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Contents

Applications of Hamilton's Compactness Theorem for Ricci flow

Peter Topping

IAS/Park City Mathematics Series
Volume XX, XXXX

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Overview

In these lectures, I will try to give an introduction to two separate aspects of Ricci flow, namely Hamilton's compactness theorem and the very neat theory of Ricci flow in 2D. The target audience consists of graduate students with some background in differential geometry and PDE theory.

Hamilton's compactness theorem is an absolutely fundamental tool in the modern theory of Ricci flow. I will spend the early part of the course explaining what this result says – roughly that given an appropriate sequence of Ricci flows, one can pass to a subsequence and get smooth convergence to a limit Ricci flow. In order to make sense of that, we will first look at the details of what it means for a sequence of Ricci flows, or simply of Riemannian manifolds, to converge in the Cheeger-Gromov sense. We will not assume any prior knowledge of this notion, although it will be almost essential to have some basic prior knowledge of Riemannian geometry, including the basic idea of the Riemannian curvature tensor.

I will then go on to illustrate the most basic application of the compactness theorem, as envisaged by Hamilton and realised fully by Perelman, by blowing up a singularity to obtain a limit ancient Ricci flow modelling the singularity. To do this we will take a brief detour to mention Perelman's 'no local collapsing theorem".

The rough idea of how this looks in practice in 3D will be explained. But to fully illustrate this application, and also some other key ideas – particularly Perelman's notion of κ-solutions and their most basic theory – we will focus on the 2D situation, and give a Perelman-style proof of the beautiful results of Hamilton and Chow explaining what Ricci flow does to an arbitrary compact surface.

From there we will consider the problem of starting a Ricci flow with a completely general metric, typically on a noncompact surface. This raises some interesting well-posedness issues as one struggles to find the right way of posing the problem to obtain both existence and uniqueness of solutions. We will resolve this problem with the notion of *instantaneously complete* Ricci flows.

The lectures will then complete a full circle by applying this 2D theory in order to understand better Hamilton's compactness theorem and the various extensions that one needs (or desires) to take the theory further. More precisely, we will use the 2D theory (including some additional examples of 'contracting cusp' Ricci flows) to construct some visual examples which violate some extended forms of Hamilton's result that were previously widely believed and used.

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK.
E-mail address: p.m.topping@warwick.ac.uk

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These lecture notes were written for a mini-course at the Park City Mathematics Institue, Utah (July 2013) based on lectures I gave the year before at ICTP, Trieste (June 2012) and at the 6th KIAS Winter School in South Korea (Feb. 2012). I would like to thank Hugh Bray, Greg Galloway, Rafe Mazzeo, Natasa Sesum, Claudio Arezzo and Jaigyoung Choe for the invitations to these events. Thanks to Neil Course and Gregor Giesen for preparing some of the pictures, which were originally used for [28] and [11], and to Michael Coffey for helping with the problem sheets that accompany the lectures.

Background reading

Hopefully you already know some Riemannian geometry. If not, you will need to read some of the basics quickly – try Fran Burstall's three lecture introduction [1] and the references therein.

You will also need some basic intuition about parabolic equations, or simply just for the linear heat equation, which you will get from many basic introductions to PDE theory.

For the fundamentals of Ricci flow, try my earlier lecture notes [28]. In particular, we will follow Chapter 7 of those, and some other parts here and there.

For the remaining parts of Perelman's theory that we will require, you will be fine with Perelman's first paper [25], although if you end up wanting to see that material with more detail, then there are some very useful resources such as the notes of Kleiner and Lott [23].

For the two-dimensional theory of Ricci flow from the viewpoint of these lectures, one can see particularly [13, 31, 15, 33], and we will also draw on elements of [12] and [14] in order to explain some subtleties of Hamilton's compactness theorem [32].

LECTURE 1

Ricci flow basics – existence and singularities

1.1. Initial PDE remarks

A simple but (as it turns out) very natural nonlinear version of the classical linear heat equation is the *logarithmic fast diffusion equation*

$$\frac{\partial u}{\partial t} = e^{-2u}\Delta u, \tag{1.1}$$

where $u : \Omega\times[0,T] \to \mathbb{R}$, for some domain $\Omega \subset \mathbb{R}^2$. This is different from the normal heat equation because the speed of diffusion is scaled by e^{-2u}, which depends on u. Historically this equation has been used extensively for modelling the evolution of the thickness of a thin colloidal film, but it also turns out to have a very geometric flavour.

This equation, being parabolic, is well-posed (in particular one has existence and uniqueness of solutions) if we specify the initial function $u(\cdot, 0)$, and also u on $\partial\Omega \times [0, T]$, just as for the ordinary heat equation. There is also an extensive literature dealing with the case $\Omega = \mathbb{R}^2$ (see, for example, [7] as a starting point) although one has extreme nonuniqueness in this case in general in the absence of any growth conditions for u as $x \to \infty$.

Given a solution u to (1.1), we can define a t-dependent Riemannian metric $g(t) = e^{2u}(dx^2 + dy^2)$, and calculate

$$\frac{\partial g}{\partial t} = 2\frac{\partial u}{\partial t}g(t) = 2e^{-2u}\Delta u\, g(t) = -2Kg(t),$$

where $K = -e^{-2u}\Delta u$ is the *Gauss curvature* of $g(t)$. In particular, we have

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}_{g(t)},$$

which is the *Ricci flow equation* of Hamilton [18].

This simple observation suggests that the right way of posing (1.1) is to take a Riemannian surface $(\mathcal{M}^2, g_0)$ as initial data, and try to find a t-dependent family $g(t)$, $t \in [0, T]$, such that $g(0) = g_0$ and

$$\frac{\partial g}{\partial t} = -2Kg(t),$$

(which forces the flow $g(t)$ to retain the same conformal structure). With respect to any local isothermal coordinate chart, such a solution induces a solution to (1.1), but different charts give different solutions and it does not make sense to discuss regions of slow diffusion (u large) or fast diffusion ($u \ll -1$). Taking this geometric viewpoint puts a collection of powerful techniques at our disposal to study (1.1), and also leads us to the right well-posedness class in full generality, as we will see in Lecture 5.

1.2. Basic Ricci flow theory

In general dimension $n \in \mathbb{N}$, a Ricci flow is a one-parameter family $g(t)$ of Riemannian metrics on an n-dimensional manifold $\mathcal{M}$, satisfying the nonlinear PDE

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}_{g(t)}.$$

Aside from trivial examples such as the real line (which has no curvature and cannot move) the simplest example is possibly the round shrinking sphere $(S^n, g^n(t))$ where $g^n(t) = (1-2(n-1)t)g_0^n$, and g_0^n is the metric of the round sphere of constant sectional curvature 1. Taking products of Ricci flows yields further Ricci flows, and thus another example is the shrinking cylinder $\mathbb{R} \times (S^2, g^2(t))$, or for a similar, compact example one could take $S^1 \times (S^2, g^2(t))$. More generally, we have:

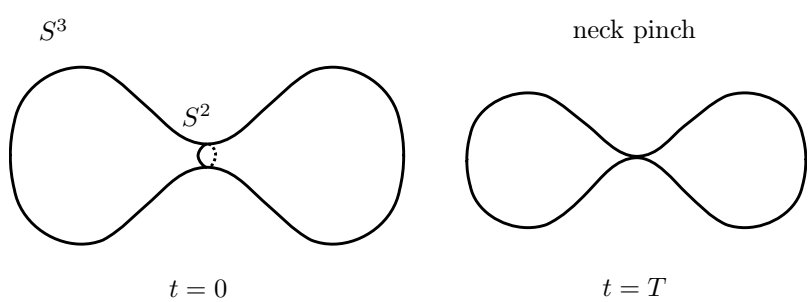
Theorem 1.1 (Hamilton [18], Shi [27], Chen-Zhu [3], Kotschwar [24])**.** *Given a complete, bounded curvature Riemannian manifold $(\mathcal{M}^n, g_0)$, there exists $T \in (0, \infty]$ and a unique complete, bounded curvature Ricci flow $g(t)$ for $t \in [0, T)$ on $\mathcal{M}^n$ such that $g(0) = g_0$.*

To clarify, such a Ricci flow is said to be *complete* if $(\mathcal{M}^n, g(t))$ is complete for all $t \in [0, T)$, and *here* the Ricci flow is said to have *bounded curvature* if for all $t_0 \in [0, T)$, we have[1]

$$\sup_{\mathcal{M}^n \times [0, t_0]} |\mathrm{Rm}| < \infty.$$

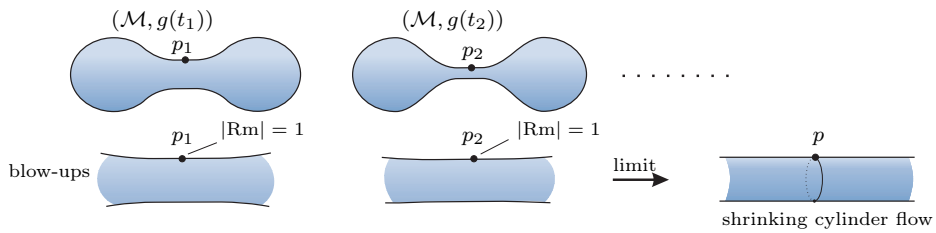
In particular we allow the curvature to blow up as $t \uparrow T$, which is possibly a little unconventional. Indeed, in Theorem 1.1 we may assume that either $T = \infty$ or $\sup_{\mathcal{M}^n} |\mathrm{Rm}|(\cdot, t) \to \infty$ as $t \uparrow T$, as was explained by Hamilton (see for example [28, §5.3]).

