# THE CANONICAL SHRINKING SOLITON ASSOCIATED TO A RICCI FLOW

Esther Cabezas-Rivas and Peter M. Topping

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**Abstract**

To every Ricci flow on a manifold $\mathcal{M}$ over a time interval $I \subset \mathbb{R}_-$, we associate a shrinking Ricci soliton on the space-time $\mathcal{M} \times I$. We relate properties of the original Ricci flow to properties of the new higher-dimensional Ricci flow equipped with its own time-parameter. This geometric construction was discovered by consideration of the theory of optimal transportation, and in particular the results of the second author [18], and McCann and the second author [12]; we briefly survey the link between these subjects.

## 1 Introduction

In 1982, Hamilton [7] introduced the study of Ricci flow, which evolves a Riemannian metric $g$ on a manifold $\mathcal{M}$ under the nonlinear evolution equation

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g(t)), \tag{1.1}$$

for $t$ in some time interval $I \subset \mathbb{R}$. Since then, the subject has developed steadily, and has become established as an effective bridge between analysis, geometry and topology (see for example [13], [14], [15] and the overview in [17]).

The initial progress relevant to the present paper was Hamilton's discovery in 1993 of the so-called Harnack quantities (see [8] for more information) and by 1995, Nolan Wallach [9, §14] had proposed that these quantities should arise as some sort of curvature of some higher-dimensional manifold or bundle associated to the Ricci flow. This idea was developed by Chow and Chu [2] who considered the space-time manifold $\mathcal{M} \times I$, and defined a pair $(\tilde{g}, \widetilde{\nabla})$ of a metric on its cotangent bundle *degenerate* in the time direction and a $\tilde{g}$-compatible torsion-free connection (which is not unique owing to the degeneracy of $\tilde{g}$) so that the derivatives in the time coordinate direction of the components of $(\tilde{g}, \widetilde{\nabla})$ resemble the formulae one can compute for the evolution of the components of the metric and its Levi-Civita connection under Ricci flow. (See [2] for more details.) It turns out that Hamilton's matrix Harnack quadratic is almost the Riemannian curvature of that space-time connection. An improved correspondence is established in the work of Chow and Knopf [4] by considering Ricci flow with a 'cosmological term'. (An example of such a flow would be $\bar{g}(\bar{t}) := \frac{1}{t} g(t)$, for $\bar{t} = \log t$, where $g(t)$ is a Ricci flow.)

In 2002, Perelman [13, §6] made a new breakthrough along these lines involving the construction of an essentially Ricci-flat manifold of dimension unbounded from above, which we now describe. The starting point is a Ricci flow $g(\cdot)$ which once seen with respect to a reverse time parameter $\tau := C - t$ (for some $C \in \mathbb{R}$) is defined for $\tau$ lying in some interval $I \subset \mathbb{R}_+$. Let $N \in \mathbb{N}$ be a large natural number, and consider the manifold $\tilde{\mathcal{M}} := \mathcal{M} \times I \times S^N$ equipped with the metric $\tilde{g}$ defined by

$$\tilde{g}_{ij} = g_{ij}; \quad \tilde{g}_{00} = \frac{N}{2\tau} + R; \quad \tilde{g}_{\alpha\beta} = \tau g_{\alpha\beta},$$

with all remaining metric coefficients $\tilde{g}_{0i}$, $\tilde{g}_{0\alpha}$ and $\tilde{g}_{i\alpha}$ equal to zero, where $i, j$ are coordinate indices on the $\mathcal{M}$ factor, $\alpha, \beta$ are those on the $S^N$ factor, 0 represents the index of the time coordinate $\tau \in I$, the scalar curvature is written $R$, and $g_{\alpha\beta}$ is the metric on the round $S^N$ of sectional curvature $\frac{1}{2N}$.

The significance of the manifold $(\tilde{\mathcal{M}}, \tilde{g})$ is that it is *Ricci-flat* up to errors of order $\frac{1}{N}$. This allowed Perelman to formally apply the Bishop-Gromov comparison theorem in order to discover his *reduced volume* [13]. By setting $\tau = -t$, Hamilton's Harnack quantities [8] can be recovered from the full curvature tensor, up to errors of order $\frac{1}{N}$, although Perelman's construction works for $\tau > 0$ while Harnack estimates hold only for $t > 0$. Indeed, even starting with a Ricci flow having positive curvature operator, the curvature operator of $\tilde{g}$ need *not* have a sign, even ignoring errors.

More recently, the theory of optimal transportation has been introduced into the study of Ricci flow, with papers by McCann and the second author [12], the second author [18] and then Lott [11]. We give more details in Section 3, but for now we mention that a notion of $\mathcal{L}$-optimal transportation was introduced in [18] which can be used to recover all the important monotonic quantities for Ricci flow that were discovered by Perelman [13] in his analysis of finite-time singularities for Ricci flow (see [18] and [11]).

The starting point for this paper is the proposal of John Lott that one might be able to make a formal justification of the results in [18] by applying optimal transportation theory developed for manifolds of positive Ricci curvature, directly to Perelman's construction $(\tilde{\mathcal{M}}, \tilde{g})$ (see also [11]). This seems to be problematic using existing optimal transportation theory. However, consideration of what alternative construction analogous to Perelman's $(\tilde{\mathcal{M}}, \tilde{g})$ could lie behind the results of [18] turns out to be fruitful; in this paper we are thus led to the following theorem in which we construct the *Canonical Shrinking Soliton* associated to a Ricci flow on $\mathcal{M}$.

