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CYCLES AND FIXED POINTS OF HAPPY FUNCTIONS

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Abstract

Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ denote the natural numbers. Given integers $e \geq 1$ and $b \geq 2$, let $x = \sum_{i=0}^{n} a_i b^i$ with $0 \leq a_i \leq b - 1$ (thus a_i are the digits of x in base b). We define the *happy function* $S_{e,b} : \mathbb{N} \longrightarrow \mathbb{N}$ by

$$S_{e,b}(x) = \sum_{i=0}^{n} a_i^e.$$

A positive integer x is then said to be (e, b)-happy if $S_{e,b}^r(x) = 1$ for some $r \geq 0$, otherwise we say it is (e, b)-unhappy. In this paper we investigate the cycles and fixed points of the happy functions $S_{e,b}$. We give an upper bound for the size of elements belonging to the cycles of $S_{e,b}$. We also prove that the number of fixed points of $S_{2,b}$ is

$$\begin{cases} 2\,\tau\left(\frac{b^2+1}{2}\right) - 1 & \text{if } b \text{ is odd,} \\ \tau(b^2+1) - 1 & \text{if } b \text{ is even,} \end{cases}$$

where $\tau(m)$ denotes the number of positive divisors of m. We use our results to determine the densities of happy numbers in a small handful of cases.

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1. Introduction

Guy [G] introduces **Problem E34 Happy numbers** as follows:

> Reg. Allenby's daughter came home from school in Britain with the concept of **happy numbers**. The problem may have originated in Russia. If you iterate the process of summing the squares of decimal digits of a number, then it's easy to see that you either reach the cycle
>
> $$4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4$$
>
> or you arrive at 1. In the latter case you started from a happy number.

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Guy asks several questions about happy numbers which can be paraphrased as follows:

(a) It seems that about $1/7$ of all numbers are happy, but what bounds on the density can be proved?

(b) How many consecutive happy numbers can you have? Can there be arbitrarily many?

(c) We define the *height* of a happy number to be the least number of iterations needed to reach 1. How big is the least happy number of height h?

(d) What if we replace squares by cubes, fourth powers, fifth powers etc., or we work in different bases?

Most of the work done so far on happy numbers concerns question (b). For example, El-Sedy and Siksek [ES] show that there are arbitrarily many consecutive happy numbers. This was generalised by others, notably Grundman and Teeple [GT1] and [GT2], Pan [P] and Zhou and Cai [ZC], to cover other bases and exponents. In all these proofs the cycles and fixed points of the iteration process play a crucial rôle. This paper is concerned with the cycles and fixed points of the iteration process.

Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ denote the natural numbers. Given integers $e \geq 1$ and $b \geq 2$, let $x = \sum_{i=0}^{n} a_i b^i$ with $0 \leq a_i \leq b-1$ (thus a_i are the digits of x in base b). We define the *happy function* $S_{e,b} : \mathbb{N} \longrightarrow \mathbb{N}$ by

$$S_{e,b}(x) = \sum_{i=0}^{n} a_i^e.$$

A positive integer x is then said to be (e, b)-happy if $S_{e,b}^r(x) = 1$ for some $r \geq 0$ (sometimes, we simply say x is happy), otherwise we say it is (e, b)-unhappy. It is usual to exclude 0 from the domain of $S_{e,b}$.

By a *cycle* of $S_{e,b}$ of length n we mean a sequence of positive integers $x_1, x_2, \ldots, x_n$ such that

$$S_{e,b}(x_1) = x_2, \quad S_{e,b}(x_2) = x_3, \ldots, \quad S_{e,b}(x_{n-1}) = x_n, \quad S_{e,b}(x_n) = x_1.$$

Where the pair (e, b) is clear from the context, we shall use the following clearer notation to denote the above cycle:

$$x_1 \to x_2 \to x_3 \to \cdots \to x_n \to x_1.$$

By a *fixed point* of $S_{e,b}$ we mean a positive integer x such that $S_{e,b}(x) = x$. Of course a fixed point is nothing more than a cycle of length 1.

Write $D_{e,b}$ for the union of all cycles of $S_{e,b}$. Thus, for example,

$$D_{2,10} = \{1\} \cup \{4, 16, 37, 58, 89, 145, 42, 20\}.$$

This paper is concerned in part with giving an upper bound for the elements of $D_{e,b}$.

