ge 1$; if $0 < \gamma < 1$ then we could replace γ by 1 since $\mathrm{Ric}_{g(t)} \ge -\gamma$ would give that $\mathrm{Ric}_{g(t)} \ge -1$. Then consider the rescaled flow $g_p(t) := \gamma g\left(\frac{t}{\gamma}\right)$ for times $0 \le t \le \gamma T$. We first observe that

$$\mathrm{Vol}\mathbb{B}_{g_p(0)}(x, 1) = \gamma^{\frac{3}{2}} \mathrm{Vol}\mathbb{B}_{g(0)}\left(x, \frac{1}{\sqrt{\gamma}}\right) \ge \gamma^{\frac{3}{2}} \gamma^{-\frac{3}{2}} v_0 = v_0. \tag{A.2}$$

Moreover, for any $0 \le t \le \gamma T$ we have both

$$\mathbb{B}_{g_p(t)}(x, 1) = \mathbb{B}_{g\left(\frac{t}{\gamma}\right)}\left(x, \frac{1}{\sqrt{\gamma}}\right) \subset\subset M \tag{A.3}$$

and for any $z \in \mathbb{B}_{g_p(t)}(x, 1)$

$$\mathrm{Ric}_{g_p(t)}(z) = \mathrm{Ric}_{\gamma g\left(\frac{t}{\gamma}\right)}(z) \ge -1 \tag{A.4}$$

since $z \in \mathbb{B}_{g\left(\frac{t}{\gamma}\right)}(x, 1)$. Therefore, by combining (A.2), (A.3) and (A.4) we have the hypotheses to be able to apply Lemma 4.1 from [ST17]. Doing so gives us constants $C_0 = C_0(v_0) \ge 1$ and $S_0 = S_0(v_0) > 0$ such that for all $0 < t \le \min\{\gamma T, S_0\}$ we have both

$$|\mathrm{Rm}|_{g_p(t)}(x) \le \frac{C_0}{t} \quad \text{and} \quad \mathrm{inj}_{g_p(t)}(x) \ge \sqrt{\frac{t}{C_0}}. \tag{A.5}$$

Both the estimates in (A.5) are preserved under rescaling back to the original flow $g(t)$. Then, by taking $S := \frac{S_0}{\gamma} > 0$, which does indeed depend only on v_0 and γ, we deduce (A.5) for the flow $g(t)$ itself and for all times $0 < t \le \min\{T, S\}$. ■

Lemma A.2 (Double bootstrap lemma 4.2 in [ST17] or Lemma 9.1 in [ST16]). *Let $\left(M^3, g(t)\right)$ be a smooth Ricci flow, for $0 \leq t \leq T$, such that for some $x \in M$ we have $\mathbb{B}_{g(0)}(x,2) \subset\subset M$, and so that*

- $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ *on* $\mathbb{B}_{g(0)}(x,2) \times (0,T]$ *for some* $c_0 \geq 1$ *and*
- $\mathrm{Ric}_{g(0)} \geq -\delta_0$ *on* $\mathbb{B}_{g(0)}(x,2)$ *for some* $\delta_0 > 0$.

Then there exists $S = S(c_0, \delta_0) > 0$ such that for all $0 < t \leq \min\{T, S\}$ we have

$$\mathrm{Ric}_{g(t)}(x) \geq -100\delta_0 c_0. \tag{A.6}$$

Theorem A.3 (Local existence theorem 1.6 in [ST17]). *Suppose $s_0 \geq 4$. Suppose $\left(M^3, g_0\right)$ is a Riemannian manifold, $x_0 \in M, \mathbb{B}_{g_0}(x_0, s_0) \subset\subset M$ and $\mathrm{Ric}_{g_0} \geq -\alpha_0$ on $\mathbb{B}_{g_0}(x_0, s_0)$ and $\mathrm{Vol}\mathbb{B}_{g_0}(x,1) \geq v_0 > 0$ for all $x \in \mathbb{B}_{g_0}(x_0, s_0 - 1)$. Then there exist constants $T = T(\alpha_0, v_0) > 0, \alpha = \alpha(\alpha_0, v_0) > 0, c_0 = c_0(\alpha_0, v_0) > 0$ and a Ricci flow $g(t)$ defined for $0 \leq t \leq T$ on $\mathbb{B}_{g_0}(x_0, s_0 - 2)$, with $g(0) = g_0$ where defined, such that for all $0 < t \leq T$ we have*

- $\mathrm{Ric}_{g(t)} \geq -\alpha$ *on* $\mathbb{B}_{g_0}(x_0, s_0 - 2)$ *and*
- $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ *on* $\mathbb{B}_{g_0}(x_0, s_0 - 2)$.

