Graphs of exceptional curves and their automorphism groups; the Segre-Manin Theorem for del Pezzo surfaces I

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Let k be a field, $\bar{k}$ an algebraic closure of k and let X be a del Pezzo surface defined over k. Recall that X is a smooth projective surface defined over k whose anticanonical divisor is ample. Moreover, $\bar{X} := X \times \operatorname{Spec}(\bar{k})$ is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or to the blow-up of $\mathbb{P}^2$ at $r \leq 8$ points in general position. In this lecture we focus on the structure of X in the case in which k is not necessarily algebraically closed. For our peace of mind, we shall assume that k is perfect. We use this assumption to deduce that a curve C on X, fixed by the action of the Galois group, is defined over k; a priori, we can only deduce that C is defined over a separable closure of k. It is a general fact that since $\mathrm{H}^1\big(X, \mathcal{O}_X\big) = (0)$, then every divisor class on X defined over a separable closure of k is in fact defined over k.

If Y is a scheme over k, we say that X *is a form of* Y *over* k to mean that the schemes $X \otimes \operatorname{Spec}(\bar{k})$ and $Y \otimes \operatorname{Spec}(\bar{k})$ are isomorphic, as schemes over $\bar{k}$.

The strategy that we follow to study X consists in determining conditions on the action of the Galois group $\mathcal{G} := Gal(\bar{k}/k)$, imposed by the fact that $\mathcal{G}$ must permute the exceptional curves on X and preserve the intersection pairing. Note that the Galois group acts on the exceptional curves defined over $\bar{k}$; thus, whenever we talk about exceptional curves, we mean curves $E \subset \bar{X}$, isomorphic to $\mathbb{P}^1_{\bar{k}}$ such that $E^2 = -1$. Since there are only a finite number of exceptional curves on X, there are only a finite number of possibilities for the action of the Galois group on the set of exceptional curves; we use this information in some cases to obtain more precise statements about the structure of X. The main ingredient is the following observation.

Lemma 1. *Let $\mathcal{C}$ be a finite set of disjoint exceptional curves on X and suppose that the action of $\mathcal{G}$ stabilizes $\mathcal{C}$. Then there is a morphism $X \to X'$, which is an isomorphism on $X \setminus \mathcal{C}$ and which contracts each curve in $\mathcal{C}$ to a single point. The surface X' is again a del Pezzo surface.*

Proof. [1]. □

Graphs of exceptional curves. A central role is played by the exceptional curves on $\bar{X}$ and their intersection numbers.

Definition 2. *The graph of exceptional curves on X is the (finite, loopless, undirected) graph G_X having the exceptional curves of $\bar{X}$ as vertices, and having $E_1 \cdot E_2$ edges between the vertices E_1 and E_2, where E_1 and E_2 are distinct exceptional curves.*

The intersection number between exceptional curves E_1, E_2 satisfies

$$E_1 \cdot E_2 \in \{-1, 0, 1, 2, 3\},$$

the product being -1 if and only if $E_1 = E_2$. Moreover,

- if X has degree at least three (that is, $\bar{X}$ is not isomorphic to the blow-up of $\mathbb{P}^2$ at seven or eight points), then $E_1 \cdot E_2 \in \{-1, 0, 1\}$, and hence G_X is a graph without multiple edges;

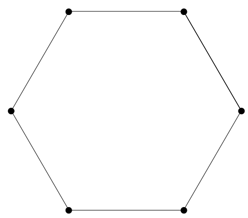
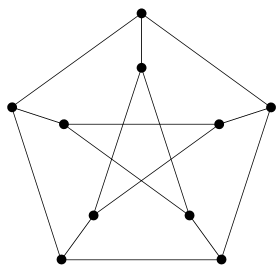
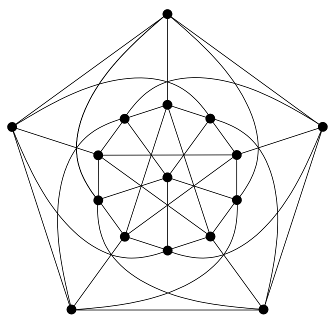
Degree of X	Graph G_X
8	
7	
6	
5	
4	

TABLE 1. Small graphs of exceptional curves

- if X has degree two (that is, $\bar{X}$ is isomorphic to the blow-up of $\mathbb{P}^2$ at seven points), then $E_1 \cdot E_2 \in \{-1, 0, 1, 2\}$.

We shall mostly be interested in del Pezzo surfaces of degree at least three. Table 1 shows the graphs of the exceptional curves on X, when $\bar{X}$ is isomorphic to the blow-up of $\mathbb{P}^2$ at one, two, three, four, or five points.

Exercise 3. *Check that the graphs above are correct and label the vertices by exceptional curves on $\bar{X}$ so that the number of edges between two distinct vertices equals the intersection number of the corresponding exceptional curves.*

The Segre-Manin Theorem. A del Pezzo surface X defined over k need not have any k-rational point. Suppose that k is a number field and v a valuation of k; let k_v denote the completion of k with respect to v. A sufficient criterion to check that a variety defined over k has no k-rational points is to show that $X(k_v) = \emptyset$. Since k_v is complete, it is often fairly easy to decide whether $X(k_v)$ is empty or not.