Clearly the shrinking sphere flow given above is an example where $T < \infty$ and the curvature blows up, but this can occur also without the whole manifold disappearing. For example, if $\mathcal{M} = S^3$, we can have a neck pinch singularity:



We want to analyse such a singularity by blowing up, i.e. rescaling the flow parabolically (see below) and trying to extract a Ricci flow that models the singularity (i.e. captures some of its information). For the neck pinch example above, the rescaling procedure should produce a shrinking cylinder Ricci flow in the limit:

[1] Rm denotes the full Riemannian curvature tensor. This condition is equivalent to upper and lower bounds (depending on t_0) on all sectional curvatures.



The rescaling alluded to above works by scaling space and time appropriately: If $g(t)$ is a Ricci flow for $t \in [a, b]$ and $\lambda > 0$, then

$$g_\lambda(t) := \lambda g(t/\lambda)$$

is a new Ricci flow for $t \in [\lambda a, \lambda b]$ as described in [28, §1.2.3]. Under this rescaling, lengths increase by a factor of $\lambda^{\frac{1}{2}}$, and sectional curvatures are scaled by a factor λ^{-1}. Therefore, we can scale up Ricci flows with large curvatures to Ricci flows with curvatures of order 1 by suitably choosing λ.

In order to make this procedure work, we will at least need a notion of convergence of Ricci flows, and before that a notion of convergence of Riemannian manifolds, and we address these points in the next lecture.

LECTURE 2

Cheeger-Gromov convergence and Hamilton's compactness theorem

The basic reference for this lecture is [28, Sections 7.1 and 7.2], although we need some slight modifications here.

Motivated by the discussion in the previous lecture, we need to find a good notion of convergence for a sequence of flows so it will make sense to pass to a limit of a sequence of scaled-up Ricci flows. We will then need Hamilton's compactness theorem to tell us that a reasonable sequence of rescaled flows will in fact converge (after passing to a subsequence). Before that we need to consider the slightly simpler problem of finding a good notion of convergence of a sequence of Riemannian manifolds.

2.1. Convergence and compactness of manifolds

It is reasonable to suggest that a sequence $\{g_i\}$ of Riemannian metrics on a manifold $\mathcal{M}$ should converge to a metric h on $\mathcal{M}$ when $g_i \to h$ as tensors. However, we would like a notion of convergence of Riemannian manifolds that is diffeomorphism invariant: it should not be affected if we modify each metric g_i by an i-dependent diffeomorphism. Once we have asked for such invariance, it is necessary to discuss convergence with respect to a point of reference on each manifold, for reasons that we will see in a moment.

Definition 2.1 (Smooth, pointed "Cheeger-Gromov" convergence of manifolds)**.** A sequence $(\mathcal{M}_i, g_i, p_i)$ of (smooth) complete, *pointed* Riemannian manifolds (that is, Riemannian manifolds $(\mathcal{M}_i, g_i)$ and points $p_i \in \mathcal{M}_i$) is said to *converge* (smoothly) to the smooth pointed Riemannian manifold $(\mathcal{N}, h, q)$ as $i \to \infty$ if there exist

(i) a sequence of domains $\Omega_i \subset\subset \mathcal{N}$, exhausting $\mathcal{N}$ (that is, so that any compact set $K \subset \mathcal{N}$ satisfies $K \subset \Omega_i$ for sufficiently large i) with $q \in \Omega_i$ for each i, and

(ii) a sequence of smooth maps $\phi_i : \Omega_i \to \mathcal{M}_i$ that are diffeomorphic onto their image and satisfy $\phi_i(q) = p_i$ for all i,

such that

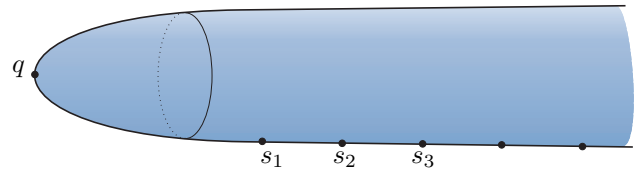
$$\phi_i^* g_i \to h$$

smoothly locally as $i \to \infty$.

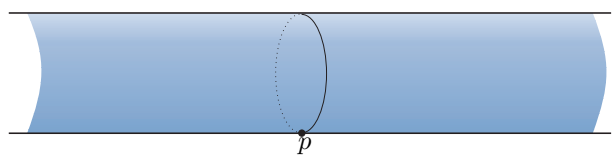
Remark 2.2. With this notion, limits will be unique in the sense that if $(\mathcal{N}_1, h_1, q_1)$ and $(\mathcal{N}_2, h_2, q_2)$ are two such limits and both are complete, then there exists an isometry $I : (\mathcal{N}_1, h_1) \to (\mathcal{N}_2, h_2)$ that sends q_1 to q_2.

Remark 2.3. To demonstrate why we need to include the points p_i in the above definition, consider the following example.

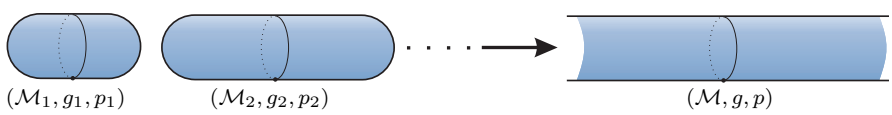
$(\mathcal{N}, h) =$



We will take the same sequence of Riemannian manifolds, with different points p_i, and get two different limits. Suppose first that for every i, $(\mathcal{M}_i, g_i)$ is equal to the $(\mathcal{N}, h)$ as shown above. Then $(\mathcal{M}_i, g_i, q) \to (\mathcal{N}, h, q)$, but $(\mathcal{M}_i, g_i, s_i)$ converges to a cylinder.



Remark 2.4. It is possible to have $\mathcal{M}_i$ compact for all i, but have the limit $\mathcal{N}$ non-compact. For example:



However, if $\mathcal{N}$ is compact, then in the Definition 2.1 of convergence, we must have $\Omega_i = \mathcal{N}$ for sufficiently large i, and the maps ϕ_i will then serve as diffeomorphisms $\mathcal{N} \to \mathcal{M}_i$ (i.e. all the $\mathcal{M}_i$ are the same as the limit) for sufficiently large i.

Two consequences of the convergence $(\mathcal{M}_i, g_i, p_i) \to (\mathcal{N}, h, q)$ are that

(i) for all $s > 0$ and $k \in \{0\} \cup \mathbb{N}$,

$$\sup_{i \in \mathbb{N}} \sup_{B_{g_i}(p_i, s)} \left|\nabla^k \mathrm{Rm}(g_i)\right| < \infty, \tag{2.1}$$

(ii)

$$\inf_i \ \mathrm{inj}(\mathcal{M}_i, g_i, p_i) > 0, \tag{2.2}$$

where $\mathrm{inj}(\mathcal{M}_i, g_i, p_i)$ denotes the injectivity radius of $(\mathcal{M}_i, g_i)$ at p_i.

In fact, conditions (i) and (ii) are *sufficient* for subconvergence. Various incarnations of the following result appear in, or can be derived from, papers of (for example) Greene and Wu [16], Fukaya [10] and Hamilton [20], all of which can be traced back to original ideas of Gromov [17] and Cheeger.

Theorem 2.5 (Compactness – manifolds)**.** *Suppose that $(\mathcal{M}_i, g_i, p_i)$ is a sequence of (smooth) complete, pointed Riemannian manifolds (all of dimension n) satisfying (2.1) and (2.2). Then there exists a (smooth) complete, pointed Riemannian manifold $(\mathcal{N}, h, q)$ (of dimension n) such that after passing to some subsequence in*

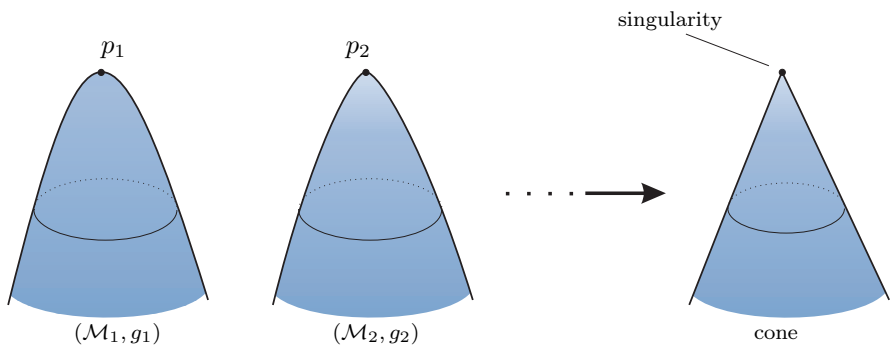
i, we have

$$(\mathcal{M}_i, g_i, p_i) \to (\mathcal{N}, h, q),$$

as $i \to \infty$.

Remark 2.6. Although we have explained that (i) and (ii) are necessary to have any hope of convergence, it may be instructive to consider what might go wrong if either condition were dropped entirely.