Theorem 1.1. *Let $e \geq 1$ and $b \geq 2$. If $b > e$, let $M = eb^e$. If $b \leq e$, let r be the smallest positive integer satisfying $e + r \leq b^r$, and let $M = b^{e+r}$. For any $x \in \mathbb{N}$, there is some k_0 such that $S_{e,b}^k(x) < M$ for all $k \geq k_0$.*

In the paper [GT2], it is shown that for every $x \in \mathbb{N}$ there is some k such that $S_{e,b}^k(x) < b^{e+1}$. Note that our result is stronger in the sense that we prove $S_{e,b}^k(x) < M$ for all sufficiently large k.

The proof of Theorem 1.1 is given in Section 2. The following corollary is immediate.

Corollary 1.1. $D_{e,b} \subseteq \{1, 2, \ldots, M-1\}$.

Again our conclusion is stronger than that in [GT2]. There, the conclusion is that every cycle contains some $x < b^{e+1}$. Our conclusion is that every x belonging to some cycle satisfies $x < M$.

In Section 3 we explain a simple and practical algorithm for computing $D_{e,b}$ based on this corollary. Guy's book [G, pages 358–359] gives partial lists of cycles for $S_{4,10}$ and $S_{5,10}$. We complete these lists using a program based on our algorithm.

In Section 4 we shall give some modest applications to densities of happy numbers. For example, as we will see, every positive integer is $(2,4)$-happy. We will also show that the $(5,4)$-happy integers are precisely those $\equiv 1 \pmod 3$, and for $b \geq 2$, that the $(1,b)$-happy numbers are precisely those $\equiv 1 \pmod{(b-1)}$. So far as we know, these are the only cases where the densities of happy numbers have been determined. In Section 5 we show that the cycles of happy functions can be arbitrarily large.

We also pay special attention to the fixed points of $S_{e,b}$. In Section 6 we show that any fixed point x of $S_{e,b}$ satisfies $x < \min\{e,b\} \cdot b^e$. In the particular case $e = 2$ we have an exact formula for the number of fixed points.

Theorem 1.2. *For a positive integer m, denote the number of positive divisors of m by $\tau(m)$.*

1. *If b is odd then $S_{2,b}$ has precisely $2\,\tau\left(\frac{b^2+1}{2}\right) - 1$ fixed points.*
2. *If b is even then $S_{2,b}$ has precisely $\tau(b^2+1) - 1$ fixed points.*

In either case, exactly one of the fixed points has a single digit in base b; this is 1. *The remaining fixed points have precisely two digits in base b.*

Theorem 1.2 is proved in Section 6. In Section 7 we deduce that the number of fixed points of $S_{2,b}$ is arbitrarily large.

It is not immediately obvious, for a given pair (e,b), whether there should exist any unhappy numbers. If $e = 2$ then the above theorem shows that there must exist unhappy numbers whenever b^2+1 is composite. This is not necessarily true if b^2+1 is prime; as we noted earlier, all integers are $(2,4)$-happy.

The paper concludes with some further questions.

2. Proof of Theorem 1.1

Before we launch into the proof of Theorem 1.1, we point out another, arguably more obvious approach, to determining $D_{e,b}$. Observe that every cycle must contain some element

x satisfying $x \leq S_{e,b}(x)$. It is easy to see that

$$x < (b-1)^e(\log_b(x) + 1).$$

There are standard (but slightly complicated) bounds for solutions to such inequalities; see for example [S, Appendix B] which gives an explicit bound $x < M'$. We have found that the bounds M' obtained via this approach are worse than our bound M given in Theorem 1.1. Moreover, all we know via this approach is that the cycle contains some element $x < M'$; we do not know that the entire cycle satisfies the bound. Thus our approach below, although longer, gives stronger results.

Theorem 1.1 is trivially true if $e = 1$ or $b = 2$, so for the remainder of this section we suppose $e \geq 2$ and $b \geq 3$. The theorem follows straightaway from the following two lemmas.

Lemma 2.1. *Let M be as in Theorem 1.1. If $y < M$ then $S_{e,b}(y) < M$.*

Lemma 2.2. *Let M be as in Theorem 1.1. For any $x \in \mathbb{N}$ there exists some ℓ such that $S^{\ell}_{e,b}(x) < M$.*

For the proofs of Lemmas 2.1 and 2.2 we shall need the following lemma.

