Exercises for Thursday April 17th

Brauer groups

Exercise 1. *Check that $\mathbb{H}_{\mathbb{R}}$ is a division algebra.*

Exercise 2. *Prove that if $c \in k^*$, then the four quaternion algebras $(a,b)_k$, $(b,a)_k$, $(ac^2,b)_k$, and $(a,bc^2)_k$ are isomorphic.*

Prove that the quaternion algebra $(a^2,b)_k$ is not a division algebra.

Prove that the quaternion algebra $(a,b)_k$ is a matrix algebra if and only if the Hilbert symbol $(a,b)_k$ equals one.

Exercise 3. *Prove that if k is algebraically closed, then $\mathrm{Br}(k) = (0)$. (Hint: It suffices to show that there are no non-trivial division algebras over k. If D is a division algebra over k, let $d \in D$ and consider $k(d)$.)*

Exercise 4. *Find an explicit isomorphism between $\mathbb{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and $M_2(\mathbb{C})$.*

Exercise 5. *Prove that the algebra $\mathbb{H}_{\mathbb{R}}$ has order two in $\mathrm{Br}(\mathbb{R})$. Prove also that if $(a,b)_k$ is a quaternion algebra over a field k, then $2[(a,b)_k] = 0$ in the Brauer group of k.*

Exercise 6. *Let v be a valuation of $\mathbb{Q}$. Show that the value of the invariant at v of the quaternion algebra $(a,b)_{\mathbb{Q}}$ is equal to the Hilbert symbol $(a,b)_v$ (with the obvious group isomorphism between $\{1,-1\}$ and $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$).*

Exercise 7. *Prove that quaternion algebras are cyclic algebras.*

Norms

Exercise 8. *Let X be a smooth projective variety over $\mathbb{Q}$. Let $f \in k(X)^\times$ be a rational function on X, and suppose that there exists a quadratic extension $k = \mathbb{Q}(\sqrt{d})$ of $\mathbb{Q}$ such that the divisor (f) is a norm from k – that is, there is some divisor $D \in \mathrm{Div}\, X_k$ such that $(f) = N_{k/\mathbb{Q}} D$.*

If K is any field containing $\mathbb{Q}$, consider the map

$$X(K) \to \frac{K^\times}{NK(\sqrt{d})^\times} \tag{1}$$

defined by f. A priori*, this only defined away from the zeros and poles of f.*

a) *Show that multiplying f by a norm does not affect this map. In other words, let $g \in k(X_k)^\times$ be a rational function on X_k and show that $f N_{k/\mathbb{Q}}(g)$ defines the same map as f.*
b) *Let $P \in X(K)$ be any point of X. Show that there exists a function $g \in k(X_k)^\times$ such that $f N_{k/\mathbb{Q}}(g)$ has neither a zero nor a pole at P. Deduce that the map (1) is well-defined on the whole of $X(K)$.*
c) *Taking $K = \mathbb{Q}_v$ in (1), we get a map which we can equally well write using the Hilbert symbol:*

$$X(\mathbb{Q}_v) \to \{\pm 1\}, \qquad x \mapsto (d, f(x))_v.$$

Multiplying these maps, we get a map from the product of all the $X(\mathbb{Q}_v)$ to $\{\pm 1\}$ as follows:

$$\prod_v X(\mathbb{Q}_v) \to \{\pm 1\}, \qquad (x_v) \mapsto \prod_v (d, f(x_v))_v.$$

Show that the diagonal image of $X(\mathbb{Q})$ in $\prod_v X(\mathbb{Q}_v)$ must lie in the kernel of this map (i.e. the inverse image of 1).

d) *Consider the example of Birch and Swinnerton-Dyer given in the lecture. The surface is defined by*

$$\begin{cases} uv = x^2 - 5y^2 \\ (u+v)(u+2v) = x^2 - 5z^2 \end{cases}.$$

Show the the function $f = u/(u+v)$ has the property described above, with $k = \mathbb{Q}(\sqrt{5})$. Write down a few other functions giving the same norm group map (1) as f.

Galois cohomology

Exercise 9. *Show that if M is a G-module, and the action of G on M is trivial, then we have $H^1(G, M) = \operatorname{Hom}(G, M)$.*

Exercise 10. *Show that a 1-cocycle $\varphi\colon G \to M$ is determined by the choice of $m = \varphi(\sigma)$. Show that there is an isomorphism $H^1(G, M) \to {}_N M/\Delta(M)$.*

Exercise 11. *Let C_n be a cyclic group of order n, acting on $\mathbb{Z}^n$ by permuting the coordinates cyclically. Find a sublattice L that G also acts on (a sub-G-module), such that $(\mathbb{Z}^n)^G \to (\mathbb{Z}^n/L)^G$ is not surjective. What is the cokernel?*

Exercise 12. *Suppose k is a number field and μ_n is the group of all n-th roots of unity in $\bar{k}$. Show that there is a natural isomorphism $k^*/(k^*)^n \to H^1(k, \mu_n)$.*

Exercise 13. *Let E be an elliptic curve of a field k without complex multiplication, i.e., the endomorphism ring of E is isomorphic to $\mathbb{Z}$. What is the automorphism group of E? Show that the twists of E are in one-to-one correspondence with the elements of $k^*/{k^*}^2$.*