We clearly need some curvature control, otherwise we may have:



There are other notions of convergence which can handle this type of limit. For example, Gromov-Hausdorff convergence allows us to take limits of metric spaces. (See, for example [17].)

The uniform positive lower bound on the injectivity radius is also necessary, since otherwise we could have, for example, degeneration of two-dimensional cylinders:

Remark 2.7. Given curvature bounds as in (2.1), the injectivity radius lower bound at p_i implies a positive lower bound for the injectivity radius at other points $q \in \mathcal{M}_i$ in terms of the distance from p_i to q, and the curvature bounds, as discussed in [20], say.

2.2. Convergence and compactness of flows

One can derive Hamilton's compactness theorem for Ricci flow from the compactness theorem for manifolds (Theorem 2.5). In order to do this, we need to make sense of what it means for a sequence of flows to converge.

Definition 2.8 (Convergence of flows)**.** Let $h(t)$ be a (smooth) one-parameter family of Riemannian metrics on a fixed underlying manifold $\mathcal{N}$, for t within some fixed time interval $I \subset \mathbb{R}$. We will call such a $(\mathcal{N}, h(t))$ a *flow* on I. Let $q \in \mathcal{N}$. Let $(\mathcal{M}_i, g_i(t))$ be a sequence of flows on I, and let $p_i \in \mathcal{M}_i$ for each i. We say that

$$(\mathcal{M}_i, g_i(t), p_i) \to (\mathcal{N}, h(t), q)$$

as $i \to \infty$ if there exist

(i) a sequence of domains $\Omega_i \subset\subset \mathcal{N}$ exhausting $\mathcal{N}$ and satisfying $q \in \Omega_i$ for each i, and

(ii) a sequence of smooth maps $\phi_i : \Omega_i \to \mathcal{M}_i$, diffeomorphic onto their image, and with $\phi_i(q) = p_i$,

such that

$$\phi_i^* g_i(t) \to h(t)$$

smoothly locally on $\mathcal{N} \times I$ as $i \to \infty$.

Remark 2.9. It also makes sense to talk about convergence on (for example) the time interval $(-\infty, 0)$, even when flows are defined only for $(-T_i, 0)$ with $T_i \to \infty$.

We now wish to specialise to the case of *Ricci* flows – i.e. flows as above which also satisfy the Ricci flow equation. Hamilton [20] showed how to combine parabolic regularity theory (via Shi's local derivative estimates, which we will not be discussing in detail) and the Cheeger-Gromov compactness of Theorem 2.5 applied only at one time $t = 0$, to prove the following result at the heart of these lectures:

Theorem 2.10 (Hamilton's compactness theorem for Ricci flows)**.** *Let $\mathcal{M}_i$ be a sequence of manifolds of dimension n, and let $p_i \in \mathcal{M}_i$ for each i. Suppose that $g_i(t)$ is a sequence of complete Ricci flows on $\mathcal{M}_i$ for $t \in (a, b)$, where $-\infty \leq a < 0 < b \leq \infty$. Suppose that*

$$(i) \qquad \sup_i \sup_{x \in \mathcal{M}_i,\ t \in (a,b)} \left|\mathrm{Rm}_{g_i(t)}\right|(x) < \infty, \qquad \textit{and}$$

$$(ii) \qquad \inf_i \ \mathrm{inj}(\mathcal{M}_i, g_i(0), p_i) > 0.$$

Then there exist a manifold $\mathcal{N}$ of dimension n, a complete Ricci flow $h(t)$ on $\mathcal{N}$ for $t \in (a, b)$, and a point $q \in \mathcal{N}$ such that, after passing to a subsequence in i, we have

$$(\mathcal{M}_i, g_i(t), p_i) \to (\mathcal{N}, h(t), q),$$

as $i \to \infty$.

Remark 2.11. In this theorem, we could equally well work on a time interval $(a, 0]$, with $a < 0$, but it is not reasonable to work on a time interval $[0, b)$ with $b > 0$, because there would be a problem obtaining such strong compactness at time $t = 0$, before the parabolic nature of Ricci flow has had a chance to assert itself by smoothing out the flow.

Remark 2.12. In the presence of condition (i), there is an alternative way of phrasing condition (ii), namely

$$(ii)^{alt} \qquad \inf_i \mathrm{Vol}_{g_i(0)}[B_{g_i(0)}(p_i, 1)] > 0.$$

The intuition here is that once the curvature is controlled, the volume of a unit ball should be roughly comparable to what it is in Euclidean space, unless the injectivity radius is small. For example, in a very thin (flat) torus $S^1_\varepsilon \times S^1$, where $0 < \varepsilon \ll 1$, the volume of a unit ball is of order ε, not of order 1, because the injectivity radius is of order ε. (See [2, Theorem 4.7].)

LECTURE 3

Applications to Singularity Analysis

3.1. The rescaled flows

Now that we have discussed the convergence and compactness of Ricci flows, we are in a better position to do the rescaling near a singularity leading to a limit Ricci flow modelling the singularity, as discussed earlier. If this material is very new to you, it would be an option to fast-forward to Theorem 3.2 on the first reading.

Let $(\mathcal{M}, g(t))$ be a Ricci flow with $\mathcal{M}$ closed, on a maximal time interval $[0, T)$ – as in Section 1.2 – and assume that $T < \infty$, so that

$$\sup_{\mathcal{M}} |\mathrm{Rm}_{g(t)}| \to \infty$$

as $t \uparrow T$. Choose points $p_i \in \mathcal{M}$ and times $t_i \uparrow T$ such that

$$Q_i := |\mathrm{Rm}_{g(t_i)}|(p_i) = \sup_{x \in \mathcal{M},\ t \in [0, t_i]} |\mathrm{Rm}_{g(t)}|(x),$$

by, for example, picking (p_i, t_i) to maximise $|\mathrm{Rm}_{g(t)}|$ over $\mathcal{M} \times [0, T - \frac{1}{i}]$. Notice in particular that $Q_i \to \infty$ as $i \to \infty$. Define rescaled (and translated) flows $g_i(t)$ by

$$g_i(t) = Q_i\, g\left(t_i + \frac{t}{Q_i}\right),$$

which as discussed in Section 1.2, will be Ricci flows on the time intervals $[-t_i Q_i, (T - t_i)Q_i)$.

Moreover, for each i, we have $|\mathrm{Rm}_{g_i(0)}|(p_i) = 1$ and for $t \in (-t_i Q_i, 0]$, we have

$$\sup_{\mathcal{M}} |\mathrm{Rm}_{g_i(t)}| \leq 1.$$

Therefore, for all $a < 0$, $g_i(t)$ is defined for $t \in (a, 0]$, for sufficiently large i, and

$$\sup_i \sup_{\mathcal{M} \times (a,0]} |\mathrm{Rm}_{g_i(t)}| < \infty.$$

By Hamilton's compactness theorem 2.10 (and Remarks 2.11 and 2.9), we can pass to a subsequence in i, and get convergence $(\mathcal{M}, g_i(t), p_i) \to (\mathcal{N}, h(t), q)$ to a "singularity model" Ricci flow $(\mathcal{N}, h(t))$, defined for $t \leq 0$, *provided* that we can establish the injectivity radius estimate

$$\inf_i \mathrm{inj}(\mathcal{M}, g_i(0), p_i) > 0, \tag{3.1}$$

or (as discussed in Remark 2.12) equivalently that

$$\inf_i \mathrm{Vol}_{g_i(0)}[B_{g_i(0)}(p_i, 1)] > 0. \tag{3.2}$$

This missing step was historically a major difficulty, except in some special cases. However, as we will now see, this issue was resolved by Perelman.

3.2. Perelman's no local collapsing theorem

The core result which solves the issue we have just discussed is:

Theorem 3.1 (Perelman [25]). *Suppose that $\mathcal{M}$ is a closed manifold and $g(t)$ is a Ricci flow on $\mathcal{M}$ for $t \in [0,T)$ with $T > 0$. Then for all $p \in \mathcal{M}$ and $t \in [0,T)$, if $r \in (0,1]$ is sufficiently small so that $|\mathrm{Rm}_{g(t)}| \leq r^{-2}$ on $B_{g(t)}(p,r)$ (we call the largest such r the* curvature scale*) then*

$$\frac{\mathrm{Vol}_{g(t)}[B_{g(t)}(p,r)]}{r^n} \geq \kappa(n, g(0), T) > 0.$$

The proof of this theorem is an appealing argument using a monotonic entropy that was discovered by Perelman, which allows one to deduce the existence of a certain log-Sobolev inequality. The details may be found in [28, §8.3].

It may be worth pausing to visualise what Theorem 3.1 is ruling out. It says that in finite time, the flow cannot collapse by shrinking a circle (which might happen without making the curvature scale shrink to nothing) although this *can* happen if we drop the hypothesis that $\mathcal{M}$ is closed, or if we consider the flow as $t \to \infty$. The flow can collapse by shrinking a 2-sphere, as we saw in the shrinking cylinder example in Section 1.2.

