

Central simple algebras and Brauer groups

Damiano Testa

In this lecture we introduce Brauer groups. For fields the classical definition involves equivalence classes of central simple algebras. Serre's Séminaire Cartan, *Applications algébriques de la cohomologie des groupes. II: théorie des algèbres simples*, exp. n. 6, 7 is a clear and thorough introduction to Brauer groups.

Brauer group of a field. Let k be a field.

Definition 1. A central simple algebra over k is a finite dimensional associative k -algebra without non-trivial two-sided ideals and whose center is the field k .

A division algebra over k is a central simple algebra over k all of whose non-zero elements are invertible.

The easiest examples of central simple algebras are matrix algebras over k : for any natural number n , the k -algebra $M_n(k)$ of $n \times n$ matrices with coefficients in k is a central simple algebra. Note that $M_n(k)$ is not a division algebra for $n \geq 2$, since the non-zero matrices of rank at most $n - 1$ are not invertible.

To find an example of a division algebra (different from $k!$), let $k = \mathbb{R}$, and let $\mathbb{H}_{\mathbb{R}}$ be the associative \mathbb{R} -algebra of quaternions generated by $1, i, j$ subject to the usual relations

$$\begin{aligned}i^2 &= -1 \\j^2 &= -1 \\ij &= -ji\end{aligned}$$

and note that the relation $(ij)^2 = -1$ is a consequence of the above relations.

Exercise 2. Check that $\mathbb{H}_{\mathbb{R}}$ is a division algebra.

The field \mathbb{R} did not play a big role in the preceding example: we needed the fact that -1 is not a square in \mathbb{R} to find a division algebra, rather than just a central simple algebra. This leads us to the definition of a quaternion algebra.

Definition 3. Let k be a field and let $a, b \in k^*$. Define $(a, b)_k$ to be the associative k -algebra generated by $1, i, j$ subject to the relations

$$\begin{aligned}i^2 &= a \\j^2 &= b \\ij &= -ji.\end{aligned}$$

As in the case of the algebra $\mathbb{H}_{\mathbb{R}}$, the relation $(ij)^2 = -ab$ follows from the definitions.

Exercise 4. Prove that if $c \in k^*$, then the four quaternion algebras $(a, b)_k$, $(b, a)_k$, $(ac^2, b)_k$, and $(a, bc^2)_k$ are isomorphic.

Prove that the quaternion algebra $(a^2, b)_k$ is not a division algebra.

Prove that the quaternion algebra $(a, b)_k$ is a matrix algebra if and only if the Hilbert symbol $(a, b)_k$ equals one.

At the moment it may seem confusing to use the same notation for a quaternion algebra and a Hilbert symbol, but we shall see that the two notions are strictly related to one another.

Theorem 5 (Wedderburn). *Let A be a central simple algebra over k . There are a unique division algebra D and a positive integer n such that A is isomorphic to $M_n(D)$.*

Wedderburn's Theorem gives a strict relation between central simple algebras and division algebras, and suggests the introduction of the following relation.

Two central simple algebras A and B over the same field k are equivalent if there are positive integers m, n such that $M_m(A) \simeq M_n(B)$. Equivalently, A and B are equivalent if A and B are matrix algebras over the same division algebra. We denote the equivalence class of the central simple algebra A over k by $[A]$, and call it a *Brauer class*.

Lemma 6. *Let A and B be central simple algebras over k . The associative k -algebra $A \otimes_k B$ is a central simple algebra over k .*

Given any k -algebra A , we denote by A^{op} the k -algebra whose underlying vector space is the same as the vector space underlying A , and whose multiplication is defined by $a \cdot_{A^{op}} b := b \cdot_A a$, that is, multiplication in A^{op} is multiplication in A in the opposite order. If A is a central simple algebra over k of dimension n , then A^{op} is also a central simple algebra over k of dimension n , and $A \otimes_k A^{op} \simeq M_n(k)$.