Lemma 2.3. *Let s, t be positive integers with $s < b$. Suppose $y < sb^t$. Then*

$$S_{e,b}(y) \leq t(b-1)^e + (s-1)^e.$$

Proof. Consider the representation of y in base b. Clearly this has at most $t+1$ digits. Moreover, if y has precisely $t+1$ digits, then the leading digit is at most $s-1$. The lemma follows from the definition of $S_{e,b}$. □

Proof of Lemma 2.1. Suppose $y < M$. We consider the cases $b > e$ and $b \leq e$ separately.

Suppose first that $b > e$. In this case $M = eb^e$. Then $y < eb^e$. By Lemma 2.3,

$$S_{e,b}(x) \leq e(b-1)^e + (e-1)^e$$

Now observe, using the Binomial Theorem, that

$$eb^e = e((b-1)+1)^e > e(b-1)^e + e^2(b-1)^{e-1} > e(b-1)^e + (e-1)^e,$$

where we have made use of $b > e$. Therefore $S_{e,b}(x) < eb^e = M$ as required.

We now turn to the case $b \leq e$. Here $M = b^{e+r}$ where r satisfies $e + r \leq b^r$. As $y < M$, by Lemma 2.3

$$S_{e,b}(y) \leq (e+r)(b-1)^e \leq b^r(b-1)^e < b^{e+r} = M,$$

as required. □

The proof of Lemma 2.2 is more complicated, and we shall need the following preparatory lemma.

Lemma 2.4. *Let $B = b^{e+1}$. If $y < B^m$ then $S_{e,b}(y) < m(B-1)$.*

Proof. Note $y < B^m = b^{m(e+1)}$; by Lemma 2.3, $S_{e,b}(y) \le m(e+1)(b-1)^e$. Note that

$$\begin{aligned} B-1 &= b^{e+1}-1 = ((b-1)+1)^{e+1}-1 \\ &= \big((b-1)^{e+1}+(e+1)(b-1)^e+\cdots+1\big)-1 \qquad \text{by the Binomial Theorem} \\ &> (e+1)(b-1)^e. \end{aligned}$$

The lemma follows. □

Proof of Lemma 2.2. Let $B = b^{e+1}$. Define

$$B_1 = B, \qquad \text{and} \qquad B_{i+1} = B^{B_i} \text{ for } i \ge 2.$$

Let $x \in \mathbb{N}$ be given. We would like to show that $S^{\ell}_{e,b}(x) < M$ for some ℓ where M is as in the statement of Theorem 1.1. Clearly

$$x < B^{1+B_k},$$

for some $k \ge 2$. Then

$$\begin{aligned} S_{e,b}(x) &\le (1+B_k)(B-1) \quad \text{by Lemma 2.4} \\ &= (1+B^{B_{k-1}})(B-1) \\ &= B^{1+B_{k-1}} + B - B^{B_{k-1}} - 1 \\ &< B^{1+B_{k-1}}. \end{aligned}$$

Thus

$$S^{k-1}_{e,b}(x) < B^{1+B}.$$

Applying Lemma 2.4 again gives

$$S^{k}_{e,b}(x) < (1+B)(B-1) < B^2,$$

and again

$$S^{k+1}_{e,b}(x) < 2B = 2b^{e+1}.$$

Hence, by Lemma 2.3,

$$S^{k+2}_{e,b}(x) \le 1+(e+1)(b-1)^e. \tag{2.1}$$

We now consider the cases $b \le e$, $b = e+1$ and $b \ge e+2$ separately. Suppose first that $b \le e$. Recall that we have defined r to be the least positive integer such that $e + r \le b^r$, and in this case $M = b^{e+r}$. Then

$$S^{k+2}_{e,b}(x) \le 1+(e+1)(b-1)^e \le 1+(e+r)(b-1)^e < b^{e+r} = M.$$

This completes the proof of the lemma for $b \le e$.

Suppose now $b = e+1$. In this case $M = eb^e$. By (2.1),

$$S^{k+2}_{e,b}(x) \le 1+(e+1)(b-1)^e = 1+b(b-1)^e < (b-1)b^e = M,$$

as required.