**Theorem 1.1.** *Suppose $g(\tau)$ is a (reverse) Ricci flow – i.e. a solution of $\frac{\partial g}{\partial \tau} = 2\,\mathrm{Ric}(g(\tau))$ – defined for $\tau$ within a time interval $(a, b) \subset (0, \infty)$, on a manifold $\mathcal{M}$ of dimension $n \in \mathbb{N}$, and with bounded curvature. Suppose $N \in \mathbb{N}$ is sufficiently large to give a positive definite metric $\hat{g}$ on $\hat{\mathcal{M}} := \mathcal{M} \times (a, b)$ defined by*

$$\hat{g}_{ij} = \frac{g_{ij}}{\tau}; \qquad \hat{g}_{00} = \frac{N}{2\tau^3} + \frac{R}{\tau} - \frac{n}{2\tau^2}; \qquad \hat{g}_{0i} = 0,$$

*where $i, j$ are coordinate indices on the $\mathcal{M}$ factor, 0 represents the index of the time coordinate $\tau \in (a, b)$, and the scalar curvature of $g$ is written as $R$.*

*Then up to errors of order $\frac{1}{N}$, the metric $\hat{g}$ is a gradient shrinking Ricci soliton on the higher dimensional space $\hat{\mathcal{M}}$:*

$$\mathrm{Ric}(\hat{g}) + \mathrm{Hess}_{\hat{g}}\left(\frac{N}{2\tau}\right) \simeq \frac{1}{2}\hat{g}, \tag{1.2}$$

*by which we mean that the quantity*

$$N\left[\mathrm{Ric}(\hat{g}) + \mathrm{Hess}_{\hat{g}}\left(\frac{N}{2\tau}\right) - \frac{1}{2}\hat{g}\right]$$

*is locally bounded independently of $N$, with respect to any fixed metric on $\hat{\mathcal{M}}$.*

It is well known (see for example [17, §1.2.2]) that a Ricci soliton metric on a manifold $\hat{\mathcal{M}}$ induces a self-similar Ricci flow on $\hat{\mathcal{M}}$. In our context this indicates that given a Ricci flow on $\mathcal{M}$ over the time interval $(a, b)$, it is then natural to introduce an additional time parameter and consider Ricci flow using $\mathcal{M} \times (a, b)$ as the underlying manifold, and we adopt this viewpoint in Section 4.

This theorem serves as an example of an application of the theory of optimal transportation to yield a new result in a different field. The route between the theorems of optimal transportation and this construction is described in Sections 3 and 4. It will become apparent, from Section 4 and [18], that our Canonical Shrinking Soliton encodes various monotonic quantities which underpin Perelman's work on Ricci flow [13, 14, 15], including entropies and quantities involving $\mathcal{L}$-length (see Section 3).

**Remark 1.2.** Our construction encodes Hamilton's Harnack quantities in the same sense as does Perelman's construction. More precisely, the components of the $(4, 0)$ curvature tensor $\tau Rm(\hat{g})$ coincide (up to errors of order $\frac{1}{N}$) with the components of the matrix Harnack expression [8, 13] after setting $t = -\tau$.

One advantage of the notion of Canonical Soliton over the construction of Perelman is that we can use it to find and prove new (and old) Harnack inequalities. Indeed, in [1] we will extend the ideas of this paper to give Canonical *Expanding* Solitons in order to achieve this. A notion of Canonical Steady Soliton, discussed in Section 5, completes the picture.

In Section 6 we investigate a refinement of this construction in which we combine the Ricci flow on $\mathcal{M} \times (a, b)$ suggested by our construction with mean curvature flow of time slices.

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## 2 The calculations

In this section, we give exact formulae for the Christoffel symbols of $\hat{g}$ from Theorem 1.1, and approximate formulae for its Ricci curvatures and for the Hessian of $\frac{N}{2\tau}$ with respect to $\hat{g}$. Theorem 1.1 will then be seen to follow easily.

**Proposition 2.1.** *In the setting of Theorem 1.1, if $\Gamma^i_{jk}$ are the Christoffel symbols of $g(\tau)$ at some point $x \in \mathcal{M}$, then the Christoffel symbols of $\hat{g}$ at $(x, \tau)$ are given by*

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk}; \quad \hat{\Gamma}^i_{j0} = R^i{}_j - \frac{\delta^i{}_j}{2\tau}; \quad \hat{\Gamma}^i_{00} = -\frac{1}{2} g^{ij} \frac{\partial R}{\partial x^j};$$

$$\hat{\Gamma}^0_{jk} = \hat{g}_{00}^{-1} \left( \frac{g_{jk}}{2\tau^2} - \frac{R_{jk}}{\tau} \right); \quad \hat{\Gamma}^0_{i0} = \frac{1}{2\tau} \hat{g}_{00}^{-1} \frac{\partial R}{\partial x^i}; \quad \hat{\Gamma}^0_{00} = \frac{1}{2} \hat{g}_{00}^{-1} \left[ -\frac{3N}{2\tau^4} - \frac{R}{\tau^2} + \frac{R_\tau}{\tau} + \frac{n}{\tau^3} \right]$$

