Exercises for Tuesday April 15th

RIEMANN-ROCH AND ADJUNCTION FORMULA

Let X be a smooth projective surface embedded by its anticanonical divisor, i.e. such that $K_X = -H$ where H is a hyperplane section. For example, X could be a cubic surface in \mathbb{P}^3 or the intersection of two quadrics in \mathbb{P}^4 .

Exercise 1. Let C be a smooth curve in X. Use the adjunction formula to show that $C^2 \ge -1$.

Exercise 2. Conversely, let D be a divisor satisfying $D^2 \ge -1$ and D.H > 0. Use the Riemann-Roch theorem to show that the linear equivalence class of D contains an effective divisor.

Exercise 3. Let D be a divisor on X such that nD is principal for some positive integer n. Show that $D^2 = 0$. Using the Riemann-Roch theorem, deduce that $D \sim 0$ and therefore that Pic X is torsion-free.

PICARD GROUP OF DEL PEZZO SURFACES

Exercise 4. We will blow up $\mathbb{A}^2(x, y)$ in the point P = (0, 0). Consider the surface X in $\mathbb{A}^2(x, y) \times \mathbb{P}^1(s, t)$ given by tx = sy. Let π denote the projection $\pi: X \to \mathbb{A}^2$. Set $E = \pi^{-1}(P)$. Show that E is isomorphic to \mathbb{P}^1 and that π induces an isomorphism from X - E to $\mathbb{A}^2 - \{P\}$. We call X the blow-up of \mathbb{A}^2 in P. Note that each direction at P is determined by a line given by $t_0x = s_0y$, so the directions in \mathbb{A}^2 are parametrized by $\mathbb{P}^1(s, t)$ and we can think of X as "the surface obtained by replacing P by all directions at P". Consider the curve $C \subset \mathbb{A}^2$ given by $y^2 = x^3 - x^2$. Show that C has a node as singularity at P. Show that $\pi^{-1}(C)$ consists of E and another component, say C'. We call C' the strict transform of C. Is C' still singular?

Exercise 5. Show that if the del Pezzo surface X is the blow-up of \mathbb{P}^2 in r points, then no six of them lie on a conic.

Exercise 6. Show that if the del Pezzo surface X is the blow-up of \mathbb{P}^2 in 8 points, then they do not lie on a singular cubic that has its singularity at one of the 8 points.

Exercise 7. Suppose X is the blow-up of \mathbb{P}^2 in $r \leq 8$ points P_1, \ldots, P_r in general position. Then the strict transforms of the following curves in \mathbb{P}^2 are all exceptional curves:

- (1) a line through 2 of the P_i ,
- (2) a conic through 5 of the P_i ,
- (3) a cubic passing through 7 of the P_i such that one of them is a double point (on that cubic),
- (4) a quartic passing through 8 of the P_i such that three of them are double points,
- (5) a quintic passing through 8 of the P_i such that six of them are double points,
- (6) a sextic passing through 8 of the P_i such that seven of them are double points and one is a triple point.

Exercise 8. Determine the number of exceptional curves on a del Pezzo surface of degree d for each d (getting two possibilities for degree 8).

Exercise 9. Let C, D be a curves and suppose that the genus of C is at most one. Where can the surface $C \times D$ appear in the classification of surfaces?

Exercise 10. Compute the Hodge numbers of \mathbb{P}^2 .

Exercise 11. Compute the Hodge numbers of $C_1 \times C_2$, where C_1 and C_2 are smooth curves of genus g_1 and g_2 respectively.

Exercise 12. Compute the Hodge numbers of smooth curves of degree d in \mathbb{P}^2 .

Exercise 13 (*). Compute the Hodge numbers of smooth surfaces in \mathbb{P}^4 that are intersections of two hypersurfaces of degree d and e.

Exercise 14. Check that the graphs in Table 1 are correct and label the vertices by exceptional curves on \bar{X} so that the number of edges between two distinct vertices equals the intersection number of the corresponding exceptional curves.

Exercise 15. Construct an example of a del Pezzo surface of degree eight defined over \mathbb{Q} , containing no rational points. (Hint: think about quadric surfaces in $\mathbb{P}^3_{\mathbb{Q}}$.)

Exercise 16. Let $X \subset \mathbb{P}^3_{\mathbb{Q}}$ be the quadric defined by

$$a_0 X_0^2 + a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 = 0$$

where $a_0, a_1, a_2, a_3 \in \mathbb{Q}^*$. Assume X contains a rational point. Show that there is

an isomorphism $X \simeq \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ defined over \mathbb{Q} if and only if $a_0a_1a_2a_3$ is a square. Find a form X of $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ with a rational point, for which the splitting is not defined over \mathbb{Q} . Determine a birational map of X to \mathbb{P}^2 defined over \mathbb{Q} .

Exercise 17. Prove that a del Pezzo surface of degree seven containing a k-rational point is k-rational, considering the contraction of the "middle" exceptional curve. (Hint: the surface obtained by the contraction is a del Pezzo surface; what is its degree?)

Exercise 18. Analyze the cases in which X is a del Pezzo surface of degree six and it has a k-rational point p lying on some exceptional curve.

Exercise 19. Let X be a del Pezzo surface of degree five and assume that $X(k) \neq \emptyset$. Show that X is birational to \mathbb{P}^2_k over k.

Exercise 20. Construct an example of a del Pezzo surface of degree four defined over \mathbb{Q} , containing no rational points.



TABLE 1. Small graphs of exceptional curves