Finally suppose that $b \geq e+2$. Again, $M = eb^e$. By (2.1),

$$S_{e,b}^{k+2}(x) \leq 1 + (e+1)(b-1)^e < (e+1)b^e.$$

Applying Lemma 2.3 we obtain

$$S_{e,b}^{k+3}(x) \leq e(b-1)^e + e^e$$

By the Binomial Theorem applied to $b^e = ((b-1)+1)^e$ we have

$$eb^e > e(b-1)^e + e^2(b-1)^{e-1} > S_{e,b}^{k+3}(x),$$

since $b \geq e+2$. This completes the proof. $\square$

3. Computing $D_{e,b}$

Corollary 1.1 gives an efficient method of computing $D_{e,b}$. Let

$$U_0 = \{1, 2, \ldots, M-1\}$$

where M is as in Theorem 1.1. Let $U_{i+1} = S_{e,b}(U_i)$. Let k be the smallest non-negative integer such that $U_{k+1} = U_k$. Then $D_{e,b} = U_k$.

3.1. Examples

We wrote a short program in `MAGMA` to compute $D_{e,b}$ using the above method. Here are two examples.

Example 1. Take $e = 2$ and $b = 10$. Here $M = 200$. We start with $U_0 = \{1, \ldots, 199\}$ and after 11 iterations find

$$U_{10} = U_{11} = \{1, 4, 16, 20, 37, 42, 58, 89, 145\}.$$

This is $D_{2,10}$ in agreement with Guy's statement in the introduction.

Example 2. Take $e = 5$ and $d = 10$. Guy [G, page 359] lists 12 non-trivial cycles found by Somjit Datta. Thus together with the trivial cycle $\{1\}$ he has found 13 cycles; no claim is made that this is the complete list of cycles. Grundman and Teeple [GT2, page 1141] list nine cycles and mention that there are an additional seven longer cycles (that are not listed). Here $M = 5 \times 10^5$. We ran our program on a 2.8 GHz Dual-Core AMD Opteron. It took our program 58 iterations and just 9 seconds to compute $D_{5,10}$. The cycles of $S_{5,10}$ are

$$C_1 : 1 \to 1, \qquad C_2 : 4150 \to 4150, \qquad C_3 : 4151 \to 4151, \qquad C_4 : 54748 \to 54748,$$
$$C_5 : 92727 \to 92727, \qquad C_6 : 93084 \to 93084, \qquad C_7 : 194979 \to 194979,$$
$$C_8 : 89883 \to 157596 \to 89883, \qquad C_9 : 58618 \to 76438 \to 58618,$$
$$C_{10} : 10933 \to 59536 \to 73318 \to 50062 \to 10933,$$
$$C_{11} : 8299 \to 150898 \to 127711 \to 33649 \to 68335 \to 44155 \to 8299,$$
$$C_{12} : \begin{cases} 8294 \to 92873 \to 108899 \to 183635 \to 44156 \to \\ 12950 \to 62207 \to 24647 \to 26663 \to 23603 \to 8294, \end{cases}$$
$$C_{13} : \begin{cases} 9044 \to 61097 \to 83633 \to 41273 \to 18107 \to 49577 \\ \to 96812 \to 99626 \to 133682 \to 41063 \to 9044, \end{cases}$$
$$C_{14} : \begin{cases} 24584 \to 37973 \to 93149 \to 119366 \to 74846 \to 59399 \to 180515 \\ \to 39020 \to 59324 \to 63473 \to 26093 \to 67100 \to 24584, \end{cases}$$
$$C_{15} : \begin{cases} 9045 \to 63198 \to 99837 \to 167916 \to 91410 \to 60075 \to 27708 \to 66414 \\ \to 17601 \to 24585 \to 40074 \to 18855 \to 71787 \to 83190 \to 92061 \to 66858 \\ \to 84213 \to 34068 \to 41811 \to 33795 \to 79467 \to 101463 \to 9045, \end{cases}$$
$$C_{16} : \begin{cases} 244 \to 2080 \to 32800 \to 33043 \to 1753 \to 20176 \to 24616 \to 16609 \\ \to 74602 \to 25639 \to 70225 \to 19996 \to 184924 \to 93898 \to 183877 \\ \to 99394 \to 178414 \to 51625 \to 14059 \to 63199 \to 126118 \to 40579 \\ \to 80005 \to 35893 \to 95428 \to 95998 \to 213040 \to 1300 \to 244. \end{cases}$$

Datta overlooked the cycles C_5, C_7 and C_8.