This is a straightforward computation from the definition of the Christoffel symbols

$$\hat{\Gamma}^a_{bc} := \frac{1}{2} \hat{g}^{ad} \left( \frac{\partial \hat{g}_{cd}}{\partial x^b} + \frac{\partial \hat{g}_{bd}}{\partial x^c} - \frac{\partial \hat{g}_{bc}}{\partial x^d} \right),$$

where $a, b, c, d$ are arbitrary indices, and the equation of Ricci flow. Using the standard formula for the coefficients of the Ricci curvature

$$\hat{R}_{ab} = \frac{\partial \hat{\Gamma}^c_{ab}}{\partial x^c} - \frac{\partial \hat{\Gamma}^c_{ac}}{\partial x^b} + \hat{\Gamma}^c_{ab}\hat{\Gamma}^d_{cd} - \hat{\Gamma}^d_{ac}\hat{\Gamma}^c_{bd},$$

the formula for the coefficients of $\mathrm{Hess}_{\hat{g}}(f)$

$$\hat{\nabla}^2_{ab}(f) = \frac{\partial^2 f}{\partial x^a \partial x^b} - \frac{\partial f}{\partial x^c}\hat{\Gamma}^c_{ab},$$

the equation for the evolution of $R$

$$R_\tau + \Delta R + 2|\mathrm{Ric}|^2 = 0,$$

(see for example [17, Proposition 2.5.4]) and the contracted Bianchi identity

$$\nabla_i R^i{}_j = \frac{1}{2}\nabla_j R,$$

one readily verifies the following:

**Proposition 2.2.** *Fixing $\tau > 0$, a time at which the Ricci flow exists, and fixing local coordinates $\{x^i\}$ in a neighbourhood $U$ of some $p \in \mathcal{M}$, then in any neighbourhood $V \subset\subset U \times (a, b)$ of $(p, \tau)$, we have*

$$\hat{R}_{ij} \simeq R_{ij}; \qquad \hat{R}_{i0} \simeq -\frac{1}{2}\nabla_i R; \qquad \hat{R}_{00} \simeq -\frac{R_\tau}{2} - \frac{R}{2\tau},$$

*where $\simeq$ denotes equality of the coefficients up to an error bounded in magnitude by $\frac{C}{N}$, with $C > 0$ a constant independent of $N$ (but depending on $V$ and the choice of coordinates). Moreover, we have*

$$\hat{\nabla}^2_{ij}(\tfrac{N}{2\tau}) \simeq \frac{g_{ij}}{2\tau} - R_{ij}; \qquad \hat{\nabla}^2_{i0}(\tfrac{N}{2\tau}) \simeq \frac{\nabla_i R}{2}; \qquad \hat{\nabla}^2_{00}(\tfrac{N}{2\tau}) \simeq \frac{N}{4\tau^3} + \frac{R}{\tau} - \frac{n}{4\tau^2} + \frac{R_\tau}{2}.$$

By combining the formulae of Proposition 2.2 and the definition of $\hat{g}$, we deduce Theorem 1.1.

**Remark 2.3.** Although each side of (1.2) evaluated on the pair $(\frac{\partial}{\partial\tau}, \frac{\partial}{\partial\tau})$ will have magnitude of order $N$, the theorem tells us that their difference does not even have any terms of order 1, only those of order $\frac{1}{N}$. All errors disappear when the original Ricci flow $g(\tau)$ is a homothetically shrinking Einstein manifold, shrinking to nothing at $\tau = 0$.

# 3 Optimal Transportation on Ricci flows

Suppose that $(\mathcal{M}, g)$ is a closed (compact, no boundary) Riemannian manifold, and $\nu_1$ and $\nu_2$ are two Borel probability measures on $\mathcal{M}$. For $p \in [1, \infty)$, we define the $p$-Wasserstein distance $W_p$ between $\nu_1$ and $\nu_2$ to be

$$W^g_p(\nu_1, \nu_2) := \left[\inf_{\pi\in\Gamma(\nu_1,\nu_2)} \int_{\mathcal{M}\times\mathcal{M}} d^p(x, y) d\pi(x, y)\right]^{\frac{1}{p}}, \tag{3.1}$$

where $d(\cdot, \cdot)$ is the Riemannian distance function induced by $g$ and $\Gamma(\nu_1, \nu_2)$ is the space of Borel probability measures on $\mathcal{M} \times \mathcal{M}$ with marginals $\nu_1$ and $\nu_2$.

A basic principle in the subject – see Sturm and von Renesse [16] and the references therein – is that two probability measures evolving under an appropriate diffusion equation should get closer in the Wasserstein sense provided the manifold satisfies some curvature condition, most famously positive Ricci curvature.

In [12], McCann and the second author showed that this type of contractivity on an *evolving* manifold $(\mathcal{M}, g(\tau))$ *characterises* super-solutions of the Ricci flow (parametrised backwards in time) by which we mean solutions to

$$\frac{\partial g}{\partial \tau} \leq 2\,\mathrm{Ric}(g(\tau)). \tag{3.2}$$

Here the relevant notion of diffusion on an evolving manifold $(\mathcal{M}, g(\tau))$ moves the probability density $u$ (with respect to the evolving Riemannian measure $\mu_{g(\tau)}$) by the parabolic equation

$$\frac{\partial u}{\partial \tau} = \Delta_{g(\tau)} u - \frac{1}{2}\mathrm{tr}\left(\frac{\partial g}{\partial \tau}\right) u,$$

and we refer to the resulting one-parameter families of measures simply as *diffusions*. This way, if we define a local top-dimensional form $\omega(\tau) := u\, dV_{g(\tau)}$, then

$$\frac{\partial \omega}{\partial \tau} = \Delta_{g(\tau)} \omega, \tag{3.3}$$

where $\Delta$ is here the connection Laplacian, and thus the evolution of the measures corresponds to Brownian motion. The following characterisation was proved for the $W_2$ distance in [12]. Tom Ilmanen has pointed out to us (via Robert McCann) that this extends to the case of $W_1$ distance, giving:

**Theorem 3.1.** *(cf. [12, Theorem 2]) Suppose that $\mathcal{M}$ is a closed manifold equipped with a smooth family of metrics $g(\tau)$ for $\tau \in [\tau_1, \tau_2] \subset \mathbb{R}$. Then the following are equivalent:*