Let us try to apply Theorem 3.1 to the Ricci flow $g(t)$ from Section 3.1. Rephrased in terms of the rescaled Ricci flows $g_i(t)$, that result tells us that for all $p \in \mathcal{M}$, $t \in [-t_iQ_i, (T-t_i)Q_i)$ and $r \in (0, Q_i^{\frac{1}{2}}]$ such that $|\mathrm{Rm}_{g_i(t)}| \leq r^{-2}$ on $B_{g_i(t)}(p,r)$, we have

$$\frac{\mathrm{Vol}_{g_i(t)}[B_{g_i(t)}(p,r)]}{r^n} \geq \kappa > 0. \tag{3.3}$$

This statement will have two applications – first it will shortly allow us to establish (3.2) in order to be able to apply Hamilton's compactness theorem to give a limit Ricci flow $h(t)$, but it will also allow us to deduce a 'non-collapsing' property of this limit.

For the first application, note that by construction, the curvature is controlled for $t \leq 0$ by $|\mathrm{Rm}_{g_i(t)}| \leq 1$ and so we are free to take $r = 1$ and $t = 0$ in (3.3) to give

$$\mathrm{Vol}_{g_i(0)}[B_{g_i(0)}(p,1)] \geq \kappa$$

for all $p \in \mathcal{M}$, which is stronger than (3.2). By the discussion in Section 3.1 this allows us to extract a limit "singularity model" Ricci flow $(\mathcal{N}, h(t))$ for $t \leq 0$. Moreover, as the second application of (3.3), after a little thought we see that we can pass to the limit $i \to \infty$ (which ultimately amounts to replacing Q_i by ∞ and $g_i(t)$ by the limit $h(t)$) to see that for all $p \in \mathcal{N}$, $t \leq 0$ and $r > 0$, if $|\mathrm{Rm}_{h(t)}| \leq r^{-2}$ on $B_{h(t)}(p,r)$, then we have

$$\frac{\mathrm{Vol}_{h(t)}[B_{h(t)}(p,r)]}{r^n} \geq \kappa > 0. \tag{3.4}$$

To summarise, we have shown the following:

Theorem 3.2. *Suppose that $\mathcal{M}^n$ is a closed n-dimensional manifold, and $g(t)$ is a Ricci flow on $\mathcal{M}$ for $t \in [0,T)$, where $T \in (0,\infty)$ is maximal. Then there exist sequences $p_i \in \mathcal{M}$ and $t_i \uparrow T$ such that*

$$Q_i := |\mathrm{Rm}_{g(t_i)}|(p_i) = \sup_{t \in [0,t_i]} \sup_{\mathcal{M}} |\mathrm{Rm}_{g(t)}|,$$

and so that if we define rescaled Ricci flows

$$g_i(t) := Q_i\, g\left(t_i + \frac{t}{Q_i}\right),$$

then there exists a complete bounded curvature Ricci flow $(\mathcal{N}^n, h(t))$ *defined for* $t \le 0$*, and* $q \in \mathcal{N}$ *such that*

$$(\mathcal{M}, g_i(t), p_i) \to (\mathcal{N}, h(t), q)$$

as $i \to \infty$*. Moreover, we have*

$$\sup_{t\le 0}\sup_{\mathcal{N}} |\mathrm{Rm}_{h(t)}| \le 1 = |\mathrm{Rm}_{h(0)}|(q),$$

and $h(t)$ *is* κ*-noncollapsed at all scales in the sense that for all* $p \in \mathcal{N}$*,* $t \le 0$ *and* $r > 0$*, if* $|\mathrm{Rm}_{h(t)}| \le r^{-2}$ *on* $B_{h(t)}(p, r)$*, then we have*

$$\frac{\mathrm{Vol}_{h(t)}[B_{h(t)}(p,r)]}{r^n} \ge \kappa > 0. \tag{3.5}$$

LECTURE 4

The case of compact surfaces – an alternative approach to the results of Hamilton and Chow

In this lecture we will apply the theory we have developed so far in the case that the underlying manifold is two-dimensional (and compact). In this dimension, and with a little more effort in three dimensions, the rescaled Ricci flows $h(t)$ we saw in the previous lecture will be so-called κ-solutions, which are special ancient solutions, that can be well understood. This dramatically restricts the types of singularity that can occur for compact $\mathcal{M}$, particularly in two dimensions when all singularities will be seen to behave asymptotically like a shrinking sphere. (This special structure does not follow in the noncompact case, as we shall see.)

The first simple ingredient in the 2D case that we will exploit is a lower bound for the Gauss curvature:

Lemma 4.1. *Given a Ricci flow $(\mathcal{M}^2, g(t))$ for $t \in (0, T]$ on a closed surface $\mathcal{M}^2$, the Gauss curvature $K_{g(t)}$ satisfies*

$$K_{g(t)} \geq -\frac{1}{2t}.$$

Proof. A computation reveals that under Ricci flow on a surface, the Gauss curvature evolves according to the PDE

$$\frac{\partial K}{\partial t} = \Delta K + 2K^2.$$

(Although we don't need it here, a similar equation holds for the scalar curvature in higher dimensions [28, Proposition 2.5.4].) The lemma is then easy to see by the maximum principle: Note that $-\frac{1}{2t}$ is a solution to the ODE $\frac{dK}{dt} = 2K^2$. (See [28, Section 3.2] for more details.) In fact, although we have assumed that $\mathcal{M}$ is compact, the result turns out to be true in much more general situations, even where the normal maximum principle fails [4], as we will see in Lemma 5.4. □

Because the curvature decreases when we scale up a metric (see the end of Section 1.2) an immediate consequence of the lemma is that the limit Ricci flows $h(t)$ arising in the previous lecture must all have weakly positive (i.e. nonnegative) curvature in this case. Combining this fact with what we already know, these limit Ricci flows $h(t)$ will then be so-called κ-solutions in the sense made precise in the following theorem.

Theorem 4.2. *When the underlying manifold $\mathcal{M}$ is a compact surface, any limit Ricci flow $h(t)$ as constructed in the previous lecture must satisfy the following properties, which define what it means to be a* **κ-solution***:*

(1) It is **Ancient***, i.e. it is defined for $-\infty < t \leq T$, for some $T \in \mathbb{R}$. (In this particular, case, we can set $T = 0$.)*

(2) *It has* **Bounded curvature***, but is not everywhere flat. (In this particular case we have* $|\mathrm{Rm}| \leq 1$ *everywhere, and* $|\mathrm{Rm}| = 1$ *at at least one point, by construction.)*
(3) *It is* **Complete***. (In this case, that was a consequence of Hamilton's compactness theorem.)*
(4) *It is* κ*-noncollapsed at all scales, i.e. there exists* $\kappa > 0$ *such that for each time* t *at which* $h(t)$ *is defined, and each point* $p \in \mathcal{N}$ *and radius* $r > 0$ *for which* $|\mathrm{Rm}| \leq r^{-2}$ *on* $B_{h(t)}(p, r)$*, we have*
$$\frac{\mathrm{Vol}_{h(t)} B_{h(t)}(p, r)}{r^n} \geq \kappa.$$
(In this case, that was a consequence of Perelman's theory.)
(5) *It has weakly positive (i.e. nonnegative) curvature operator. (In this case, that simply means that the Gauss curvature is weakly positive, which was the most recent thing we established.)*

The notion of κ-solutions was introduced by Perelman [25, §11] principally to study Ricci flow of three-dimensional manifolds. By virtue of the so-called Hamilton-Ivey pinching result [21] (which we will not discuss here) it is also possible to establish that limit Ricci flows $h(t)$ have (weakly) positive curvature in that case too. The true significance of κ-solutions is that in low dimensions, they are restrictive enough that we can understand what they look like. In two-dimensions, this is particularly clean:

Theorem 4.3 (Perelman [25, §11.3])**.** *The only oriented 2D* κ*-solution is the round shrinking sphere* $(S^2, g^2(t))$*, as found in Section 1.2.*

We may therefore deduce that for a Ricci flow on an oriented, compact surface, whenever a finite-time singularity occurs, it can be rescaled to give the Ricci flow that is a round shrinking sphere. But we already saw in Remark 2.4 that the only way a Cheeger-Gromov limit can be a compact manifold is if the original sequence of manifolds is (eventually) diffeomorphic to this limit. That means that we can only have a finite-time singularity in a Ricci flow on an oriented, compact surface if that surface is S^2. Note that we then have $\mathrm{Vol}_{g(t_i)} \mathcal{M} \to 0$ as $i \to \infty$. On the other hand, if the underlying manifold is S^2, then we can make a short calculation (see [28, (2.5.7) and (2.5.8)]) to see that the volume measure evolves according to
$$\frac{\partial}{\partial t} d\mu_{g(t)} = -2K d\mu_{g(t)},$$
and hence by the Gauss-Bonnet theorem we have
$$\frac{d}{dt} \mathrm{Vol}_{g(t)} \mathcal{M} = -2 \int_{\mathcal{M}} K d\mu_{g(t)} = -8\pi,$$
and the Ricci flow must shrink to nothing in a specific finite time $T = \frac{\mathrm{Vol}_{g_0} \mathcal{M}}{8\pi}$. Note that at the times t_i as which we are blowing up, the flow must look more and more like a round sphere of volume $8\pi(T - t_i)$, i.e. of curvature $1/2(T - t_i)$. We have proved the following result, with very different techniques to those originally used.