The following is a variant of Lemma 2.3 in [ST16]. We replace the required compactness of a time t ball by compactness of a time 0 ball. Moreover, we now obtain volume estimates for unit balls within a later time t ball, rather than just for a single fixed unit ball at later times t. Again this makes explicit ideas implicitly used in both [ST16] and [ST17].

Lemma A.4 (Variant of lower volume control lemma 2.3 in [ST16]). *Suppose that $(M^n, g(t))$ is a smooth Ricci flow over the time interval $t \in [0,T)$ and that for some $R \geq 2$ we have that $\mathbb{B}_{g(0)}(x_0, R) \subset\subset M$ for some $x_0 \in M$. Moreover assume that*

- $\mathrm{Ric}_{g(t)} \geq -K$ *on* $\mathbb{B}_{g(0)}(x_0, R)$, *for some* $K > 0$ *and all* $t \in [0,T)$,
- $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ *on* $\mathbb{B}_{g(0)}(x_0, R)$, *for some* $c_0 > 0$ *and all* $t \in (0,T)$,
- $\mathrm{Vol}\mathbb{B}_{g(0)}(x_0, 1) \geq v_0 > 0$.

Then there exists $\varepsilon_R = \varepsilon_R(v_0, K, R, n) > 0$ and $\hat{T} = \hat{T}(v_0, c_0, K, n, R) > 0$ such that for all $t \in [0,T) \cap \left[0, \hat{T}\right)$ we have $\mathbb{B}_{g(t)}(x_0, R-1) \subset \mathbb{B}_{g(0)}(x_0, R)$, and that for all $x \in \mathbb{B}_{g(t)}(x_0, R-2)$, we have $\mathrm{Vol}\mathbb{B}_{g(t)}(x,1) \geq \varepsilon_R$.

Proof. Lemma A.5 yields a $\beta = \beta(n) \geq 1$ for which $\mathbb{B}_{g(0)}(x_0, R) \supset \mathbb{B}_{g(t)}(x_0, R - \beta\sqrt{c_0 t})$ for all $t \in [0,T)$. Therefore, for $0 \leq t \leq \min\left\{T, \frac{1}{\beta^2 c_0}\right\}$ we have $\mathbb{B}_{g(t)}(x_0, R-1) \subset \mathbb{B}_{g(0)}(x_0, R)$, so $\mathbb{B}_{g(t)}(x_0, R-1) \subset\subset M$ and the assumed curvature estimates hold on $\mathbb{B}_{g(t)}(x_0, R-1)$ for all such times t. Lemma 2.3 in [ST16] with $\gamma = 1$ yields $\varepsilon_0 = \varepsilon_0(v_0, K, n) > 0$ and $\tilde{T} = \tilde{T}(v_0, c_0, K, n) > 0$ such that $\mathrm{Vol}\mathbb{B}_{g(t)}(x_0, 1) \geq \varepsilon_0 > 0$ for all times $0 \leq t \leq \min\left\{T, \frac{1}{\beta^2 c_0}, \tilde{T}\right\}$. Set $\hat{T} := \min\left\{\tilde{T}, \frac{1}{\beta^2 c_0}\right\} > 0$, which depends only on v_0, K, c_0, n and R. Given any $t \in [0, \min\{T, \hat{T}\}]$, the Ricci lower bound $\mathrm{Ric}_{g(t)} \geq -K$ throughout $\mathbb{B}_{g(t)}(x_0, R-1)$ allows us, via Bishop-Gromov, to reduce ε_0 to a constant $\varepsilon_R = \varepsilon_R(v_0, K, n, R) > 0$ such that for all $x \in \mathbb{B}_{g(t)}(x_0, R-2)$, we have $\mathrm{Vol}\mathbb{B}_{g(t)}(x,1) \geq \varepsilon_R > 0$. ■