Example 3. Let $e = 4$ and $b = 10$. The cycles are

$$C_1 \;:\; 1 \to 1, \qquad C_2 \;:\; 1634 \to 1634, \qquad C_3 \;:\; 8208 \to 8208,$$
$$C_4 \;:\; 9474 \to 9474, \qquad C_5 \;:\; 2178 \to 6514 \to 2178,$$
$$C_6 \;:\; 1138 \to 4179 \to 9219 \to 13139 \to 6725 \to 4338 \to 4514 \to 1138.$$

Guy's book [G, page 358] mentions C_1, C_3, C_5, C_6 but not C_2 and C_4.

Example 4. We used our program to compute $D_{2,b}$ for all $b \le 1000$ and its decomposition into cycles. The entire computation took just under four hours on our 2.8 GHz Dual-Core AMD Opteron. The largest $D_{2,b}$ found was $D_{2,816}$ which has 1676 elements. This decomposes into 11 cycles of lengths 1, 2, 2, 3, 5, 7, 33, 47, 210, 630, 736.

The largest cycle for $S_{2,b}$ with $b \le 1000$ is for $b = 974$. This cycle has length 1005 and has least element 1858 and largest element 1792145. It is interesting to note that precisely 495 elements of this cycle satisfy the inequality $x < S_{2,974}(x)$ and the remainder satisfy $x > S_{2,974}(x)$.

We also took the opportunity test to how good our bound for the elements of $D_{e,b}$ is. By Theorem 1.1 and Corollary 1.1 we know that $\max D_{2,b} \le 2b^2 - 1$. For each $3 \le b \le 1000$ we computed the value

$$m(b) = \frac{\max D_{2,b}}{2b^2 - 1}.$$

The largest value for $m(b)$ we found was for $b = 981$. Here $2b^2 - 1 = 1924721$ and $\max D_{2,981} = 1916884$, giving

$$m(981) = \frac{1916884}{1924721} = 0.9959\cdots.$$

This shows the our bound M is indeed sharp.

4. A Remark on the Densities of Happy Numbers

As quoted in the introduction, Guy asks for the density of happy numbers in the case $e = 2$, $b = 10$. This appears to us to be a difficult problem. However, there are a few pairs (e, b) where have been able to compute the precise density of happy numbers.

Let $e \geq 1$ and $b \geq 2$. Let

$$\mathcal{A} = \{p \text{ prime } : \ p \mid (b-1) \text{ and } (p-1) \mid (e-1)\}, \qquad P = \prod_{p \in \mathcal{A}} p.$$

By convention, if $\mathcal{A} = \emptyset$ then $P = 1$. As observed in [ZC], for all positive integers n, we have $S_{e,b}(n) \equiv n \pmod{P}$; this congruence is easy to prove using Fermat's Little Theorem. Let

$$D'_{e,b} = \{x \in D_{e,b} : x \equiv 1 \pmod{P}\}.$$

If $D'_{e,b} = \{1\}$, then it is easy to see that the (e, b)-happy integers are precisely those that are $\equiv 1 \pmod{P}$. In this case the density of happy numbers is $1/P$. We can only find a handful of examples where this observation is useful.

Example 5. Let $b = 2$ (and $e \geq 1$ arbitrary). Then $D_{2,e} = \{1\}$ and so every positive integer is $(2, e)$-happy.

Example 6. Let $e = 1$ (and $b \geq 2$ arbitrary). It is easy to see that $S_{1,b}(n) \equiv n \pmod{b-1}$ for all n. Moreover,

$$D_{1,b} = \{1, 2, \ldots, b-1\}.$$

Thus the happy numbers are precisely those $\equiv 1 \pmod{b-1}$, and the density of happy numbers is $1/(b-1)$.

Example 7. Let $(e, b) = (2, 4)$. We found that $D_{2,4} = \{1\}$. Hence every number is $(2, 4)$-happy.

Example 8. Let $(e, b) = (5, 4)$. Then

$$D_{5,4} = \{1, 3, 32, 33, 243, 245, 246, 276, 308, 309, 488, 519, 729\}.$$

However $P = 3$ in this case and so $D'_{5,4} = \{1\}$. By the above observation we know that the $(5, 4)$-happy numbers are precisely the ones $\equiv 1 \pmod{3}$. Hence exactly $1/3$ of numbers are $(5, 4)$-happy.