Theorem 4.4 (Hamilton [19], Chow [5])**.** *Let* $\mathcal{M}^2$ *be a closed, oriented surface and* g_0 *any smooth metric. Then there exists a unique Ricci flow* $g(t)$ *on* $\mathcal{M}$*, for* $t \in [0, T)$ *so that* $g(0) = g_0$*. We may assume that* $T = \infty$ *unless* $\mathcal{M} = S^2$*, in*

which case we have $T = \frac{\mathrm{Vol}_{g_0}\mathcal{M}}{8\pi}$*, and there exist sequences of times* $t_i \uparrow T$ *and diffeomorphisms* $\varphi_i : \mathcal{M} \to \mathcal{M}$ *such that*

$$\frac{\varphi_i^*(g(t_i))}{2(T-t_i)} \to g_{+1}$$

as $i \to \infty$*, where* g_{+1} *is a round metric of constant curvature* $+1$*.*

Remark 4.5. With a little more work [19, 5, 11], one can show in fact that

$$\frac{g(t)}{2(T-t)} \to g_{+1}$$

uniformly as $t \uparrow T$ in Theorem 4.4.

The higher genus case is possibly a little easier to deal with, and one can prove:

Theorem 4.6 (Hamilton [19])**.** *Suppose the surface in Theorem 4.4 is a torus* T^2*. Then the Ricci flow* $g(t)$ *satisfies* $g(t) \to g_0$ *smoothly as* $t \to \infty$*, where* g_0 *is a flat metric.*

If instead the surface in Theorem 4.4 is orientable, of genus $\gamma > 1$*, then* $\frac{g(t)}{2t} \to g_{-1}$ *smoothly, where* g_{-1} *is the unique hyperbolic metric that shares a conformal class with* $g(t)$*.*

The easiest way of deriving the 2D compact Ricci flow theory appeals to the Uniformisation theorem, which tells us that any Riemann surface admits a compatible Riemannian metric (i.e. inducing the same conformal sturcture) which has constant curvature $+1$, 0 to -1, depending on its genus. More recently, Chen, Lu and Tian [6] have shown that in this case of compact underlying manifold, one can develop the Ricci flow theory independently of the Uniformisation theorem, thus yielding a new proof of that fundamental result in the special case of compact underlying manifold.

To motivate the material of the next lecture, one might ask the question of whether Ricci flow will try to perform the same task of finding a constant curvature metric in the case that the underlying manifold is noncompact. Now, even posing the Ricci flow is an issue, as we shall see.

LECTURE 5

The 2D case in general – Instantaneously Complete Ricci flows

5.1. How to pose the Ricci flow in general

We have just seen the complete theory, including asymptotics, in the compact 2D case. We also have short-time well-posedness for Ricci flows starting with complete, bounded-curvature metrics by virtue of Theorem 1.1. In this lecture we wish to understand the case that the underlying manifold $\mathcal{M}$ is noncompact, but the starting metric g_0 could have *unbounded curvature* or even be *incomplete*.

To illustrate the difficulties involved here, we consider the following simple example:

> *How should we Ricci flow the flat disc (D^2, g_0), where $g_0 = dx^2 + dy^2$ is the Euclidean metric?*

To get a feel for this, let's return to the discussion of the logarithmic fast diffusion equation at the beginning of these lectures, and write the Ricci flow as $g(t) = e^{2u}(dx^2 + dy^2)$ where u must solve

$$\begin{cases} \dfrac{\partial u}{\partial t} = e^{-2u}\Delta u \\ u(\cdot, 0) \equiv 0. \end{cases} \tag{5.1}$$

Of course, there is an obvious solution, given that the initial metric is flat, which is the flow that does not move, and is the flat disc for all time. That is, the solution $u \equiv 0$ throughout D and for all times. However, this sort of trivial solution would not exist even for a small perturbation of the initial data above, and now that we have written out the PDE (5.1) we see that even amongst solutions u that are continuous up to the boundary of D, we are free to *prescribe* $u(\cdot, t)$ on ∂D for each positive t. We thus need an extra 'boundary'-type condition to kill this extreme nonuniqueness. The central idea of the theory we are about to see [29] is that in full generality for Ricci flow on surfaces, a geometric condition which leads to uniqueness (whilst preserving existence of solutions!) is that of *instantaneous completeness*, that is:

> *We consider Ricci flows $g(t)$ for $t \in [0, T)$ that are complete for all $t \in (0, T)$.*

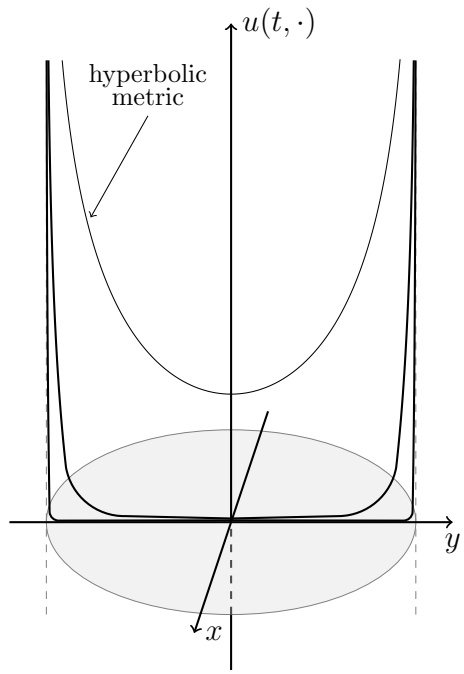
In particular, in the case of Ricci flow starting with a disc, we have the following result which is a variant of a very special case of [29].

Theorem 5.1. *There exists a Ricci flow $g(t)$ on the disc D^2 for $t \in [0, \infty)$ such that*

(1) $g(0) = g_0$, the standard flat metric,
(2) $g(t)$ is complete for all $t > 0$.

Any other Ricci flow $\tilde{g}(t)$ on D^2 over any time interval $[0,T)$ (with $T \in (0,\infty]$) that satisfies both (1) and (2) above must agree with $g(t)$ on $[0,T)$ where they are both defined.

Given that the unit disc is not complete, something unusual must be happening! This is easiest to see by considering the conformal factor u: For arbitrarily small $t > 0$, the conformal factor u must blow up near the boundary at a fast enough rate to be sure that the metric is complete. That means that the integral of e^u along any curve escaping to 'infinity' in the disc must be infinite. The graph of the conformal factor looks something like:[1]



Initially the flow will look more or less like the flat disc on the interior, but in an expanding layer around the boundary, the flow will have curvature approximately its minimum possible value $-\frac{1}{2t}$ (recall Lemma 4.1). Indeed, if we write $g_{-1} := \frac{4g_0}{(1-x^2-y^2)^2}$ for the (unique) complete hyperbolic metric on the disc D that is a conformal deformation of g_0, then the Ricci flow $g(t)$ will resemble $2t\,g_{-1}$ on the expanding layer around the boundary, and eventually on the whole disc.

In fact, there is a much more general existence and uniqueness theory, which we outline in the following section.

5.2. The existence and uniqueness theory

The original existence theory for instantaneously complete Ricci flows [29] was the first to allow the initial curvature to be unbounded (below) and also allowed the initial metric to be incomplete. However, more recently, the full existence and uniqueness theory has been completed, which handles a completely general surface, and gives the optimal existence time. Moreover, the flow $g(t)$ we find has the

[1]Thanks to Gregor Giesen for this figure.

additional property of being 'maximally stretched,' by which we mean that any other (conformal) Ricci flow $\tilde{g}(t)$ with $\tilde{g}(0) \leq g(0)$ must satisfy $\tilde{g}(t) \leq g(t)$ for all t in any time interval $[0, a)$ during which both flows are defined (irrespective of whether or not $\tilde{g}(t)$ is complete or has bounded curvature).

Theorem 5.2 (Existence: joint with Giesen [13]; uniqueness from [33])**.** *Let $(\mathcal{M}, g_0)$ be a smooth Riemannian surface which need not be complete, and could have curvature unbounded below and/or above. Depending on the conformal type, we define $T \in (0, \infty]$ by*

$$T := \begin{cases} \frac{1}{4\pi\chi(\mathcal{M})} \operatorname{Vol}_{g_0} \mathcal{M} & \text{if } (\mathcal{M}, g_0) \cong \mathcal{S}^2, \mathbb{C} \text{ or } \mathbb{R}P^2, \\ \infty & \text{otherwise.}^2 \end{cases}$$

Then there exists a unique smooth Ricci flow $(\mathcal{M}, g(t))$ for $t \in [0, T)$ such that

(1) $g(0) = g_0$;
(2) $g(t)$ is instantaneously complete;
(3) $g(t)$ is maximally stretched.

This flow is unique in the sense that if $\tilde{g}(t)$ is any Ricci flow on $\mathcal{M}$, for $t \in [0, \tilde{T})$, satisfying (1) and (2), then $\tilde{T} \leq T$ and $\tilde{g}(t) = g(t)$ for all $t \in [0, \tilde{T})$.

If $T < \infty$, then we have

$$\operatorname{Vol}_{g(t)} \mathcal{M} = 4\pi\chi(\mathcal{M})(T - t) \to 0 \text{ as } t \nearrow T,$$

and in particular, T is the maximal existence time.