Lemma A.5 (The shrinking balls lemma; Corollary 3.3 in [ST16]). *Suppose $(M^n, g(t))$ is a Ricci flow for $0 \leq t \leq T$ on an n-dimensional manifold M. Then there exists a $\beta = \beta(n) \geq 1$ such that the following is true. Suppose $x_0 \in M$ and that $\mathbb{B}_{g(0)}(x_0, r) \subset\subset M$ for some $r > 0$, and $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$, or more generally $\mathrm{Ric}_{g(t)} \leq (n-1)\frac{c_0}{t}$, on $\mathbb{B}_{g(0)}(x_0, r) \cap \mathbb{B}_{g(t)}(x_0, r - \beta\sqrt{c_0 t})$ for each $t \in (0,T]$ and some $c_0 > 0$. Then for all $0 \leq t \leq T$*

$$\mathbb{B}_{g(t)}\left(x_0, r - \beta\sqrt{c_0 t}\right) \subset \mathbb{B}_{g(0)}(x_0, r). \tag{A.7}$$

More generally, for $0 \leq s \leq t \leq T$, we have

$$\mathbb{B}_{g(t)}\left(x_0, r - \beta\sqrt{c_0 t}\right) \subset \mathbb{B}_{g(s)}\left(x_0, r - \beta\sqrt{c_0 s}\right). \tag{A.8}$$

Lemma A.6 (The expanding balls lemma; see Lemma 3.1 in [ST16] and Lemma 2.1 in [ST17])**.** *Suppose $K > 0$ and $(M, g(t))$ is a Ricci flow for $t \in [-T, 0]$, $T > 0$, on a manifold M of any dimension. Suppose $x_0 \in M$ and that $\mathbb{B}_{g(0)}(x_0, R) \subset\subset M$ and $\mathrm{Ric}_{g(t)} \geq -K$ on $\mathbb{B}_{g(0)}(x_0, R) \cap \mathbb{B}_{g(t)}\left(x_0, Re^{Kt}\right) \subset \mathbb{B}_{g(t)}(x_0, R)$ for each $t \in [-T, 0]$. Then for all $t \in [-T, 0]$*

$$\mathbb{B}_{g(0)}(x_0, R) \supset \mathbb{B}_{g(t)}\left(x_0, Re^{Kt}\right). \tag{A.9}$$

Lemma A.7 (Bi-Hölder Distance Estimates; Lemma 3.1 in [ST17])**.** *Suppose $(M^n, g(t))$ is a Ricci flow for $t \in (0, T]$, not necessarily complete, such that for some $x_0 \in M$, and all $t \in (0, T]$, we have $\mathbb{B}_{g(t)}(x_0, 2r) \subset\subset M$. Suppose further that for some $c_0, \alpha > 0$, and for each $t \in (0, T]$, we have*

$$-\alpha \leq \mathrm{Ric}_{g(t)} \leq \frac{(n-1)c_0}{t} \tag{A.10}$$

throughout $\mathbb{B}_{g(t)}(x_0, 2r)$. Define $\Omega_T := \bigcap_{0<t\leq T} \mathbb{B}_{g(t)}(x_0, r)$. Then for any $x, y \in \Omega_T$ the distance $d_{g(t)}(x, y)$ is unambigious for all $t \in (0, T]$ and must be realised by a minimising geodesic lying within $\mathbb{B}_{g(t)}(x_0, 2r)$. Then, for any $0 < t_1 \leq t_2 \leq T$, we have

$$d_{g(t_1)}(x, y) - \beta\sqrt{c_0}\left(\sqrt{t_2} - \sqrt{t_1}\right) \leq d_{g(t_2)}(x, y) \leq e^{\alpha(t_2 - t_1)} d_{g(t_1)}(x, y), \tag{A.11}$$

where $\beta = \beta(n) > 0$. In particular, $d_{g(t)}$ converges uniformly to a distance metric d_0 on Ω_T as $t \downarrow 0$, and