We ran a search with $1 \leq e \leq 6$ and $2 \leq b \leq 10$ for (e, b) satisfying the condition $D'_{e,b} = \{1\}$. The above examples were the only ones found.

5. Large Cycles

As we saw, $S_{2,10}$ has two cycles, one of length 1 and the other of length 8. An obvious question is how large can the cycles of happy functions be. In fact they can be arbitrarily large as we shall see now.

Let r, $e \geq 1$. Let $b = 2^{e^{r-1}} - 1$. Consider the cycle starting with 2:

$$2 \to 2^e \to 2^{e^2} \to \cdots \to 2^{e^{r-1}}.$$

But $2^{e^{r-1}} = b + 1$, so in base b this has the digits 11, hence $S_{e,b}(2^{e^{r-1}}) = 2$. This shows that we obtain a cycle of length r starting with 2, where r is as large as we please.

6. Fixed Points of Happy Functions

As we saw before, fixed points are cycles of length 1. We will see that fixed points are much easier to analyse than arbitrary cycles and so we can say much more about them. For $e = 2$ we show that there can be arbitrarily many fixed points.

It is easy to see that if $e = 1$ then the only fixed points are the single digit numbers $1, 2, \ldots, b-1$. Also, if $b = 2$ then the only fixed point is 1. We shall restrict to $e \geq 2$ and $b \geq 3$. Of course, the only single-digit fixed point is 1.

6.1. Fixed points for exponent $e = 2$

We now restrict to $e = 2$ which turns out to be much easier to analyse than the general case.

Lemma 6.1. *Let $b \geq 3$. Then the happy function $S_{2,b}$ has no fixed points with at least three digits.*

Proof. Suppose that x is a non-zero fixed point of $S_{2,b}$. By Theorem 1.1, $x < 2b^2$. We may write $x = ub^2 + vb + w$ where $0 \leq u, v, w \leq (b-1)$ and $u = 0$ or 1. It is sufficient to show that $u = 0$.

Suppose $u = 1$. As x is a fixed point,

$$b^2 + vb + w = 1 + v^2 + w^2.$$

Thus

$$(b-1)^2 \geq w^2 = b^2 + v(b-v) + w - 1 \geq b^2 - 1 = (b-1)(b+1),$$

giving a contradiction. □

6.2. Proof of Theorem 1.2

Here $e = 2$. The theorem is obvious for $b = 2$ so we suppose $b \geq 3$. By Lemma 6.1, $S_{2,b}$ has no fixed points with three or more digits in base b, and so every fixed point has at most two digits.

A non-negative number with at most two digits in base b has the form $ub + v$ with $0 \leq u,\ v < b$. This is a fixed point for $S_{2,b}$ if and only if $u^2 + v^2 = ub + v$. Let

$$U = \{(u,v) \in \mathbb{Z}^2 \quad : \quad 0 < u,\ v < b \quad \text{and} \quad u^2 + v^2 = ub + v\}.$$

It is easy to see that $1+\#U$ is the number of fixed points of $S_{2,b}$. Thus to prove Theorem 1.2 we need to show that

$$\#U = \begin{cases} 2\,\tau\left(\frac{b^2+1}{2}\right) - 2 & \text{if } b \text{ is odd,} \\ \tau(b^2+1) - 2 & \text{if } b \text{ is even.} \end{cases}$$

Starting with $u^2 + v^2 = ub + v$ and completing the squares we obtain

$$(2u-b)^2 + (2v-1)^2 = b^2 + 1.$$

Let $n = b^2 + 1$ and let

$$X = \{(x,y) \in \mathbb{Z}^2 \quad : \quad y \geq 1 \text{ and odd}, \quad x \neq \pm b \quad \text{and} \quad x^2 + y^2 = n\}.$$

As we will see, the number of elements of X can be determined by relating it to the number of representations of n as the sum of two squares. For now we would like to show that $\#U = \#X$. Define $\phi : U \to X$ and $\psi : X \to U$ by

$$\phi(u,v) = (2u-b, 2v-1), \qquad \psi(x,y) = \left(\frac{x+b}{2}, \frac{y+1}{2}\right).$$

It is clear that ϕ and ψ are mutual inverses (and thus bijections) provided they are well-defined. Thus to show that $\#U = \#X$ all we have to do is to show that ϕ and ψ are well-defined.