Several aspects of this theorem resemble the results we have seen of Hamilton and Chow in the compact case (which have been absorbed into this result). However, there are some significant departures from that theory, and some shocking examples of flows fitting into this theorem, as we survey in a moment.

Our Ricci flow can start with a completely general initial surface, but in the special case that this initial surface is both complete and of curvature bounded above and below, there is a competing flow, namely that of Shi from Theorem 1.1, existing on some time interval $[0, \tilde{T})$, and by the uniqueness assertion of Theorem 5.2, these flows agree while Shi's exists.

As we discussed in Section 1.2, if $\tilde{T} < \infty$, then we can assume that the curvature of $\tilde{g}(t)$ blows up as $t \uparrow \tilde{T}$. It may be then hard to imagine how one could have $T > \tilde{T}$ since this would imply that the curvature could blow up in our flow, but that we could then keep on flowing. However, this is exactly what can happen, for example:

Theorem 5.3 (Proved with Giesen [15])**.** *There exist a complete immortal Ricci flow $\big(g(t)\big)_{t \in [0,\infty)}$ on $\mathbb{C}$ arising from Theorem 5.2, and a time $t_1 \in [1, 3)$ such that*

$$\sup_{\mathcal{M}} \left| K_{g(t)} \right| \begin{cases} < \infty & \text{for all } t \in [0, 1) \\ = \infty & \text{for all } t \in \left(t_1, t_1 + \frac{1}{100}\right) \\ < \infty & \text{for all } t \in [4, \infty). \end{cases}$$

In fact, using the techniques we develop to prove this theorem, we can prescribe with great generality the set of times at which the curvature is unbounded. Note that this feature now means that the term 'maximal existence time' needs to be defined carefully in each noncompact situation. Our maximal existence time is

[2]Note that also $T = \infty$ if $\operatorname{Vol}_{g_0} \mathbb{C} = \infty$.

greater than, and in general *strictly* greater than the traditional maximal existence time.

When we have unbounded curvature in these examples, it is unbounded above. It can never be unbounded below for positive times, and indeed we have the following curvature estimates. Formally they follow via the maximum principle, although that does not apply directly in this general situation. In practice, they follow from the actual *construction* made in the proof of Theorem 5.2, or via the theory in [29, 4].

Lemma 5.4. *In Theorem 5.2, the flow satisfies the lower curvature bound*

$$K_{g(t)} \geq -\frac{1}{2t}$$

for all $t \in (0, T)$. If in addition we have an initial upper curvature bound $K_{g_0} \leq K_0 \geq 0$ for the initial metric, then the flow $g(t)$ has the upper curvature bound

$$K_{g(t)} \leq \frac{K_0}{1 - 2K_0 t}$$

for all $t \geq 0$ if $K_0 = 0$, and for all $t \in [0, \frac{1}{2K_0})$ otherwise. (Lower bounds for the existence time T are implicit here.) Irrespective of any initial upper curvature bound, if we have a lower curvature bound $K_{g_0} \geq k_0 \leq 0$ for the initial metric, and the initial metric is **complete***, then the flow $g(t)$ has the lower curvature bound*

$$K_{g(t)} \geq \frac{k_0}{1 - 2k_0 t}$$

for all $t \in [0, T)$.

It may be clear from the discussion above that while our flow from Theorem 5.2 takes a general metric and makes it immediately complete and of curvature bounded below, it need not immediately transform the metric to one of curvature bounded above. That is, we do not immediately find ourselves in the classical situation. This is illustrated in generality by:

Theorem 5.5 (Proved with Giesen [14])**.** *For all noncompact Riemann surfaces $\mathcal{M}^2$, there exists a Ricci flow $g(t)$ on $\mathcal{M}^2$ for $t \in [0, \infty)$ (respecting the complex structure of the surface) such that*

$$\sup_{\mathcal{M}} K_{g(t)} = \infty$$

for all $t \in [0, \infty)$.

5.3. Asymptotics

One of the features of the Ricci flow theory on compact surfaces that we saw in Lecture 4, was that it 'geometrised' a surface – i.e. up to scaling, it converged to a constant curvature metric. In the noncompact case, so far we only fully understand the asymptotics in the hyperbolic case:

Theorem 5.6 (Proved with Giesen [13])**.** *Suppose we have a Ricci flow in Theorem 5.2 on a surface which supports a complete hyperbolic metric H conformally equivalent to the Ricci flow (in which case $T = \infty$ automatically). Then we have convergence of the rescaled solution*

$$\frac{1}{2t} g(t) \longrightarrow H \quad \textit{smoothly locally as } t \to \infty.$$

If additionally there exists a constant $M > 0$ such that $g_0 \leq MH$ then the convergence is global: For any $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\eta \in (0,1)$ there exists a constant $C = C(k, \eta, M) > 0$ such that for all $t \geq 1$ we have

$$\left\| \frac{1}{2t} g(t) - H \right\|_{C^k(\mathcal{M},H)} \leq \frac{C}{t^{1-\eta}} \quad \overset{t\to\infty}{\longrightarrow} \quad 0.$$

In fact, in this latter case, for all $t > 0$ we have $\left\| \frac{1}{2t} g(t) - H \right\|_{C^0(\mathcal{M},H)} \leq \frac{C}{t}$, and even

$$0 \leq \frac{1}{2t} g(t) - H \leq \frac{M}{2t} H.$$

Related results were proved by Ji-Mazzeo-Sesum [22] under the additional hypotheses that the initial data was complete, and of bounded curvature, and conformally equivalent to a compact surface with finitely many punctures, and with all ends asymptotic to hyperbolic cusps.

This theorem has to be reconciled with Theorem 5.5, which told us that there exists a Ricci flow with unbounded curvature for all time on any noncompact Riemann surface. In particular on any noncompact Riemann surface that supports a hyperbolic metric, for example the disc D, there exists such an unbounded curvature Ricci flow. On the other hand, in such a case Theorem 5.6 tells us that the curvature is converging to -1, which is bounded! Ricci flow resolves this paradox by sending the bad regions of high curvature out to spatial infinity, the convergence of Theorem 5.6 being only smooth local convergence in general.

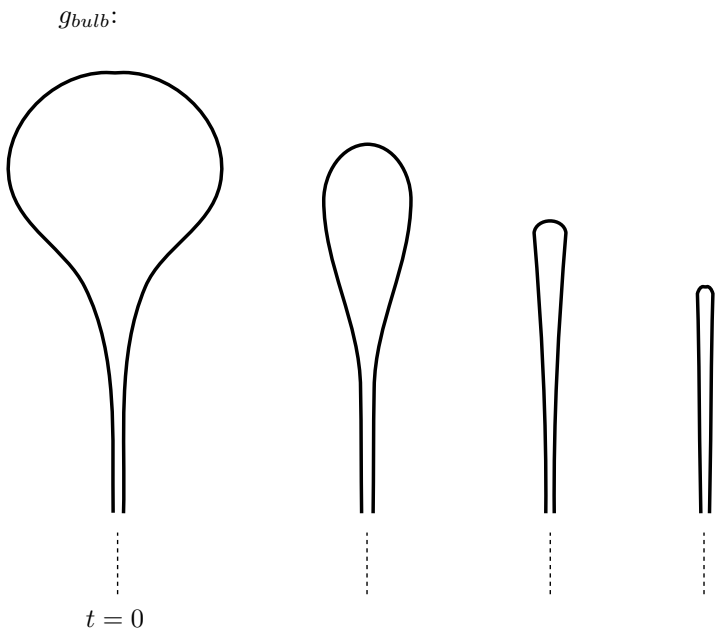
5.4. Singularities not modelled on shrinking spheres

We take the opportunity to give an additional example of Theorem 5.2 in action that not only illustrates a distinction compared with Lecture 4 where all singularities were modelled on shrinking spheres, but will also be useful in Lecture 7.

The starting metric will be conformally the complex plane, and look geometrically like a spherical bulb with a hyperbolic cusp attached. Such a metric could be constructed by bare hands, or would naturally arise by taking initial data in Theorem 5.2 consisting of a punctured sphere, and flowing for a moment.

By scaling up or down, we assume our bulb metric g_{bulb} has area exactly equal to 8π, and that way Theorem 5.2 will make it flow for exactly time 2.

According to the theory developed in [8], [9], and the references therein, the end of this flow will always look like a hyperbolic cusp. On the other hand, the bulb part will gradually shrink under the flow, and just before the surface disappears (along with all its area) it will look like a 'cigar' (see [28, §1.2.2]) tapering into a hyperbolic cusp.



Even though one can blow up this finite-time singularity similarly to as in Lecture 3, this situation does not fit into the discussion of Lecture 4 because we are not on a compact manifold, and Perelman's no local collapsing theorem 3.1 fails, and we cannot deduce that the blow-up is a κ-solution and thus a shrinking sphere as in Theorem 4.3. Indeed, as indicated above, the blow-ups in this case would converge to a cigar metric [28, §1.2.2].