*(A) $g(\tau)$ is a super Ricci flow (i.e. satisfies (3.2));*

*(B) whenever $\tau_1 < a < b < \tau_2$ and $\nu_1(\tau)$, $\nu_2(\tau)$ are diffusions (as defined above) for $\tau \in (a, b)$, the function $\tau \mapsto W_1^{g(\tau)}(\nu_1(\tau), \nu_2(\tau))$ is weakly decreasing in $\tau \in (a, b)$;*

*(C) whenever $\tau_1 < a < b < \tau_2$ and $f : \mathcal{M} \times (a, b) \to \mathbb{R}$ is a solution to $-\frac{\partial f}{\partial \tau} = \Delta_{g(\tau)} f$, the Lipschitz constant of $f(\cdot, \tau)$ with respect to $g(\tau)$ is weakly increasing in $\tau$.*

*Proof.* All implications in this theorem are proved exactly as in [12] except for $(A) \implies (B)$ whose proof, pointed out by Ilmanen, we now describe. Suppose that $g(\tau)$ is a super Ricci flow, and that $\nu_1(\tau)$ and $\nu_2(\tau)$ are two diffusions, all defined for $\tau$ in a neighbourhood of $\tau_0 \in (a, b)$. By Kantorovich-Rubinstein duality (see e.g. [19, §1.2.1]) we have

$$W_1^{g(\tau)}(\nu_1(\tau), \nu_2(\tau)) := \max\left\{ \int_{\mathcal{M}} \varphi d\nu_1(\tau) - \int_{\mathcal{M}} \varphi d\nu_2(\tau) \,\middle|\, \varphi : \mathcal{M} \to \mathbb{R} \text{ is Lipschitz} \right.$$
$$\left. \text{and } \|\varphi\|_{Lip} \leq 1 \text{ with respect to } g(\tau) \right\}. \tag{3.4}$$

Let $\varphi_0 : \mathcal{M} \to \mathbb{R}$ be a function which achieves the maximum in this variational problem at time $\tau_0$, and extend $\varphi_0$ to a function $\varphi : \mathcal{M} \times [\tau_0 - \varepsilon, \tau_0] \to \mathbb{R}$ for some $\varepsilon > 0$ by solving the equation

$$-\frac{\partial \varphi}{\partial \tau} = \Delta_{g(\tau)} \varphi.$$

By the implication $(A) \implies (C)$ of the theorem (proved in [12]) for all $\tau \in [\tau_0 - \varepsilon, \tau_0]$ we have $\|\varphi(\cdot, \tau)\|_{Lip} \leq 1$ and therefore $\varphi(\cdot, \tau)$ can be used as a competitor in the variational problem (3.4) to see that

$$W_1^{g(\tau)}(\nu_1(\tau), \nu_2(\tau)) \geq \int_{\mathcal{M}} \varphi(\cdot, \tau) d\nu_1(\tau) - \int_{\mathcal{M}} \varphi(\cdot, \tau) d\nu_2(\tau).$$

But by an integration by parts formula [17, §6.3] we know that the functions

$$\tau \mapsto \int_{\mathcal{M}} \varphi(\cdot, \tau) d\nu_1(\tau) \qquad \text{and} \qquad \tau \mapsto \int_{\mathcal{M}} \varphi(\cdot, \tau) d\nu_2(\tau)$$

are each independent of $\tau$, so we deduce that

$$W_1^{g(\tau)}(\nu_1(\tau), \nu_2(\tau)) \geq W_1^{g(\tau_0)}(\nu_1(\tau_0), \nu_2(\tau_0)).$$

□

In the next section, we will demonstrate how the manifold $(\hat{\mathcal{M}}, \hat{g})$ of Theorem 1.1 arises naturally by trying to reconcile Theorem 3.1 with the main result proved by the second author in [18], which we now rephrase into the most suggestive form for our present purposes. The idea of $\mathcal{L}$-optimal transportation [18] is to transport a probability measure from one time slice of a Ricci flow to another, using a cost function derived from Perelman's $\mathcal{L}$-length. More precisely, given a time interval $[\tau_1, \tau_2] \subset (0, \infty)$ in the domain of definition of the Ricci flow, we consider the cost function $c : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ induced by the Lagrangian $L(x, v, \tau) := \sqrt{\tau}(R(x, \tau) + |v|^2 - \frac{n}{2\tau})$ which gives

$$c(x, y) = \inf_{\gamma} \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left( R(\gamma(\tau), \tau) + |\gamma'(\tau)|^2 - \frac{n}{2\tau} \right) d\tau,$$

where the infimum is taken over all $C^1$ curves $\gamma : [\tau_1, \tau_2] \to \mathcal{M}$ for which $\gamma(\tau_1) = x$ and $\gamma(\tau_2) = y$. Using the definitions from [13, §7] and [18]

$$\mathcal{L}(\gamma) = \int_{\tau_1}^{\tau_2} \sqrt{\tau}(R(\gamma(\tau), \tau) + |\gamma'(\tau)|^2) d\tau; \qquad Q(x, \tau_1; y, \tau_2) = \inf_{\gamma} \mathcal{L}(\gamma),$$

where the infimum is again over curves $\gamma$ as above, we can write

$$c(x, y) = Q(x, \tau_1; y, \tau_2) - n(\sqrt{\tau_2} - \sqrt{\tau_1}).$$

This cost function then induces a distance from one Borel probability measure $\nu_1$ (viewed as existing at time $\tau_1$) to another $\nu_2$ (viewed at time $\tau_2$) via the formula

$$\begin{aligned} \mathcal{D}(\nu_1, \tau_1; \nu_2, \tau_2) &= \inf_{\pi \in \Gamma(\nu_1, \nu_2)} \int_{\mathcal{M} \times \mathcal{M}} Q(x, \tau_1; y, \tau_2) d\pi(x, y) - n(\sqrt{\tau_2} - \sqrt{\tau_1}) \\ &=: V(\nu_1, \tau_1; \nu_2, \tau_2) - n(\sqrt{\tau_2} - \sqrt{\tau_1}), \end{aligned} \tag{3.5}$$

using $V$ as defined in [18]. From the following theorem, one can recover [18] most of Perelman's monotonic quantities (both involving entropies and $\mathcal{L}$-length) which are central in his work on Ricci flow [13, 14, 15].