There is no difficulty in seeing that ϕ is well-defined. Let us show that ψ is well-defined. Suppose $(x,y) \in X$. As y is odd and $x^2 + y^2 = b^2 + 1$, we see that x and b have the same parity. Thus $\psi(x,y) \in \mathbb{Z}^2$. Let $(u,v) = \psi(x,y)$. From $x^2 + y^2 = b^2 + 1$ we deduce that $u^2 + v^2 = ub + v$. Moreover, $-b \leq x \leq b$ and as $x \neq \pm b$, we have $-b < x < b$ so that $0 < u < b$. Also, $1 \leq y \leq b$, so that $1 \leq v \leq (b+1)/2 < b$ as $b \geq 3$. Hence $(u,v) \in U$. This shows that ψ is well-defined. Thus $\#U = \#X$.

Now let

$$Y = \{(x,y) \in \mathbb{Z}^2 \quad : \quad y \geq 1 \text{ and odd}, \quad \text{and} \quad x^2 + y^2 = n\}.$$

Clearly $\#Y = 2 + \#X = 2 + \#U$. Hence to complete the proof of Theorem 1.2 it is sufficient to show that

$$\#Y = \begin{cases} 2\,\tau\left(\frac{n}{2}\right) & \text{if } b \text{ is odd,} \\ \tau(n) & \text{if } b \text{ is even.} \end{cases}$$

To do this we introduce the following set which contains Y,

$$Z = \{(x,y) \in \mathbb{Z}^2 : x^2 + y^2 = n\}.$$

The number of representations of an a positive integer n as the sum of two squares is well-known:

$$\#Z = 4 \sum_{d \mid n} \left(\frac{-4}{d}\right);$$

see for example [C, Theorem 5.4.15]. However, all odd divisors d of $n = b^2 + 1$ satisfy $d \equiv 1 \pmod 4$. Thus, in this case, $\#Z$ is 4 times the number of odd divisors of n. If b is even then n is odd. If b is odd then $n/2$ is odd. Hence

$$\#Z = \begin{cases} 4\,\tau(n/2) & \text{if } b \text{ is odd,} \\ 4\,\tau(n) & \text{if } b \text{ is even.} \end{cases}$$

To complete the proof of Theorem 1.2 we must relate $\#Y$ to $\#Z$.

Suppose first that b is odd, and so $n \equiv 2 \pmod 4$. Every solution to $x^2 + y^2 = n$ must have x and y odd. Thus Y is the subset of pairs (x, y) in Z with $y \geq 1$. Since there are no solutions with $y = 0$, we see that

$$\#Y = \frac{\#Z}{2} = 2\,\tau(n/2),$$

as required. This completes the proof of Theorem 1.2 for b odd.

Suppose now that b is even. Then $n \equiv 1 \pmod 4$. If (x, y) is any solution to $x^2 + y^2 = n$ then precisely one of x, y is odd and the other is even. Considering the fixed-point free involutions

$$(x, y) \mapsto (y, x), \qquad (x, y) \mapsto (x, -y)$$

on Z, we see that precisely $1/4$ of the elements of Z belong to Y. Hence

$$\#Y = \frac{\#Z}{4} = \tau(n),$$

completing the proof of Theorem 1.2.

7. Arbitrarily Many Fixed Points

In this section we use Theorem 1.2 to deduce that the number of fixed points for $S_{2,b}$ can be arbitrarily large.

Corollary 7.1. *$S_{2,b}$ can have arbitrarily many fixed points.*

Proof. Let $r \geq 1$ be given. We will show that there exists a value of b with $b^2 + 1$ having at least r odd prime factors. Then by Theorem 1.2, $S_{2,b}$ has at least $2^r - 1$ fixed points. Letting r be arbitrarily large proves the theorem.

Let $p_1, \dots, p_r$ be distinct primes $\equiv 1 \pmod 4$. There exist integers m_i with $m_i^2 \equiv -1 \pmod{p_i}$ for $i = 1, \dots, r$. By the Chinese Remainder Theorem, there is some positive integer b such that $b \equiv m_i \pmod{p_i}$ for $i = 1, \dots, r$. Then $b^2 + 1$ is divisible by $p_1, p_2, \dots, p_r$ as required. $\square$

8. Specific Fixed Points

By Theorem 1.2 we know that if b is odd then $S_{2,b}$ has at least three fixed points. If b is even then we can conclude that there is at least one fixed point, but of course 1 is always a fixed point, so the conclusion in this case does not add anything new. It is natural to wonder whether for odd b there is a formula for the other two fixed points that are guaranteed to exist. This is indeed the case.