If ℓ/k is a field extension and if A is a central simple algebra over k , then $A \otimes_k \ell$ is a central simple algebra over ℓ , called the extension of A to ℓ .

We are in a position to define the Brauer group of the field k .

Definition 7. *The Brauer group $\text{Br}(k)$ of the field k is the abelian group whose elements are the equivalence classes of central simple algebras over k , and such that $[A] \cdot [B] := [A \otimes_k B]$. The identity in $\text{Br}(k)$ is the class of the field k and the inverse of the element $[A]$ is the element $[A^{op}]$.*

If the field k is algebraically closed, then $\text{Br}(k)$ is trivial.

Exercise 8. *Prove that if k is algebraically closed, then $\text{Br}(k) = (0)$. (Hint: It suffices to show that there are no non-trivial division algebras over k . If D is a division algebra over k , let $d \in D$ and consider $k(d)$.)*

It follows easily from the preceding exercise that if k is any field, then for any element $[A]$ of $\text{Br}(k)$ there is a finite extension ℓ/k such that $A \otimes_k \ell$ is a matrix algebra over ℓ . This property is sometimes stated as "any Brauer class becomes trivial after a finite extension of the base field".

Corollary 9. *The dimension of a central simple algebra over k is a square.*

Proof. Exercise. □

Exercise 10. *Find an explicit isomorphism between $\mathbb{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and $M_2(\mathbb{C})$.*

Example 11. *The Brauer group of the real numbers \mathbb{R} is $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$, the non-trivial element corresponding to the algebra $\mathbb{H}_{\mathbb{R}}$.*

Note that \mathbb{C} does not represent an element of the Brauer group of \mathbb{R} (why?).

Exercise 12. *Prove that the algebra $\mathbb{H}_{\mathbb{R}}$ has order two in $\text{Br}(\mathbb{R})$. Prove also that if $(a, b)_k$ is a quaternion algebra over a field k , then $2[(a, b)_k] = 0$ in the Brauer group of k .*

Example 13. Let p be a prime number. The Brauer group of the p -adic numbers \mathbb{Q}_p is \mathbb{Q}/\mathbb{Z} .

In fact, there is a canonical isomorphism $inv_p: \text{Br}(\mathbb{Q}_p) \rightarrow \mathbb{Q}/\mathbb{Z}$, called the invariant at p . We denote by $inv_{\mathbb{R}}: \text{Br}(\mathbb{R}) \rightarrow \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ the unique isomorphism.

Exercise 14. Let v be a valuation of \mathbb{Q} . Show that the value of the invariant at v of the quaternion algebra $(a, b)_{\mathbb{Q}}$ is equal to the Hilbert symbol $(a, b)_v$ (with the obvious group isomorphism between $\{1, -1\}$ and $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$).

There is a standard exact sequence

$$(0.1) \quad 0 \rightarrow \text{Br}(\mathbb{Q}) \rightarrow \bigoplus_v \text{Br}(\mathbb{Q}_v) \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

where the morphism $\text{Br}(\mathbb{Q}) \rightarrow \bigoplus_v \text{Br}(\mathbb{Q}_v)$ is the evaluation of all the invariants and the morphism $\bigoplus_v \text{Br}(\mathbb{Q}_v) \rightarrow \mathbb{Q}/\mathbb{Z}$ is the sum of all the invariants. A similar sequence exists also for general number fields.

The sequence (0.1) shows a very important property: the local invariants of a Brauer class satisfy a global relation. This observation gives a compatibility among the local invariants; it is a relation that cannot be detected working with a single prime at a time.

Brauer group of a variety. Let X be a smooth projective variety defined over a field k . Our goal is to define a subgroup of the Brauer group of X , that contains “concrete” elements that we can use to analyze arithmetic properties of X . The starting point is the definition of the Brauer group of X , but we are going to do this only vaguely, and we concentrate on more concrete algebras, called cyclic algebras.