Let $\mathcal{V}$ be a class of varieties defined over a number field k, for instance $\mathcal{V}$ could be the class of del Pezzo surfaces of given degree defined over k, the class of Fano varieties, or the class of forms of projective space. We say that $\mathcal{V}$ *satisfies the Hasse principle* or that the *Hasse principle holds for* $\mathcal{V}$ if for all $V \in \mathcal{V}$ the statements

- $V(k) \neq \emptyset$, and
- $V(k_v) \neq \emptyset$, for all valuations v of k,

are equivalent. The Hasse principle holds for forms of projective space of any dimension, for degree two hypersurfaces of projective space, for del Pezzo surfaces of degree at least five. The Hasse principle is known not to hold for cubic curves in $\mathbb{P}^2$ and for del Pezzo surfaces of degree four, three, or two. Since a del Pezzo surface of degree one always has a k-rational point, the Hasse principle holds trivially for such surfaces.

Thus it is sometimes possible to decide whether a del Pezzo surface X has or not a k-rational checking for k_v-rational points for all valuations v (at least when k is a number field). Once we established that $X(k) \neq \emptyset$, we may be interested in understanding the structure of the set of k-rational points of X. This question is addressed by the Segre-Manin Theorem. Recall that a surface S is k-unirational if there is a surjective morphism from a variety birational to $\mathbb{P}^2_k$ to S.

Theorem 4 (Segre-Manin)**.** *Let X be a del Pezzo surface defined over a field k. Suppose that* $\deg(X) \geq 2$*, that $X(k) \neq \emptyset$, and, if* $\deg(X) = 2$*, that X contains a k-rational point not lying on any exceptional curve. Then X is k-unirational.*

We prove below some of the cases of the Segre-Manin Theorem. Sometimes the assumption $X(k) \neq \emptyset$ is automatically satisfied; unless $\deg(X) \leq 4$ the surface X is actually k-rational. Even when X is k-rational and X is a form of a blow-up of $\mathbb{P}^2_k$, it is not necessarily the case that X is the blow-up of $\mathbb{P}^2_k$ at a Galois invariant set of points.

Degree 9: $\bar{X} \simeq \mathbb{P}^2_{\bar{k}}$.

In this case X becomes isomorphic to the projective plane after an extension of the base-field. It is possible that $X(k) = \emptyset$; but if $X(k) \neq \emptyset$, then in fact $X \simeq \mathbb{P}^2_k$. Here is an argument, that requires the following facts:

- if X is a form of $\mathbb{P}^2_k$, then there is a scheme $X^\vee$ defined over k that is a form of $(\mathbb{P}^2_k)^\vee$, the dual projective plane; iterating this construction brings us back to X;
- if X is a form of $\mathbb{P}^2_k$ having a *line* defined over k, then X is in fact isomorphic to $\mathbb{P}^2_k$ over k.

The first statement follows easily from a cohomological interpretation of forms of $\mathbb{P}^2$ over k: each form corresponds to a cohomology class, and the "dual" form corresponds to the opposite cohomology class. The second statement is a consequence of the fact that the line bundle associated to a line induces a morphism to $\mathbb{P}^2_k$ and that this morphism is an isomorphism.

Thus we argue as follows. If X has a k-rational point p, then the set of all lines in $\bar{X}$ containing p is a subscheme of $X^\vee$ defined over k; it is clearly isomorphic to a line. Thus it follows that $X^\vee \simeq \mathbb{P}^2_k$, and in particular $X^\vee$ has a k-rational point. Applying the same reasoning to $X^\vee$ we deduce that $X \simeq (X^\vee)^\vee$ is isomorphic to $\mathbb{P}^2_k$.

Degree 8: Case 1 – $\bar{X} \simeq Bl_p(\mathbb{P}^2_{\bar{k}})$.

In this case, $\bar{X}$ contains a unique exceptional curve E; the Galois group therefore fixes E and hence E is defined over k. Contracting E we obtain a form of $\mathbb{P}^2_k$ defined over k, together with a k-rational point, corresponding to the image of E. By the previous case we know that such a variety is isomorphic to $\mathbb{P}^2_k$ and therefore X is isomorphic over k to the blow-up of $\mathbb{P}^2_k$ at a k-rational point.

Degree 8: Case 2 – $\bar{X} \simeq \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$.

The del Pezzo surface X need not have a k-rational point.