Corollary 8.1. *Let $b \geq 3$ be odd and write $b = 2u+1$. Then 1, $2u^2+2u+1$ and $2(u+1)^2$ are fixed points of $S_{2,b}$, which respectively have the digits 1, $(u, u+1)$ and $(u+1, u+1)$ in base b. If b^2+1 is twice a prime then these are the only fixed points.*

Proof. Note that $2u^2 + 2u + 1$ and $2(u + 1)^2$ respectively have digits $(u, u + 1)$ and $(u + 1, u + 1)$ in base b. So

$$S_{2,b}(2u^2 + 2u + 1) = u^2 + (u + 1)^2 = 2u^2 + 2u + 1.$$
$$S_{2,b}(2(u + 1)^2) = 2(u + 1)^2.$$

If $b^2 + 1$ is twice a prime then by Theorem 1.2 we know there are exactly three fixed points, and this completes the proof. □

If $e = 2$ and b is odd then, in the notation of Section 4, $P = 2$ and we know that every happy number must be odd. Note by the above that $2u^2 + 2u + 1$ is an unhappy number which is odd. The following corollary is interesting in view of Section 4.

Corollary 8.2. *If $b \geq 3$ is odd then the $(2, b)$-happy numbers are a proper subset of the odd numbers.*

9. Further Questions

The following two questions are very interesting and deserve further attention:

(I) Can one obtain sharp estimates for the number of fixed points of $S_{e,b}$ (for $e \geq 3$) in terms of e and b?

(II) Can one obtain sharp estimates for the number and sizes of the cycles of $S_{e,b}$ in terms of e and b?

Of course by Corollary 1.1 we know that $\#D_{e,b} < M$. But the experiments described in Section 3 indicate that this bound is very far from being sharp. Can one obtain better bounds? In fact for $e = 2$ we can do somewhat better. Recall in this case that $M = 2b^2$ and that every element of $x \in D_{2,b}$ satisfies $x < 2b^2$. However, if $x \in D_{2,b}$ then $x = S_{2,b}(y)$ for some $y \in D_{2,b}$. As $y < 2b^2$, it has at most three digits in base b, and if it has exactly three digits then the leading digit is 1. Hence $x = S_{2,b}(y)$ has the form $u^2 + v^2$ or $1 + u^2 + v^2$. Denote by $N_2(X)$ the number of positive integers $n \leq X$ that can be written as the sum of two squares. A well-known theorem of Landau (see [C, Proposition 5.4.10] and [M]) states that

$$N_2(X) = (C + o(1)) \frac{X}{\sqrt{\log X}} \qquad \text{as } X \to \infty,$$

where

$$C = \frac{1}{\sqrt{2}} \prod_{p \equiv 3 \mod 4} \left(1 - \frac{1}{p^2}\right)^{-1/2} = \frac{\pi}{4} \prod_{p \equiv 1 \mod 4} \left(1 - \frac{1}{p^2}\right)^{1/2} = 0.764223653 \cdots .$$

As every integer in $D_{2,b}$ is at most $2b^2$ and has the form $u^2 + v^2$ or $1 + u^2 + v^2$ we conclude that

$$\#D_{2,b} < \left(2^{3/2} C + o(1)\right) \frac{b^2}{\sqrt{\log b}} \qquad \text{as } b \to \infty. \tag{9.1}$$

In Example 4 we found that the largest $\#D_{2,b}$ with $b \leq 1000$ was $\#D_{2,816} = 1676$. However

$$2^{3/2}C \cdot \frac{816^2}{\sqrt{\log 816}} = 555859.9\cdots .$$

This shows that (9.1) is still a very poor upper bound for $\#D_{2,b}$ despite being much better than M.

Now keep $e \geq 2$ fixed. Regarding cycle lengths, it seems reasonable to guess that any cycle for $S_{e,b}$ has length at most $O(b^{e/2})$ as $b \to \infty$. This is what one would expect if $S_{e,b}$ is a 'random walk' on the set $\{1, 2, \ldots, eb^e\}$ which contains the cycles.

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