$$d_0(x, y) - \beta\sqrt{c_0 t} \leq d_{g(t)}(x, y) \leq e^{\alpha t} d_0(x, y), \tag{A.12}$$

for all $t \in (0, T]$. Moreover, there exists $\gamma > 0$, depending only on n, c_0 and upper bounds for T and r, such that

$$d_{g(t)}(x, y) \geq \gamma \left[d_0(x, y)\right]^{1+2(n-1)c_0} \tag{A.13}$$

for all $t \in (0, T]$. Finally, for all $t \in (0, T]$ and $R < R_0 := re^{-\alpha T} - \beta\sqrt{c_0 T} < r$, we have

$$\mathbb{B}_{g(t)}(x_0, R_0) \subset \Omega_T \quad \text{and} \quad \mathbb{B}_{d_0}(x_0, R) \subset\subset \mathcal{O} \tag{A.14}$$

where $\mathcal{O}$ is the component of Interior (Ω_T) containing x_0.

Lemma A.8 (Distance function convergence under local convergence; Lemma 6.1 in [ST17])**.** *Suppose (M_i, g_i) is a sequence of smooth $n-$dimensional Riemannian manifolds, possibly incomplete, and $x_i \in M_i$ for each i. Suppose there exist a smooth, possibly incomplete $n-$dimensional Riemannian manifold $(\mathcal{N}, \hat{g})$ and a point $x_0 \in \mathcal{N}$ with $\mathbb{B}_{\hat{g}}(x_0, 2r) \subset\subset \mathcal{N}$ for some $r > 0$, and a sequence of smooth maps $\varphi_i : \mathcal{N} \to M_i$, diffeomorphic onto their images, with $\varphi_i(x_0) = x_i$, such that $\varphi_i^* g_i \to \hat{g}$ smoothly on $\overline{\mathbb{B}_{\hat{g}}(x_0, 2r)}$. Then*

1. *If $0 < a \leq 2r$, and $a < b$, then $\varphi_i\left(\mathbb{B}_{\hat{g}}(x_0, a)\right) \subset \mathbb{B}_{g_i}(x_i, b)$ for sufficiently large i.*
2. *If $0 < a < b \leq 2r$, then $\mathbb{B}_{g_i}(x_i, a) \subset\subset \varphi_i\left(\mathbb{B}_{\hat{g}}(x_0, b)\right)$ for sufficiently large i.*
3. *For every $s \in (0, r)$, we have*
$$d_{g_i}\left(\varphi_i(x), \varphi_i(y)\right) \to d_{\hat{g}}(x, y)$$
as $i \to \infty$, uniformly for x and y in $\mathbb{B}_{\hat{g}}(x_0, s)$.

B. Appendix - Shi's Estimates and Compactness

A useful variant of Shi's derivative estimates, that is implicit in Section 5 of [ST17], is the following result.

Lemma B.1 (Local Shi decay)**.** *Let $(M^n, g(t))$ be a smooth Ricci flow for $t \in [0,T]$, and assume for some $R > 0$ and $x_0 \in M$ that $\mathbb{B}_{g(0)}(x_0,R) \subset\subset M$. Moreover, suppose that for all $0 < t \leq T$ we have $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ throughout $\mathbb{B}_{g(0)}(x_0,R)$ for some $c_0 > 0$. Then for any $\varepsilon \in (0,R)$, there exists $\hat{T} = \hat{T}(c_0,n,\varepsilon) > 0$ and, for $l \in \mathbb{N}$, there exists $C_l = C_l(l,c_0,n,\varepsilon) > 0$ such that if $0 < \tau \leq \min\{T,\hat{T}\}$ then we have $\mathbb{B}_{g(\tau)}(x_0,R-\varepsilon) \subset \mathbb{B}_{g(0)}(x_0,R)$ and*

$$\left|\nabla^l \mathrm{Rm}\right|_{g(t)} \leq \frac{C_l}{t^{1+\frac{l}{2}}} \tag{B.1}$$

throughout $\mathbb{B}_{g(\tau)}(x_0,R-\varepsilon) \times (0,\tau]$.