Exercise 5. *Construct an example of a del Pezzo surface of degree eight defined over $\mathbb{Q}$, containing no rational points. (Hint: think about quadric surfaces in $\mathbb{P}^3_{\mathbb{Q}}$.)*

Suppose that $X(k) \neq \emptyset$, and let p be a k-rational point. The two rulings $L_1 := \{p\} \times \mathbb{P}^1_{\bar{k}}$ and $L_2 := \mathbb{P}^1_{\bar{k}} \times \{p\}$ (defined over $\bar{k}$) are at worst exchanged by the Galois group, since they are the unique divisors in their equivalence class containing p. Thus the sum $L_1 + L_2$ is certainly invariant under $\mathcal{G}$. The morphism associated to the divisor $L_1 + L_2$ induces an isomorphism of X with a quadric surface in $\mathbb{P}^3_k$, containing the image of p as a k-rational point. Projection away from p induces a birational map $X \dashrightarrow \mathbb{P}^2_k$. Note that the splitting of X as a product of two product of $\mathbb{P}^1_k$ need not be defined over k.

Exercise 6. *Let $X \subset \mathbb{P}^3_{\mathbb{Q}}$ be the quadric defined by*

$$a_0X_0^2 + a_1X_1^2 + a_2X_2^2 + a_3X_3^2 = 0$$

where $a_0, a_1, a_2, a_3 \in \mathbb{Q}^*$. Show that there is an isomorphism $X \simeq \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ defined over $\mathbb{Q}$ if and only if $a_0a_1a_2a_3$ is a square.

Find a form X of $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ with a rational point, for which the splitting is not defined over $\mathbb{Q}$. Determine a birational map of X to $\mathbb{P}^2$ defined over $\mathbb{Q}$.

Degree 7: $\bar{X} \simeq Bl_{p,q}(\mathbb{P}^2_{\bar{k}})$**.** Looking at the graph of exceptional curves on X, we immediately find that the Galois group has at most two orbits on the set of exceptional curves: the two external vertices, or the middle vertex (see Table 1). Contracting the pair of "external" curves gives a morphism $X \to X'$ whose image is a form of $\mathbb{P}^2$. But, the image of the "middle" exceptional curve is also defined over k and its image in X' is a line, defined over k. This implies that X' is isomorphic to $\mathbb{P}^2_k$, and therefore X is the blow-up of $\mathbb{P}^2_k$ at a pair of (possibly conjugate) points.

Exercise 7. *Obtain the same result, considering the contraction of the "middle" exceptional curve. (Hint: the surface obtained by the contraction is a del Pezzo surface; what is its degree?)*

Degree 6: $\bar{X} \simeq Bl_{p,q,r}(\mathbb{P}^2_{\bar{k}})$, p, q, r not collinear. The automorphism group of the graph G_X acts transitively on the vertices (and on the edges): we should not expect to obtain many results without further assumptions. Indeed there are forms of $Bl_{p,q,r}(\mathbb{P}^2_{\bar{k}})$ without k-rational points.

Suppose that X has a k-rational point p not contained in any exceptional curve. Blowing up p we obtain a del Pezzo surface of degree five, whose graph is the Petersen graph (Table 1). The set $\mathcal{C}$ consisting of the three exceptional curves adjacent to the exceptional curve obtained by blowing up p is stable under the Galois group, and these three curves are disjoint (see Table 1). Therefore contracting $\mathcal{C}$ we obtain a del Pezzo surface of degree eight, together with a rational point, and we conclude as above.

Exercise 8. *Analyze the cases in which the k-rational point p lies on some exceptional curve.*

Degree 5: $\bar{X} \simeq Bl_{p,q,r,s}(\mathbb{P}^2_{\bar{k}})$, p, q, r, s not collinear in triples.

Exercise 9. *Assuming that $X(k) \neq \emptyset$, show that X is birational to $\mathbb{P}^2_k$ over k.*

It is a non-trivial result, stated by Enriques and proved by Swinnerton-Dyer, that $X(k) \neq \emptyset$.

All del Pezzo surfaces of degree at least five defined over a number field satisfy the Hasse principle.

Degree 4: $\bar{X} \simeq Bl_{p,q,r,s,t}(\mathbb{P}^2_{\bar{k}})$, p, q, r, s, t not collinear in triples. All del Pezzo surfaces of degree four are isomorphic to complete intersection of two quadrics in $\mathbb{P}^4_k$.

Exercise 10. *Construct an example of a del Pezzo surface of degree four defined over $\mathbb{Q}$, containing no rational points.*

Note that Martin will present an example of such a surface in his Wednesday lecture. The surface S that he considers is a counter-example to the Hasse principle: S has $\mathbb{Q}_v$-rational points for all valuations v of $\mathbb{Q}$, but $S(\mathbb{Q}) = \emptyset$.

This is the first case in which X need not be k-rational. The argument involves a bit more of geometry reasoning than the previous ones: here is a sketch. As always we suppose that there is a point $p \in X(k)$ not lying on any exceptional curve; recall that X is an intersection of two quadrics in $\mathbb{P}^4$. For simplicity, we are going to assume that the field k is infinite. First, projecting away from p we obtain a birational morphism of X to a smooth cubic X' in $\mathbb{P}^3$. The "image" of the point is a line L lying on the surface X'. Any plane containing L intersects X' in a reducible curve consisting of L and a conic. Whenever one of these conics contains one of the (many!) rational points of the line L, the conic itself contains many rational points. This is enough to establish the claim.

References

[1] Yu. I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.