MA135—VECTORS AND MATRICES
EXAMPLE SHEET 4

All questions in Sections A and B must be handed in to your supervisor via the supervisor's pigeon loft by **3pm Monday, Week 5.** Section C questions should be **attempted** by students who hope to get a 1st or 2:1 degree, but are not to be handed in.

Section A

A1 Let Π be the plane in $\mathbb{R}^3$ given in standard form by $x + y + z = 1$. Write the vector equation and the point-normal form equation for Π.

A2 Let Π_1, Π_2 be the two planes in $\mathbb{R}^3$ given by

$$\Pi_1 \; : \; x + y + z = 1, \qquad \Pi_2 \; : \; x - 2y - z = 0.$$

Find the vector equation of the straight line given by the intersection of Π_1 and Π_2.

A3 Find the vector equation of the plane in $\mathbb{R}^3$ passing through the point $(0, 1, 1)$ and containing the line $L \; : \; \mathbf{x} = (1, 0, 0) + t(0, 0, 1)$.

A4 Calculate $B = (b_{ij})_{3\times 4}$ where $b_{ij} = \begin{cases} ij & \text{if } i \leq j \\ i + j & \text{otherwise.} \end{cases}$

A5 Let

$$A = \begin{pmatrix} -2 & 5 & -3 \\ 4 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 3 \\ -2 & 2 \end{pmatrix},$$

$$C = \begin{pmatrix} -3 & 2 & 0 \\ 4 & 5 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Which of the following operations is defined; if defined, give the result.

$$A + 2B, \quad 2C - A, \quad C + 0_{2\times 2}, \quad B\,\mathrm{diag}(2, 4), \quad AB,$$
$$A^t B, \quad D^{-1}, \quad B^{-1}D, \quad D/B, \quad \det(B^{100})$$

A6 Write the system of linear equations

$$5x - 2y = 7, \qquad -3x + 7y = 19$$

in matrix form and solve. What do the two equations and their solution mean geometrically?

A7 Prove (directly from the definition of determinant) that if A, B are 2×2 matrices then $\det(AB) = \det(A)\det(B)$.

Section B

B1 Write the system of linear equations

$$x + \lambda y = 1, \qquad \lambda x + 4y = 3$$

in matrix form. For which values of λ does this system have a unique solution? Express this unique solution in terms of λ.

B2 For each of the following statements, either give a proof to show that its is true, or a counterexample to show that it is false:

(i) If A, B are invertible $n \times n$ matrices then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(ii) If A, B, C are 2×2 matrices and $AB = AC$ then either $A = 0_{2\times 2}$ or $B = C$.

(iii) If A is a non-zero 2×2 matrix then A^2 is non-zero.

B3 Let $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Use induction to show $A^n = \begin{pmatrix} 3^n & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$ for every positive integer n.

B4 Suppose A, B, C, D are square matrices having the same size and satisfying: $(A^{-1}D + BC)A + B^2 = (CA + B)^2$. If A is invertible, express D in terms of A, B, C, simplifying as much as possible.

Section C

C1 Which matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ commute with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

C2 Suppose A and P are $n \times n$ matrices, and that P is invertible, and n is a positive integer. Show that $(P^{-1}AP)^n = P^{-1}A^nP$. Is this true for negative n?

C3 An $n \times n$ matrix is called symmetric is $A^t = A$ and called skew-symmetric if $A^t = -A$. Let B be an $n \times n$ matrix. Show that

(i) $B + B^t$ and BB^t are symmetric.

(ii) $B - B^t$ is skew-symmetric.

(iii) Write B as a sum of a symmetric matrix and a skew-symmetric matrix.

C4 An $n \times n$ matrix A is said to be *orthogonal* if $A^tA = I_n$. How many 6×6 matrices are simultaneously diagonal and orthogonal?