Proof of Lemma B.1. Let $\beta = \beta(n) \geq 1$ be the constant arising in the shrinking balls lemma A.5. Define $\hat{T} := \frac{\varepsilon^2}{9\beta^2 c_0} > 0$ and let $0 < \tau \leq \min\{T,\hat{T}\}$. The c_0/t curvature bound means that from Lemma A.5 we deduce that $\mathbb{B}_{g(\tau)}(x_0,R-\varepsilon) \subset \mathbb{B}_{g(0)}\left(x_0,R-\frac{2\varepsilon}{3}\right) \subset \mathbb{B}_{g(0)}(x_0,R)$.

Let $x \in \mathbb{B}_{g(\tau)}(x_0,R-\varepsilon)$, $t \in (0,\tau]$, and consider $\mathbb{B}_{g\left(\frac{t}{2}\right)}\left(x,\frac{\varepsilon}{3}\right)$. We have $x \in \mathbb{B}_{g(0)}\left(x_0,R-\frac{2\varepsilon}{3}\right)$, as we have just shown, hence via the shrinking balls lemma A.5 we have $\mathbb{B}_{g\left(\frac{t}{2}\right)}\left(x,\frac{\varepsilon}{3}\right) \subset \mathbb{B}_{g(0)}(x,\frac{2\varepsilon}{3}) \subset \mathbb{B}_{g(0)}(x_0,R) \subset\subset M$.

Thus $\overline{\mathbb{B}_{g\left(\frac{t}{2}\right)}\left(x,\frac{\varepsilon}{4}\right)} \subset\subset \mathbb{B}_{g\left(\frac{t}{2}\right)}\left(x,\frac{\varepsilon}{3}\right)$ and $|\mathrm{Rm}|_{g(s)} \leq \frac{2c_0}{t}$ throughout $\mathbb{B}_{g\left(\frac{t}{2}\right)}\left(x,\frac{\varepsilon}{3}\right)$ for all $s \in \left[\frac{t}{2},t\right]$. We can apply Theorem 14.14 in [Cho07] to the Ricci flow $s \mapsto g(s-t/2)$ for $s \in [t/2,t]$, with $r := \frac{\varepsilon}{4}$, $K := \frac{2c_0}{t}$ and $\alpha := c_0$ to deduce that for a constant $C = C(l,c_0,n,\varepsilon) > 0$ we have

$$\left|\nabla^l \mathrm{Rm}\right|_{g(s)}(x) \leq \frac{2c_0 C}{(s-t/2)^{\frac{l}{2}} t} \tag{B.2}$$

for all $s \in \left(\frac{t}{2},t\right]$. Here we have used the observation in [ST17] that the constant $C(\alpha,K,r,m,n)$ arising in Theorem 14.14 in [Cho07] can be written in the form $C(\alpha,r,m,n)K$. Restricting to $s = t$ then gives (B.1) as required. ■

The following lemma localises the well-known Hamilton-Cheeger-Gromov compactness lemma. The proof carries over more or less verbatim, and details can be found in [McL18].

Lemma B.2 (Local compactness)**.** *Suppose (M_i,g_i) is a sequence of smooth n-dimensional Riemannian manifolds, not necessarily complete, and that $x_i \in M_i$ for each i. Suppose that, for some $R,v > 0$, we have $\mathbb{B}_{g_i}(x_i,R) \subset\subset M_i$ and $\mathrm{Vol}\mathbb{B}_{g_i}(x_i,R) \geq v$ for each i, and that (for each l) we have $|\nabla^l \mathrm{Rm}|_{g_i} \leq C$ throughout $\mathbb{B}_{g_i}(x_i,R)$, where C is independent of i, but allowed to depend on l.*

Then after passing to an appropriate subsequence in i, there exist a smooth, typically incomplete n-dimensional Riemannian manifold $(\mathcal{N},g_\infty)$, a point $x_0 \in \mathcal{N}$ with $\mathbb{B}_{g_\infty}(x_0,r) \subset\subset \mathcal{N}$ for every $r \in (0,R)$, and a sequence of smooth maps $\varphi_i : \mathbb{B}_{g_\infty}(x_0,\frac{i}{i+1}R) \to M_i$, diffeomorphic onto their images and mapping x_0 to x_i, such that $\varphi_i^* g_i \to g_\infty$ smoothly locally on $\mathbb{B}_{g_\infty}(x_0,R)$.