**Theorem 3.2.** *(Equivalent to [18, Theorem 1.1].) Suppose that $g(\tau)$ is a Ricci flow on a closed manifold $\mathcal{M}$ over an open time interval containing $[\bar{\tau}_1, \bar{\tau}_2]$, and suppose that $\nu_1(\tau)$ and $\nu_2(\tau)$ are two diffusions (in the same sense as in Theorem 3.1) defined for $\tau$ in neighbourhoods of $\bar{\tau}_1$ and $\bar{\tau}_2$ respectively. Then the distance between the diffusions decays in the sense that for $s \geq 1$ sufficiently close to* 1,

$$\mathcal{D}(\nu_1(s\bar{\tau}_1), s\bar{\tau}_1; \nu_2(s\bar{\tau}_2), s\bar{\tau}_2) \leq s^{-\frac{1}{2}} \mathcal{D}(\nu_1(\bar{\tau}_1), \bar{\tau}_1; \nu_2(\bar{\tau}_2), \bar{\tau}_2).$$

This formulation of the theorem indicated to us that we should look for a context in which the result arises as an application of Theorem 3.1. This leads to Theorem 1.1, and we explain the connection in the next section.

# 4 The relationship between $(\hat{\mathcal{M}}, \hat{g})$ and $\mathcal{L}$-optimal transportation

In this section, as opposed to all others in this paper, we allow ourselves to make purely heuristic arguments. We present a formal argument to recover Theorem 3.2 by applying Theorem 3.1 to a flow starting at the specific manifold $(\hat{\mathcal{M}}, \hat{g})$ of Theorem 1.1. While this is the simplest way to explain the link between our new manifold $(\hat{\mathcal{M}}, \hat{g})$ and optimal transportation, we stress that the main interest here is the reverse flow of ideas: the Ricci soliton $(\hat{\mathcal{M}}, \hat{g})$ was *discovered* by trying to find a context in which Theorem 3.1 formally implied Theorem 3.2.

We begin this section by arguing formally that shortest paths in $(\hat{\mathcal{M}}, \hat{g})$ correspond to $\mathcal{L}$-geodesics for the original Ricci flow. Suppose $x, y \in \mathcal{M}$ and $[\tau_1, \tau_2] \subset (0, \infty)$ lies within the time domain on which the Ricci flow is defined. Consider paths $\Gamma : [\tau_1, \tau_2] \to \hat{\mathcal{M}}$ connecting $(x, \tau_1)$ and $(y, \tau_2)$ in $\hat{\mathcal{M}}$ of the form $\Gamma(\tau) = (\gamma(\tau), \tau)$, where $\gamma : [\tau_1, \tau_2] \to \mathcal{M}$ satisfies $\gamma(\tau_1) = x$ and $\gamma(\tau_2) = y$. Then

$$\begin{aligned} Length(\Gamma) &= \int_{\tau_1}^{\tau_2} \left| \gamma'(\tau) + \frac{\partial}{\partial \tau} \right|_{\hat{g}} d\tau \\ &= \int_{\tau_1}^{\tau_2} \left[ \frac{|\gamma'|^2_{g(\tau)}}{\tau} + \frac{N}{2\tau^3} + \frac{R}{\tau} - \frac{n}{2\tau^2} \right]^{\frac{1}{2}} d\tau \\ &= \int_{\tau_1}^{\tau_2} \left[ \frac{N}{2\tau^3} \right]^{\frac{1}{2}} \left( 1 + \frac{\tau^2 |\gamma'|^2}{N} + \frac{\tau^2 R}{N} - \frac{\tau n}{2N} + O(\tfrac{1}{N^2}) \right) d\tau \\ &= \sqrt{2N}(\tau_1^{-\frac{1}{2}} - \tau_2^{-\frac{1}{2}}) + \frac{1}{\sqrt{2N}} \left[ \mathcal{L}(\gamma) - n(\sqrt{\tau_2} - \sqrt{\tau_1}) \right] + O(\tfrac{1}{N^{3/2}}). \end{aligned} \tag{4.1}$$

Just as in [13, §6], this indicates that minimising paths $\Gamma$ should essentially arise from $\mathcal{L}$-geodesics $\gamma$ for large $N$, and suggests the formula

$$d_{\hat{g}}((x, \tau_1), (y, \tau_2)) = \sqrt{2N}(\tau_1^{-\frac{1}{2}} - \tau_2^{-\frac{1}{2}}) + \frac{1}{\sqrt{2N}} \left[ Q(x, \tau_1; y, \tau_2) - n(\sqrt{\tau_2} - \sqrt{\tau_1}) \right] + O(\tfrac{1}{N^{3/2}}). \tag{4.2}$$