The general set-up is as follows. The Brauer group $\text{Br}(X)$ of X is a subgroup of $\text{Br}(k(X))$, the Brauer group of the function field of X . A Brauer class α in $\text{Br}(k(X))$ is a form of a matrix algebra over $k(X)$. Choose a representative A for the Brauer class α ; thus A is a central simple algebra over $k(X)$ such that $[A] = \alpha$. In particular it is a finite dimensional vector space over $k(X)$ and we can choose a basis A_1, \dots, A_n for A . Thus the algebra A is specified by the basis A_1, \dots, A_n together with the “structure constants” $\{a_{ij}^l\}_{i,j,l \in \{1, \dots, n\}}$, where $a_{ij}^l \in k(X)$ are such that

$$A_i \cdot A_j = \sum_l a_{ij}^l A_l.$$

Moreover, there is an open subset $U \subset X$ such that all the structure constants a_{ij}^l are regular on U , and we can “evaluate” the central simple algebra A at all points of U . This means that we can associate to each point $p \in U$ the algebra over k with basis $A_1(p), \dots, A_n(p)$ and structure constants $\{a_{ij}^l(p)\}$, and this algebra is a central simple algebra over k . The Brauer group $\text{Br}(X)$ of X is the subgroup of $\text{Br}(k(X))$ consisting of all the Brauer classes of $k(X)$ that admit a covering of X by open subsets U with the properties above.

Cyclic algebras. A general class of central simple algebras is the class of *cyclic algebras*. Let ℓ/k be a finite cyclic extension of fields of degree n . Given $b \in k^*$ and a generator σ of $\text{Gal}(\ell/k)$, we construct a central simple algebra over k as follows: take the “twisted” polynomial ring $\ell[x]_{\sigma}$, where $lx = x\sigma(l)$ for all $l \in \ell$ and quotient out by the two-sided ideal generated by $x^n - b$. This algebra is usually denoted $(\ell/k, b)$. The algebra depends on the choice of σ , though the notation does

not show this. If X is a geometrically integral k -variety, then the cyclic algebra $(k(X_\ell)/k(X), f)$ is also denoted $(\ell/k, f)$; this should not cause confusion because $\text{Gal}(k(X_\ell)/k(X)) \simeq \text{Gal}(\ell/k)$. More explicitly, the k -algebra $(\ell/k, f)$ is also a vector space over ℓ , with basis $1, x, x^2, \dots, x^{n-1}$ over ℓ , and multiplication defined by

$$\begin{aligned} x^i \cdot x^j &= \begin{cases} x^{i+j} & \text{if } i+j < n \\ x^{i+j-n}b & \text{if } i+j \geq n \end{cases} \\ lx &= x\sigma(l) \quad \text{for all } l \in \ell. \end{aligned}$$

Exercise 15. *Prove that quaternion algebras are cyclic algebras.*

Given a smooth projective variety X defined over a field k , a Galois extension ℓ/k , and a divisor D on $X \times \text{Spec}(\ell)$, define

$$N_{\ell/k}(D) := \sum_{g \in \text{Gal}(\ell/k)} g(D);$$

we call $N_{\ell/k}(D)$ the *norm of D* . The following is a criterion for testing whether or not a cyclic algebra is in the image of the map $\text{Br}(X) \rightarrow \text{Br}(k(X))$. For a proof, see [?, Prop. 2.2.3] [Cor05] or [?, Prop. 4.17] [Bri02].

Proposition 16. *Let X be a smooth, geometrically integral k -variety, ℓ/k a finite cyclic extension and $f \in k(X)^*$. The cyclic algebra $(\ell/k, f)$ is in the image of the natural map $\text{Br}(X) \rightarrow \text{Br}(k(X))$ if and only if $(f) = N_{\ell/k}(D)$, for some $D \in \text{Div}(X_\ell)$. If k is a number field and $X(k_v) \neq \emptyset$ for all valuations v of k , then $(\ell/k, f)$ comes from $\text{Br}(k)$ if and only if we can take D to be principal.*