The local version of Hamilton-Cheeger-Gromov compactness for flows, which is already implicit in [ST17], is the following.

Lemma B.3 (Local Ricci flow compactness)**.** *Suppose $(M_i,g_i(t))$ is a sequence of smooth n-dimensional Ricci flows, not necessarily complete, each defined for $t \in [0,T]$, and that $x_i \in M_i$ for each i. Suppose that, for some $R > 0$, we have $\mathbb{B}_{g_i(0)}(x_i,R) \subset\subset M_i$ for each i, that $\mathrm{Vol}\mathbb{B}_{g_i(0)}(x_i,R) \geq v > 0$, and throughout $\mathbb{B}_{g_i(0)}(x_i,R)$ that $\mathrm{Ric}_{g_i(t)} \geq -\alpha < 0$ for all $t \in [0,T]$ and $|\mathrm{Rm}|_{g_i(t)} \leq c_0/t$ for all $t \in (0,T]$, for positive constants v, α and c_0 that are independent of i.*

Then for all $\eta \in (0,R/2)$, there exists $S > 0$ depending only on R, n, v, α, c_0 and η such that after passing to an appropriate subsequence in i, there exist a smooth n-dimensional manifold $\mathcal{N}$, a point $x_0 \in \mathcal{N}$ and a Ricci flow $g(t)$ on $\mathcal{N}$ for $t \in (0,\tau]$, where $\tau := \min\{T,S\}$, with the following properties.

First, $\mathbb{B}_{g(t)}(x_0,R-\eta) \subset\subset \mathcal{N}$ for all $t \in (0,\tau]$. Second, if we define Ω to be the connected component of the interior of

$$\bigcap_{s\in(0,\tau]} \mathbb{B}_{g(s)}(x_0,R-\eta) \subset \mathcal{N}$$

containing x_0, then for all $t \in (0,\tau]$ we have $\mathbb{B}_{g(t)}(x_0, R-2\eta) \subset \Omega$. Third, there exists a sequence of smooth maps $\varphi_i : \Omega \to \mathbb{B}_{g_i(0)}(x_i, R) \subset M_i$, diffeomorphic onto their images and mapping x_0 to x_i, such that $\varphi_i^* g_i(t) \to g(t)$ smoothly uniformly on $\Omega \times [\delta, \tau]$ for every $\delta \in (0,\tau)$.

Finally, throughout Ω we have $\mathrm{Ric}_{g(t)} \geq -\alpha$ and $|\mathrm{Rm}|_{g(t)} \leq c_0/t$ for all $t \in (0,\tau]$.

Proof. We begin by applying the *local Shi decay lemma* B.1 to each $g_i(t)$, with $\varepsilon := \eta/3$. This ensures that there exists $S > 0$ depending only on n, c_0 and η such that for $0 < t \leq \min\{T, S\}$ we have $\mathbb{B}_{g_i(t)}(x_i, R-\varepsilon) \subset \mathbb{B}_{g_i(0)}(x_i, R)$ and that for $0 < t \leq \tau \leq \min\{T, S\}$ we have

$$\left|\nabla^l \mathrm{Rm}\right|_{g_i(t)} \leq \frac{C_l}{t^{1+\frac{l}{2}}} \tag{B.3}$$

throughout $\mathbb{B}_{g_i(\tau)}(x_i, R-\varepsilon)$, where C_l depends on l, c_0, n and η. Next with a view to later applying the expanding and shrinking balls lemmas, we reduce $S > 0$ further, depending now also on α, so that

$$R(1-e^{-\alpha S}) < \varepsilon \qquad \text{and} \qquad S \leq \frac{\eta^2}{\beta^2 c_0}. \tag{B.4}$$

where $\beta = \beta(n) \geq 1$ comes from Lemma A.5. A final reduction of $S > 0$, depending now also on v, ensures that by Lemma 2.3 of [ST16] (which is in a more appropriate form than the variant Lemma A.4) and Bishop-Gromov, we have $\mathrm{Vol}\mathbb{B}_{g_i(s)}(x_i, R-\varepsilon) \geq \tilde{v} > 0$ for all $s \in [0, \min\{T, S\}]$, where $\tilde{v}$ depends only on v, α, R and n.