LECTURE 6

Contracting Cusp Ricci flows

There is one further type of well-posedness issue to discuss, and a new type of Ricci flow in 2D fitting into this framework that will be needed in the next lecture. Up until now, we have posed a Ricci flow by fixing an underlying manifold and specifying an initial metric. Thus the initial metric is achieved as a limit of the tensors $g(t)$ as $t \downarrow 0$. However, given the discussion of Cheeger-Gromov convergence that we saw in Lecture 2, it could be considered more natural to ask for the initial manifold to be achieved as a Cheeger-Gromov-type limit of $(\mathcal{M}, g(t))$ as $t \downarrow 0$.

Definition 6.1. ([31].) We say that a complete Ricci flow $(\mathcal{M}^n, g(t))$ for $t \in (0, T]$ has a complete Riemannian manifold $(\mathcal{N}^n, g_0)$ as initial condition if there exists a smooth map $\varphi : \mathcal{N} \to \mathcal{M}$, diffeomorphic onto its image, such that

$$\varphi^*(g(t)) \to g_0$$

smoothly locally on $\mathcal{N}$ as $t \downarrow 0$.

The usual way of posing Ricci flow fits into this framework by setting $\mathcal{N} = \mathcal{M}$ and taking φ to be the identity.

In practice, we will be interested in the case that $(\mathcal{N}, g_0)$ has bounded curvature but $g(t)$ is allowed to have curvature with no uniform upper bound. In this way, $\mathcal{M}$ and $\mathcal{N}$ may not be diffeomorphic since parts of $\mathcal{M}$ may be shot out to infinity as $t \downarrow 0$ resulting in a change of topology in the limit.

This generalised notion of initial condition permits some new types of solution which do not fit into the classical framework. In particular, we show that a bounded-curvature Riemannian surface with a hyperbolic cusp need not be obliged to flow forwards in time retaining the cusp (as it would in the solution of Shi or of Theorem 5.2) but can add in a point at infinity, removing the puncture in the surface, and let the cusp contract in a controlled way. More generally we have:

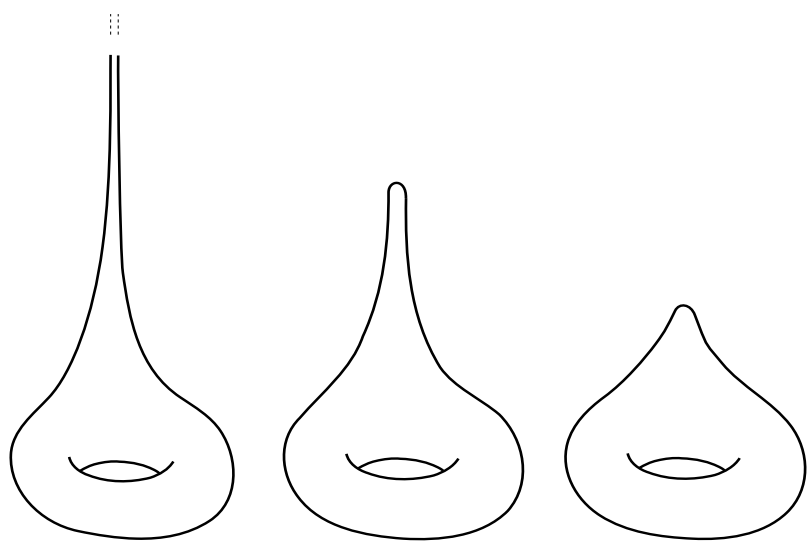
Theorem 6.2 ([31])**.** *Suppose $\mathcal{M}$ is a compact Riemann surface and $\{p_1, \ldots, p_n\} \subset \mathcal{M}$ is a finite set of distinct points. If g_0 is a complete, bounded-curvature, smooth, conformal metric on $\mathcal{N} := \mathcal{M} \backslash \{p_1, \ldots, p_n\}$ with uniformly strictly negative curvature in a neighbourhood of each point p_i, then there exists a Ricci flow $g(t)$ on $\mathcal{M}$ for $t \in (0, T]$ (for some $T > 0$) having $(\mathcal{N}, g_0)$ as initial condition in the sense of Definition 6.1. We can take the map φ there to be the natural inclusion of $\mathcal{N}$ in $\mathcal{M}$.*

Moreover, the cusps contract logarithmically in the sense that for some $C < \infty$ and all $t \in (0, T]$ sufficiently small, we have

$$\frac{1}{C}(-\ln t) \leq diam\,(\mathcal{M}, g(t)) \leq C(-\ln t). \tag{6.1}$$

Furthermore, the curvature of $g(t)$ is bounded below uniformly as $t \downarrow 0$.

Let us consider one specific example where this theorem applies, that will be required in the next lecture. Consider a punctured torus $\mathcal{N} := T^2 \backslash \{p\}$ (the torus having any complex structure) and let g_0 be g_{hyp}, the unique complete conformal hyperbolic metric on $\mathcal{N}$. The metric will have a cusp at the puncture. One Ricci flow continuation would be the homothetically expanding flow (which coincides with the solution constructed by Shi and that of Theorem 5.2) but another continuation would see the cusp contract with the subsequent Ricci flow living on the whole torus $\mathcal{M}$.



One characteristic of these nonuniqueness examples is that the initial condition $(\mathcal{N}, g_0)$ does not have a lower bound for its injectivity radius, or equivalently that one can find unit balls of arbitrarily small area. In fact, this is a necessary condition for nonuniqueness.

Theorem 6.3 ([31]). *Suppose that $(\mathcal{N}, g_0)$ is a complete Riemannian surface with bounded curvature which is noncollapsed in the sense that for some $r_0 > 0$ we have*

$$\mathrm{Vol}_{g_0}(B_{g_0}(x, r_0)) \geq \varepsilon > 0 \tag{6.2}$$

for all $x \in \mathcal{N}$. If for $i = 1, 2$ we have complete Ricci flows $(\mathcal{M}_i, g_i(t))$ for $t \in (0, T_i]$ (some $T_i > 0$) with $(\mathcal{N}, g_0)$ as initial condition in the sense of Definition 6.1, then these two Ricci flows must agree over some nonempty time interval $t \in (0, \delta]$ in the sense that there exists a diffeomorphism $\psi : \mathcal{M}_1 \to \mathcal{M}_2$ with $\psi^*(g_2(t)) = g_1(t)$ for all $t \in (0, \delta]$.

Despite the nonuniqueness implied by Theorem 6.2, that construction throws up a quite different uniqueness issue: Does there exist more than one flow that does the same job of contracting the cusps? The next result shows that there does not.

Theorem 6.4 ([31]). *In the situation of Theorem 6.2 (in which φ is the natural inclusion of $\mathcal{N}$ into $\mathcal{M}$) if $\tilde{g}(t)$ is a smooth Ricci flow on $\mathcal{M}$ for some time interval $t \in (0, \delta)$ ($\delta \in (0, T]$) such that $\tilde{g}(t) \to g_0$ smoothly locally on $\mathcal{N}$ as $t \downarrow 0$ and the*

Gauss curvature of $\tilde{g}(t)$ is uniformly bounded below, then $\tilde{g}(t)$ agrees with the flow $g(t)$ constructed in Theorem 6.2 for $t \in (0, \delta)$.

LECTURE 7

Subtleties of Hamilton's compactness theorem

Having started with Hamilton's compactness theorem 2.10 and applied it to 2D Ricci flow theory, we now complete the circle by applying our 2D Ricci flow theory to understand the subtleties concerning extensions of Hamilton's compactness theorem that are required in applications.

Up until now, the compactness theorem was used to blow up a Ricci flow that is becoming singular, at points of maximum curvature (see Lecture 3). However, in practice, it is essential to be able to analyse other parts of the Ricci flow where the curvature may be blowing up, even though the curvature might be much larger elsewhere. Moreover, there is an entirely different situation where one needs to apply the compactness theorem, namely when making contradiction arguments. Loosely speaking, in order to prove that a certain fact about Ricci flow is true to some degree, one assumes otherwise – i.e. that there exists a sequence of Ricci flows which violates the assertion to a greater and greater degree. One then tries to appeal to compactness to get a limit Ricci flow with contradictory features.

For all these applications, Hamilton's compactness theorem 2.10 is not quite enough, because the curvature hypothesis

$$(i) \qquad \sup_i \sup_{x\in\mathcal{M}_i,\ t\in(a,b)} \left|\mathrm{Rm}_{g_i(t)}\right|(x) < \infty$$

is too restrictive. A modification of the proof yields the following modified result where only local curvature bounds are assumed:

Theorem 7.1 (Extended Hamilton compactness theorem)**.** *Suppose that $(\mathcal{M}_i, g_i(t))$ is a sequence of complete Ricci flows on n-dimensional manifolds $\mathcal{M}_i$ for $t \in (a,b)$, where $-\infty \le a < 0 < b \le \infty$. Suppose that $p_i \in \mathcal{M}_i$ for each i, and that*

(i) for all $r > 0$, there exists $M = M(r) < \infty$ such that for all $t \in (a,b)$ and for all i, there holds

$$\sup_{B_{g_i(0)}(p_i,r)} \left|\mathrm{Rm}_{g_i(t)}\right| \le M, \qquad \textit{and}$$

(ii)

$$\inf_i \ \mathrm{inj}(\mathcal{M}_i, g_i(0), p_i) > 0.$$

Then there exist a manifold $\mathcal{N}$ of dimension n, a Ricci flow $h(t)$ on $\mathcal{N}$ for $t \in (a,b)$ and a point $q \in \mathcal{N}$, such that $(\mathcal{N}, h(0))$ is complete, and after passing to a subsequence in i, we have

$$(\mathcal{M}_i, g_i(t), p_i) \to (\mathcal{N}, h(t), q), \tag{7.1}$$

as $i \to \infty$.