We now wish to apply Theorem 3.1 to certain diffusions on a reverse Ricci flow $G(s)$ on *space-time* starting at time $s = 1$ with the metric $G(1) = \hat{g}$ on $\hat{\mathcal{M}} = \mathcal{M} \times (a, b)$. Since $\hat{g}$ satisfies the (approximate) Ricci soliton equation (1.2), the theory of Ricci solitons (see [17, §1.2.2]) tells us that (modulo errors) we can take $G(s)$ to be

$$G(s) := s\, \psi_s^*(\hat{g}) \tag{4.3}$$

where $\psi_1 : \hat{\mathcal{M}} \to \hat{\mathcal{M}}$ is the identity, and $\psi_s$ is the family of maps obtained by integrating the vector field $X_s := -\frac{1}{s}\hat{\nabla}(\frac{N}{2\tau})$ with $\hat{\nabla}$ representing the gradient with respect to $\hat{g}$. We may compute

$$X_s = \frac{N}{2s\tau^2} \hat{g}_{00}^{-1} \frac{\partial}{\partial \tau} = \frac{\tau}{s} \frac{\partial}{\partial \tau} + O(\tfrac{1}{N}), \tag{4.4}$$

and by neglecting the error of order $\frac{1}{N}$ (but see also Section 6) this integrates to

$$\psi_s(x, \tau) \simeq (x, s\tau).$$

In particular, the (approximate, reverse) Ricci flow $G(s)$ operates by pulling back $\hat{g}$ in time, and scaling appropriately.

Applying Theorem 3.1, we find that two diffusions on the evolving manifold with metric $G(s)$ should get closer in the $W_1$ sense as $s$ increases. The Ricci flow $g(\tau)$ we use to generate $(\hat{\mathcal{M}}, \hat{g})$ will be that in the hypotheses of Theorem 3.2. The measures we wish to put into Theorem 3.1 will be derived from the measures $\nu_1(\tau)$ and $\nu_2(\tau)$ appearing in the hypotheses of Theorem 3.2, together with the corresponding $\bar{\tau}_1$ and $\bar{\tau}_2$. Let $F_\tau : \mathcal{M} \to \hat{\mathcal{M}}$ be defined to be the embedding $F_\tau(x) = (x, \tau)$; then the initial measures we wish to put into Theorem 3.1 are $(F_{\bar{\tau}_k})_\# \nu_k(\bar{\tau}_k)$, for $k = 1, 2$, which are each supported on a time-slice in $\hat{\mathcal{M}}$. Because of the extreme stretching of the $\tau$ direction in the metric $\hat{g}$ (recall that $\hat{g}_{00}$ is of order $N$) there is essentially no diffusion of the measures in the $\tau$ direction, and we view them as remaining supported in the time slices $\mathcal{M} \times \{\bar{\tau}_k\}$. They then evolve mainly under diffusion in the $\mathcal{M}$ factor, under the Laplacian induced by $G_{ij}(s)$. By (4.3) and the definition of $\hat{g}$ from Theorem 1.1, on the time-slice $\mathcal{M} \times \{\bar{\tau}_k\}$ we have (approximately)

$$G_{ij}(s) = s[\psi_s^*(\hat{g})]_{ij} = s[\hat{g}|_{\mathcal{M}\times\{s\bar{\tau}_k\}}]_{ij} = \frac{g_{ij}(s\bar{\tau}_k)}{\bar{\tau}_k}.$$

Therefore, the measures evolve by diffusion in the $\mathcal{M}$ factor under

$$\Delta_{\frac{g(s\bar{\tau}_k)}{\bar{\tau}_k}} = \bar{\tau}_k \Delta_{g(s\bar{\tau}_k)},$$

which is also the evolution of $s \mapsto \nu_k(s\bar{\tau}_k)$. In other words, we have (formally) deduced that

$$s \mapsto W_1^{G(s)}((F_{\bar{\tau}_1})_\# \nu_1(s\bar{\tau}_1), (F_{\bar{\tau}_2})_\# \nu_2(s\bar{\tau}_2))$$

is weakly decreasing.

If we now push forward this whole construction under the maps $\psi_s$, and adopt the abbreviation

$$\hat{\nu}_k(\tau) := (F_\tau)_\# \nu_k(\tau),$$

then we find that

$$s \mapsto W_1^{s\hat{g}}(\hat{\nu}_1(s\bar{\tau}_1), \hat{\nu}_2(s\bar{\tau}_2)) = s^{\frac{1}{2}} W_1^{\hat{g}}(\hat{\nu}_1(s\bar{\tau}_1), \hat{\nu}_2(s\bar{\tau}_2)) \tag{4.5}$$

is weakly decreasing.

Now the expansion (4.2) of the Riemannian distance on $(\hat{\mathcal{M}}, \hat{g})$ suggests that

$$W_1^{\hat{g}}(\hat{\nu}_1(\tau_1), \hat{\nu}_2(\tau_2)) \simeq \sqrt{2N}(\tau_1^{-\frac{1}{2}} - \tau_2^{-\frac{1}{2}}) + \frac{1}{\sqrt{2N}} \mathcal{D}(\nu_1(\tau_1), \tau_1; \nu_2(\tau_2), \tau_2).$$

Therefore, the monotonicity in (4.5) implies that

$$s \mapsto \sqrt{2N}(\bar{\tau}_1^{-\frac{1}{2}} - \bar{\tau}_2^{-\frac{1}{2}}) + \frac{s^{\frac{1}{2}}}{\sqrt{2N}} \mathcal{D}(\nu_1(s\bar{\tau}_1), s\bar{\tau}_1; \nu_2(s\bar{\tau}_2), s\bar{\tau}_2)$$

is monotonically decreasing, and hence so is

$$s \mapsto s^{\frac{1}{2}} \mathcal{D}(\nu_1(s\bar{\tau}_1), s\bar{\tau}_1; \nu_2(s\bar{\tau}_2), s\bar{\tau}_2),$$

which is the content of Theorem 3.2 as desired.