At this point we fix S, and the corresponding $\tau := \min\{T, S\}$, and apply the local compactness lemma B.2 to the sequence $g_i(\tau)$ with R there equal to $R-\varepsilon$ here. The conclusion is that after passing to an appropriate subsequence in i, there exist a smooth n-dimensional Riemannian manifold $(\mathcal{N}, g_\infty)$, a point $x_0 \in \mathcal{N}$ with $\mathbb{B}_{g_\infty}(x_0, r) \subset\subset \mathcal{N}$ for every $r \in (0, R-\varepsilon)$, and a sequence of smooth maps $\varphi_i : \mathbb{B}_{g_\infty}(x_0, \frac{i}{i+1}(R-\varepsilon)) \to M_i$, diffeomorphic onto their images and mapping x_0 to x_i, such that $\varphi_i^* g_i(\tau) \to g_\infty$ smoothly locally on $\mathbb{B}_{g_\infty}(x_0, R-\varepsilon)$.

By Part 1 of Lemma A.8, applied with g_i and $\hat{g}$ there equal to $g_i(\tau)$ and g_∞ here, respectively, and with $a = 2r$ and b there equal to $R-2\varepsilon$ and $R-\varepsilon$ here, respectively, we find that after dropping finitely many terms, we have

$$\varphi_i\left(\mathbb{B}_{g_\infty}(x_0, R-2\varepsilon)\right) \subset \mathbb{B}_{g_i(\tau)}(x_i, R-\varepsilon) \subset \mathbb{B}_{g_i(0)}(x_i, R)$$

for every i (where the second inclusion here was established at the beginning of the proof).

By Hamilton's original argument [Ham95] we can pass to a further subsequence and find a Ricci flow $g(t)$ on $\mathbb{B}_{g_\infty}(x_0, R-\varepsilon)$, $t \in (0,\tau]$ so that $g(\tau) = g_\infty$ on $\mathbb{B}_{g_\infty}(x_0, R-\varepsilon)$ and so that $\varphi_i^* g_i(t) \to g_\infty$ smoothly locally on $\mathbb{B}_{g_\infty}(x_0, R-\varepsilon) \times (0,\tau]$ as $i \to \infty$. In particular, we can pass our curvature hypotheses to the limit to obtain that $\mathrm{Ric}_{g(t)} \geq -\alpha$ and $|\mathrm{Rm}|_{g(t)} \leq c_0/t$, for all $t \in (0,\tau]$ and throughout $\mathbb{B}_{g_\infty}(x_0, R-2\varepsilon)$.

Next, our constraint (B.4) implies that $(R-2\varepsilon)(1-e^{-\alpha S}) < \varepsilon$, i.e. that $R - 3\varepsilon < (R-2\varepsilon)e^{-\alpha S}$, and hence by the expanding balls lemma A.6, we know that for all $t \in (0,\tau]$ we have

$$\mathcal{N} \supset\supset \mathbb{B}_{g(\tau)}(x_0, R-2\varepsilon) \supset \mathbb{B}_{g(t)}(x_0, (R-2\varepsilon)e^{\alpha(t-\tau)}) \supset \mathbb{B}_{g(t)}(x_0, (R-2\varepsilon)e^{-\alpha S}) \supset \mathbb{B}_{g(t)}(x_0, R-3\varepsilon),$$

and hence (recalling that $\varepsilon = \eta/3$) we have $\mathbb{B}_{g(t)}(x_0, R-\eta) \subset \mathbb{B}_{g(\tau)}(x_0, R-2\varepsilon) \subset\subset \mathcal{N}$ as required. One consequence is that our curvature bounds hold within each $\mathbb{B}_{g(t)}(x_0, R-\eta)$, for all $t \in (0,\tau]$. Moreover, if we reduce $\mathcal{N}$ to $\mathbb{B}_{g_\infty}(x_0, R-\varepsilon) \subset \mathcal{N}$, then we still have $\mathbb{B}_{g(t)}(x_0, R-\eta) \subset\subset \mathcal{N}$ for all $t \in (0,\tau]$, and now the Ricci flow is defined throughout $\mathcal{N}$.