In the literature this result is often stated (and used!) with the additional conclusion that the limit Ricci flow $h(t)$ is *complete* – that is, $h(t)$ is complete

for every $t \in (a, b)$. It was noted recently [23, Appendix E] following questions raised by Cabezas-Rivas and Wilking that this completeness 'does not immediately follow'. Our objective in this lecture is to demonstrate that in fact the completeness fails in general, and some modification of earlier applications is required.

7.1. Intuition behind the construction

In this section we sketch an intuitive construction that indicates that a limiting Ricci flow arising in Theorem 7.1 can be complete over an open time interval containing $t = 0$, but incomplete beyond a certain time. The precise construction we make in [32] is a little different to reduce the amount of technology required to make it rigorous. (That technology was later developed in [15].)

At the core of the construction are the 'contracting cusp' examples of Ricci flows we constructed in Lecture 6, and in particular, the alternative way of flowing the torus $T^2\backslash\{p\}$ with its complete hyperbolic manifold g_{hyp} (which has a hyperbolic cusp as its end). Recall that one way of flowing this initial manifold is simply to dilate (analogously to the shrinking sphere example from Section 1.2) giving

$$G_1(t) := (1 + 2t)g_{hyp},$$

for all $t \in [0, \infty)$. In addition, we have the alternative flow from Lecture 6 where one imagines capping off the hyperbolic cusp infinitely far out, and letting it contract down. Let us write $G_2(t)$ for this alternative flow, which we view either as a complete Ricci flow on T^2 for $t > 0$, or as a Ricci flow on $T^2\backslash\{p\}$ for $t \geq 0$ which is incomplete for $t > 0$ but equal to g_{hyp} for $t = 0$.

From this discussion, we see that a perfectly valid smooth Ricci flow $h(t)$ on $T^2\backslash\{p\}$ would consist of $G_1(t+1)$ for $t \in [-1, 1]$, followed by an appropriate scaling of $G_2(t)$, restricted to $T^2\backslash\{p\}$. Precisely, that scaled flow would be the restriction to $T^2\backslash\{p\}$ of $5G_2\left((t-1)/5\right)$ for $t \in (1, \infty)$. Beyond $t = 1$, the flow would be incomplete because we have removed the point p.

The core principle of this lecture is:

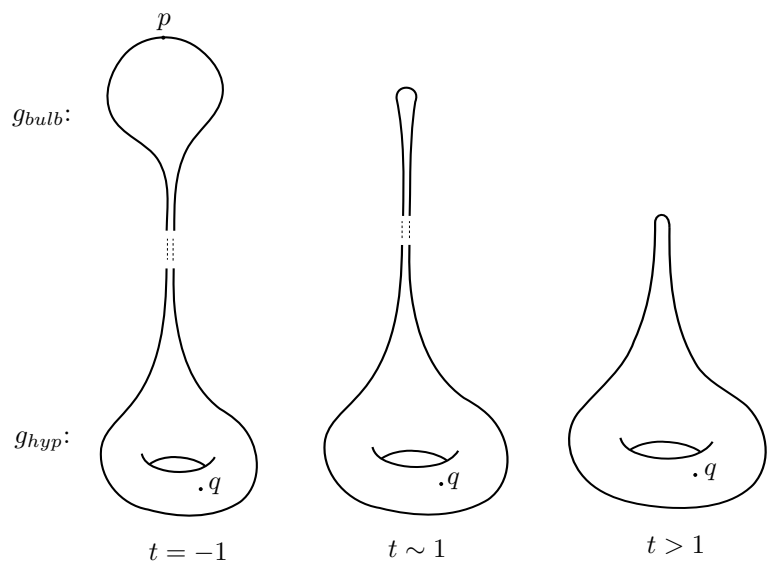
> *Such flows $h(t)$ can arise as Hamilton-Cheeger-Gromov limits of complete Ricci flows within the extension of Hamilton's compactness theorem given in Theorem 7.1.*

A precise statement can be found in [32]; here we sketch how one could hope to construct a sequence of complete, bounded-curvature Ricci flows satisfying the hypotheses of Theorem 7.1, with a limit flow as given above.

The basic building blocks are the complete hyperbolic metric g_{hyp} on $T^2\backslash\{p\}$ and a complete metric g_{bulb} on the punctured 2-sphere (equivalently on the plane) whose end is asymptotic to a hyperbolic cusp, and whose area is exactly 8π, as considered in Section 5.4. In particular, the Ricci flow of Theorem 5.2 would flow g_{bulb} for exactly time 2 before all area was sucked out of the bulb part of the manifold, and at each time the flow would have a cusp-like end.

The Ricci flows $g_i(t)$ we wish to imagine putting into Theorem 7.1 will exist on the whole of T^2. They will be the Ricci flows whose initial data at $t = -1$ arises by chopping off the ends of the cusps of both the metrics g_{hyp} and g_{bulb}, and gluing them together. The larger i becomes, the further out we wish to make our truncations. If we fix a base-point $q \in T^2\backslash\{p\}$, and consider p to correspond to some fixed point in the bulb surface for each i, then the distance $d_{g_i(-1)}(p, q)$ converges to infinity as $i \to \infty$.

The idea then is that the two distinct parts of $g_i(-1)$ will evolve largely independently of each other within the time interval $[-1, 1)$, but then at time $t = 1$, the area within the bulb part of the flow will have been exhausted, and the cusp end of the torus part of the flow should start contracting. The effect of this is that during the initial time interval $[-1, 1)$, the Ricci flows $(T^2, g_i(t), q)$ will converge to a homothetically expanding hyperbolic metric on $T^2 \backslash \{p\}$ in the large i limit (with the bulb part too far away to see) but for $t > 1$, the manifolds $(T^2, g_i(t), q)$ should have uniformly controlled diameter (independent of i) and will converge to a smooth metric on the whole torus in the Cheeger-Gromov sense. Conversely, an observer at p will (in the limit $i \to \infty$) be living in a separate 'bulb' universe to q until time $t \sim 1$ when a big crunch occurs and the point q flies into view as p appears in the 'torus' universe. The construction is illustrated in the following figure.



It is not too difficult to argue precisely, using pseudolocality technology (see [25]) that one obtains the expanding hyperbolic metric flow as the limit on an initial time interval. In order to show that the cusp will start collapsing at a uniformly controlled rate before some uniformly bounded time we require some more involved *a priori* estimates. In [32], we prove these estimates in a slightly different situation to the one outlined above, notably replacing the bulb metric by long thin cigars with k-dependent geometry, but with area uniformly bounded above and below.

7.2. Fixing proofs requiring completeness in the extended form of Hamilton's compactness theorem

We have seen in this lecture that under the traditional hypotheses of the extended Hamilton compactness theorem 7.1, we cannot rely on the limit being complete for all times. However, in applications, this completeness is typically essential. We would therefore like to end by commenting on some additional hypotheses that will deliver this completeness. In the major applications of this theory, such as in

Perelman's work on Ricci flow [25], one or other of these hypotheses can typically be seen to be available. More details can be found in [32].

The first fix will work in many applications where the dependency of the upper bound for the curvature on the radius is really hiding a dependence on i. One important example of this is in the proof of Perelman's pseudolocality theorem [25]. More precisely, the limit $h(t)$ will be complete for all $t \in (a, b)$ in Theorem 7.1 if we replace hypothesis (i) with

> (i)$'$ For all $r > 0$, there exist $K \in \mathbb{N}$ dependent on r, and $M < \infty$ **independent** of r, such that for all $t \in (a, b)$ and for all $i \geq K$, there holds
> $$\sup_{B_{g_i(0)}(p_i, r)} \left|\mathrm{Rm}_{g_i(t)}\right| \leq M.$$

Perhaps the greatest use of Ricci flow compactness in the work of Perelman is in his development of the theory of κ-solutions in [25, §11], as defined in Lecture 4, which is critical in applications. In this situation, one has *nonnegative curvature*, which prevents distances from increasing as time advances. This implies that the flow cannot turn an incomplete manifold into a complete one as t increases, and therefore that the limit flow must be complete for all negative times. The following result illustrates the idea, and can be generalised in many different directions (cf. [23]).

Theorem 7.2 (Compactness of Ricci flows with a lower Ricci bound)**.** *In the situation of Theorem 7.1, if each flow $g_i(t)$ has Ricci curvature uniformly bounded below (also uniformly in i) then the limit Ricci flow $(\mathcal{N}, h(t))$ is complete for all $t \in (a, 0]$.*

Note that the example whose construction we have sketched in Section 7.1 has a loss of completeness beyond some *positive* time. In the special case of two-dimensional Ricci flows, one can *never* have a loss of completeness for negative times in Theorem 7.1, because we know ([4], cf. (4.1) and (5.4)) that the Gauss curvature of each flow is bounded below by $-\frac{1}{2(t-a)}$, and one can apply Theorem 7.2 over arbitrarily smaller time intervals $(a + \varepsilon, b)$, for $\varepsilon \in (0, |a|)$.

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