## 5 Construction of a space-time steady soliton

In this section we make a steady soliton construction analogous to the shrinking soliton construction of Theorem 1.1. There is also an expanding soliton construction similar to that of Theorem 1.1 which we will use in [1] to prove Harnack inequalities. Only the shrinking case is adapted to Perelman's $\mathcal{L}$-length and his monotonic quantities which are so important in the study of finite-time singularities [13, 14, 15]. However, the steady case is the simplest construction of all and has its own potential applications.

**Theorem 5.1.** *Suppose $g(\tau)$ is a (reverse) Ricci flow – i.e. a solution of $\frac{\partial g}{\partial \tau} = 2\,\mathrm{Ric}(g(\tau))$ – defined for $\tau$ within a time interval $(a, b) \subset \mathbb{R}$, on a manifold $\mathcal{M}$ of dimension $n \in \mathbb{N}$, and with bounded curvature. Suppose $N \in \mathbb{N}$ is sufficiently large to give a positive definite metric $\bar{g}$ on $\hat{\mathcal{M}} := \mathcal{M} \times (a, b)$ defined by*

$$\bar{g}_{ij} = g_{ij}; \qquad \bar{g}_{00} = N + R; \qquad \bar{g}_{0i} = 0,$$

*where $i, j$ are coordinate indices on the $\mathcal{M}$ factor and $0$ represents the index of the time coordinate $\tau \in (a, b)$.*

*Then up to errors of order $\frac{1}{N}$, the metric $\bar{g}$ is a gradient steady Ricci soliton on $\hat{\mathcal{M}}$:*

$$\mathrm{Ric}(\bar{g}) + \mathrm{Hess}_{\bar{g}}(-N\tau) \simeq 0, \tag{5.1}$$

*in the same sense as in Theorem 1.1.*

For this metric, the Christoffel symbols can be computed to be

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk}; \quad \bar{\Gamma}^i_{j0} = R^i{}_j; \quad \bar{\Gamma}^i_{00} = -\frac{1}{2} g^{ik} \frac{\partial R}{\partial x^k};$$

$$\bar{\Gamma}^0_{jk} = -\frac{1}{N+R} R_{jk}; \quad \bar{\Gamma}^0_{j0} = \frac{1}{2} \frac{\partial}{\partial x^j} \ln(N+R); \quad \bar{\Gamma}^0_{00} = \frac{1}{2} \frac{\partial}{\partial \tau} \ln(N+R).$$

The Ricci coefficients are, up to errors of order $\frac{1}{N}$,

$$\bar{R}_{ij} \simeq R_{ij}; \qquad \bar{R}_{i0} \simeq -\frac{\nabla_i R}{2}; \qquad \bar{R}_{00} \simeq -\frac{R_\tau}{2},$$

and the coefficients of $\mathrm{Hess}_{\bar{g}}(-N\tau)$ are

$$\bar{\nabla}^2_{ij}(-N\tau) \simeq -R_{ij}; \qquad \bar{\nabla}^2_{i0}(-N\tau) \simeq \frac{\nabla_i R}{2}; \qquad \bar{\nabla}^2_{00}(-N\tau) \simeq \frac{R_\tau}{2},$$

which yields Theorem 5.1.

Just as the $(\hat{\mathcal{M}}, \hat{g})$ of Theorem 1.1 encodes Perelman's $\mathcal{L}$-length in its geodesic distance, in the sense of (4.1), the manifold $(\hat{\mathcal{M}}, \bar{g})$ of Theorem 5.1 encodes Li-Yau's length [10]

$$\mathcal{L}_0(\gamma) := \int_{\tau_1}^{\tau_2} (R(\gamma(\tau), \tau) + |\gamma'(\tau)|^2) d\tau$$

which predates $\mathcal{L}(\gamma)$, via the formula (using the same notation as in (4.1))

$$Length(\Gamma) = \sqrt{N}(\tau_2 - \tau_1) + \frac{1}{2\sqrt{N}} \mathcal{L}_0(\gamma) + O(\tfrac{1}{N^{3/2}}).$$

If one considers optimal transportation on this alternative $(\hat{\mathcal{M}}, \bar{g})$, then one recovers the variant of $\mathcal{L}$-optimal transportation using the $\mathcal{L}_0$-length of Li-Yau, as studied in [11].

# 6 Mean curvature of time-slices

In the framework of Theorem 1.1, the underlying manifold $\mathcal{M}$ of our original Ricci flow can be regarded as a hypersurface $\mathcal{M} \times \{\tau\}$ of the ambient $(\hat{\mathcal{M}}, \hat{g})$. Its mean curvature is $H = H^{\hat{g}}_{\mathcal{M}\times\{\tau\}} = \hat{g}_{00}^{-\frac{1}{2}}(R - \frac{n}{2\tau})$; in fact, the mean curvature vector $\vec{H}^{\hat{g}}_{\mathcal{M}\times\{\tau\}} := -H\nu =$

$-H\hat{g}_{00}^{-\frac{1}{2}}\frac{\partial}{\partial\tau}$ gives precisely the term we neglected in the approximate computation of $X_s$ from (4.4). More precisely, we have the exact formula

$$-\hat{\nabla}\left(\frac{N}{2\tau}\right)=\tau\frac{\partial}{\partial\tau}+\vec{H}^{\hat{g}}_{\mathcal{M}\times\{\tau\}}. \tag{6.1}$$