It remains to show that $\mathbb{B}_{g(t)}(x_0, R-2\eta) \subset \Omega$, and for that it suffices to prove that

$$\mathbb{B}_{g(t)}(x_0, R-2\eta) \subset \mathbb{B}_{g(s)}(x_0, R-\eta) \qquad \text{for each } s, t \in (0,\tau]. \tag{B.5}$$

In the case $s < t$ this follows from the shrinking balls lemma A.5 applied with time zero there equal to time s here, and r there equal to $R-\eta$ here. That lemma tells us that $\mathbb{B}_{g(t)}(x_0, R-\eta-\beta\sqrt{c_0(t-s)}) \subset \mathbb{B}_{g(s)}(x_0, R-\eta)$, by (A.7) not (A.8), and because $\beta\sqrt{c_0(t-s)} \leq \beta\sqrt{c_0 S} \leq \eta$, by (B.4), this implies (B.5).

Meanwhile, in the case $s > t$, (B.5) follows from the expanding balls lemma A.6 applied with time zero

there equal to time s here, and R there equal to $R-\eta$ here. That lemma tells us that $\mathbb{B}_{g(s)}(x_0, R-\eta) \supset \mathbb{B}_{g(t)}(x_0, (R-\eta)e^{\alpha(t-s)})$, and so we will have proved (B.5) if we can prove that $(R-\eta)e^{\alpha(t-s)} \geq R-2\eta$, or more generally that $(R-\eta)e^{-\alpha S} \geq R-2\eta$, which is equivalent to $(R-\eta)(1-e^{-\alpha S}) \leq \eta$. This is turn follows from the first part of (B.4). ■

C. Appendix - Smooth Manifold Construction

Theorem C.1 (Smooth Manifold Construction)**.** *Assume that for each $i \in \mathbb{N}$ we have a smooth n-manifold M_i and a point $x_i \in M_i$, and that each M_i is contained in the next in the sense that for each $i \in \mathbb{N}$ there exists a smooth map $\psi_i : M_i \to M_{i+1}$, mapping x_i to x_{i+1} and diffeomorphic onto its image. Then there exists a smooth n-manifold M, containing a point x_0, and there exist smooth maps $\theta_i : M_i \to M$, all mapping x_i to x_0, diffeomorphic onto their image, and satisfying that $\theta_i(M_i) \subset \theta_{i+1}(M_{i+1})$, and further that*

$$M = \bigcup_{i=1}^{\infty} \theta_i(M_i). \tag{C.1}$$

Moreover, we have that

$$\psi_i = \theta_{i+1}^{-1} \circ \theta_i : M_i \to M_{i+1}. \tag{C.2}$$

Proof. Define $M := \bigsqcup_{i=1}^{\infty} M_i / \sim$, equipped with the quotient topology, where $\sim$ is the equivalence relation generated by identifying points x and y if $y = \psi_i(x)$ for some $i \in \mathbb{N}$. Let $x_0 \in M$ be the equivalence class generated by the points $x_i \in M_i$. For each $i \in \mathbb{N}$ define $\theta_i : M_i \to M$ to be the map sending a point x to the equivalence class $[x]$. Thus $\theta_i(x_i) = x_0$, and θ_i is a homeomorphism onto its image, while $M = \bigcup_{i=1}^{\infty} \theta_i(M_i)$ which is (C.1). Moreover, for each $x \in M_i$ we have $\theta_i(x) = [x_i] = [\psi_i(x)] = \theta_{i+1}(\psi_i(x))$, which gives (C.2). Since ψ_i is a diffeomorphism onto its image, (C.2) allows us to combine the smooth atlases for each M_i into a smooth atlas for M by composing with the maps θ_i^{-1}. Hence we simultaneously establish both that M is a smooth n-manifold, and that each θ_i is a diffeomorphism onto its image as claimed. ■

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