If we suppose that $\tau_0\in U\subset\subset(a,b)$, we can pick $\varepsilon>0$ sufficiently small so that for $s\in(1-\varepsilon,1+\varepsilon)$ the maps $\psi_s$ arising from integrating the vector field

$$X_s:=-\frac{1}{s}\hat{\nabla}(\frac{N}{2\tau})$$

starting with $\psi_1:\hat{\mathcal{M}}\to\hat{\mathcal{M}}$ the identity, restrict to well-defined maps $\mathcal{M}\times U\mapsto\hat{\mathcal{M}}$, diffeomorphic onto their images. It then makes sense at a rigorous level to define $G(s):=s\,\psi_s^*(\hat{g})$ as a flow on $\mathcal{M}\times U$ for $s\in(1-\varepsilon,1+\varepsilon)$, which is then approximately a reverse Ricci flow as in Section 4. By reducing $\varepsilon>0$ further, we can also be sure that $\mathcal{M}\times\{s\tau_0\}\subset\psi_s(\mathcal{M}\times U)$ for $s\in(1-\varepsilon,1+\varepsilon)$.

We then have the following exact statement (i.e. involving no errors at all) relating reverse mean curvature flow to certain one-parameter families of time-slices $\mathcal{M}\times\{\tau\}$, which is in a similar spirit to [3, Lemma 3.2].

**Theorem 6.1.** *Given $U$, $\tau_0$, $\varepsilon$, $\psi_s$ and the flow $G(s)$ as above, if we define $\mathcal{F}:\mathcal{M}\times(1-\varepsilon,1+\varepsilon)\to\mathcal{M}\times U$ by $\mathcal{F}(x,s)=\psi_s^{-1}(x,s\tau_0)$, then the one-parameter family of submanifolds*

$$\mathcal{N}_s:=\mathcal{F}(\mathcal{M}\times\{s\})=\psi_s^{-1}(\mathcal{M}\times\{s\tau_0\})$$

*is a reverse mean curvature flow within the flow $G(s)$ in the sense that*

$$-\frac{\partial\mathcal{F}(x,s)}{\partial s}=\vec{H}^{G(s)}_{\mathcal{N}_s}(\mathcal{F}(x,s)).$$

Note that here both the (approximate) Ricci flow $G(s)$ and the mean curvature flow $\mathcal{N}_s$ evolve in the same direction – backwards with respect to $s$.

*Proof.* By differentiating the expression $\psi_s\circ\psi_s^{-1}=$ *identity* with respect to $s$, we find that

$$\begin{aligned}(\psi_s)_*\frac{\partial\psi_s^{-1}}{\partial s}(x,s\tau_0)&=-\frac{\partial\psi_s}{\partial s}(\psi_s^{-1}(x,s\tau_0))=-X_s(x,s\tau_0)=\frac{1}{s}\hat{\nabla}\left(\frac{N}{2\tau}\right)(x,s\tau_0)\\&=-\tau_0\frac{\partial}{\partial\tau}-\frac{1}{s}\vec{H}^{\hat{g}}_{\mathcal{M}\times\{s\tau_0\}}(x,s\tau_0),\end{aligned} \tag{6.2}$$

where we have used (6.1). Therefore, we may compute

$$\begin{aligned}\frac{\partial}{\partial s}\left(\psi_s^{-1}(x,s\tau_0)\right)&=\frac{\partial\psi_s^{-1}}{\partial s}(x,s\tau_0)+(\psi_s^{-1})_*\left(\tau_0\frac{\partial}{\partial\tau}\right)=-(\psi_s^{-1})_*\left[\frac{1}{s}\vec{H}^{\hat{g}}_{\mathcal{M}\times\{s\tau_0\}}(x,s\tau_0)\right]\\&=-\frac{1}{s}\vec{H}^{\psi_s^*(\hat{g})}_{\mathcal{N}_s}(\psi_s^{-1}(x,s\tau_0))=-\vec{H}^{G(s)}_{\mathcal{N}_s}(\mathcal{F}(x,s)).\end{aligned}$$

□

We conclude with an observation about mean curvature flow of one dimension less. Suppose that $(\mathcal{M},g(\tau))$ is a reverse Ricci flow for $\tau\in(a,b)$ and that for some $(n-1)$-dimensional manifold $P$, the map $\sigma:P\times(a,b)\to\mathcal{M}$ is a reverse mean curvature flow in the sense that $P_\tau:=\sigma(P\times\{\tau\})$ is a family of smooth hypersurfaces of $\mathcal{M}$ and

$$-\frac{\partial\sigma}{\partial\tau}(x,\tau)=\vec{H}^{g(\tau)}_{P_\tau}(\sigma(x,\tau)).$$

Then the space-time track, which is the image of $\Sigma : P \times (a, b) \to \hat{\mathcal{M}}$ defined by

$$\Sigma(x, \tau) := (\sigma(x, \tau), \tau),$$

is $\phi$-minimal in $(\hat{\mathcal{M}}, \hat{g})$ for $\phi = \frac{N}{2\tau}$, up to errors of order $\frac{1}{N}$, in the sense that

$$H_\phi := H^{\hat{g}}_{\Sigma(P\times(a,b))} - \langle \hat{\nabla}\phi, \nu_\Sigma \rangle = O(N^{-1}),$$

where $\nu_\Sigma$ is the unit outward normal to $\Sigma(P \times (a, b))$ and we have used the natural generalization of mean curvature introduced in [6]. (See also Chapter 11 §3.10 of [5] for the corresponding property of Perelman's metric $\tilde{g}$ described in Section 1.)

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MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL, UK
http://www.warwick.ac.uk/staff/E.Cabezas-Rivas
http://www.warwick.ac.uk/~maseq