Exercises for Monday April 14th

Picard groups

Exercise 1. *Let Z be a prime divisor in a smooth variety X, and let U denote the complement $X \setminus Z$. Show that the sequence*

$$\mathbb{Z} \to \operatorname{Pic} X \to \operatorname{Pic} U \to 0,$$

where the first map is $1 \mapsto Z$ and the second $D \mapsto D \cap U$, is exact.

Exercise 2. *Use the result of Exercise 1 to show that $\operatorname{Pic} \mathbb{P}^n \cong \mathbb{Z}$, for any positive integer n.*

Exercise 3. *Let X be the projective quadric surface $xy = zw$, and let U be the open subset defined by $w \neq 0$.*

(1) *Show that U is isomorphic to $\mathbb{A}^2$, and deduce that $\operatorname{Pic} U = 0$.*
(2) *Show that $X \setminus U$ consists of two straight lines. Using the exact sequence of Exercise 1, show that $\operatorname{Pic} X \cong \mathbb{Z}^2$, generated by the classes of these two straight lines.*
(Hint: to show that the two lines are not equivalent, you may like to use intersection numbers.)

Canonical divisors

Exercise 4. *Consider the differential dx/y on the affine curve C in $\mathbb{A}^2(x, y)$ given by $y^2 = f(x)$ for some separable polynomial f. Show that ω is regular at every point of C. Show that this is consistent with Proposition 1.12. Show that if f has degree at least 3, then ω is in fact regular on the entire projective closure of C in $\mathbb{P}^3$.*

Exercise 5. *Show that if X is a hypersurface in $\mathbb{P}^n$ for $n \geq 3$, then $\Gamma(\Omega_X) = 0$.*

Exercise 6. *Show that if X is a complete intersection in $\mathbb{P}^n$ of dimension at least 2, then $\Gamma(\Omega_X) = 0$.*

Exercise 7. *Compute the divisor (dt) on $\mathbb{P}^1(x, y)$ with $t = x/y$.*

Exercise 8. *Compute the divisor (dx/y) on the projective closure in $\mathbb{P}^2$ of the affine curve given by $y^2 = f(x)$ with f a separable polynomial of degree 2, 3, 4, general d.*

Exercise 9. *Compute the divisor $(dt_1 \wedge \ldots \wedge dt_n)$ on $\mathbb{P}^n(x_0, x_1, \ldots, x_n)$ with $t_i = x_i/x_0$.*

Exercise 10. *Let X be a hypersurface in $\mathbb{P}^n_k(x_0, \ldots, x_n)$ given by the homogeneous polynomial F of degree d, let L be any linear form in $k[x_0, \ldots, x_n]$, and set*

$$\omega = \frac{x_0^n L^{-n-1+d}}{\partial F/\partial x_0} dt_1 \wedge \ldots \wedge dt_{n-1}$$

with $t_i = x_i/x_0$. After checking that all degrees work out to make ω a well-defined element of $\bigwedge^{n-1} \Omega_{k(X)/k}$, show that we have $(\omega) = (-n-1+d)(H \cap X)$, where H is the hyperplane given by $L = 0$.

Exercise 11. *Let $X \subset \mathbb{A}^n(x_1,)$ be a smooth complete intersection of dimension $n-k$, defined by the polynomials $f_1, \ldots, f_k \in k[x_1, \ldots, x_n]$. Let J be a sequence as above, and let I be the increasing sequence of the elements of $\{1, \ldots, n\} \setminus J$. Then up to sign the differential $\omega_J = M_J^{-1} dx_{i_1} \wedge \ldots \wedge dx_{i_{n-k}}$ is independent of the choice of J.*

Exercise 12. *Use the notation as in the previous exercise, and assume P is a point on X. Then there is a particular sequence J as in that exercise such that $M_J(P) \neq 0$ and for the corresponding sequence I, the elements $x_i - x_i(P)$ with $i \in I$ form a set of local parameters at P. Conclude that $(\omega_J) = 0$ on $X \subset \mathbb{A}^n$.*

Exercise 13. *Homogenize the previous exercises to find out the contribution to (ω) of the hyperplane at infinity of the projective closure of X. Check that your answer agrees with Proposition 3.6.*

Exercise 14. *Suppose X is a smooth complete intersection as in Proposition 3.6, and assume that X is a surface. Compute the self-intersection of a canonical divisor on X.*

Exercise 15. *Let $\mathbb{P}(w_0, w_1, \ldots, w_n)$ be weighted projective n-space with coordinates $x_0, \ldots, x_n$ such that x_i has weight w_i, and assume $w_0 = 1$. Let X be a smooth hypersurface in $\mathbb{P}(w_0, w_1, \ldots, w_n)$ of (weighted) degree d. Set $D = X \cap H$ where H is the hyperplane given by $x_0 = 0$. Then any canonical divisor on X is linearly equivalent to $(d - \sum_i w_i)D$.*

Exercise 16. *Find an example of a variety X of dimension n for which the map $\bigwedge^n(\Gamma(\Omega_X)) \to \Gamma(\omega_X)$ is not surjective.*

Exercise 17. *Let C be the curve in $\mathbb{P}^3(x, y, z, w)$ parametrized by $(u^4 : u^3t : ut^3 : t^4)$. Let H be the hyperplane given by $w = 0$ and set $D = C \cap H$. Show that the functions $1, x/w, y/w, z/w$ do not generate $\Gamma(\mathcal{L}(D))$. (Hint: find an isomorphism from C to $\mathbb{P}^1$ and find what divisor D corresponds to on $\mathbb{P}^1$.)*

Exercise 18. *Use Proposition 4.3 to show that the geometric genus of a hypersurface in $\mathbb{P}^n$ of degree d equals $\binom{d-1}{n}$.*

Exercise 19. *Show that the g_i in Example 4.4 are global sections of $\mathcal{L}(D)$ with $D = (\omega_0)$. More precisely, show that for any $\omega \in \bigwedge^n \Omega_{k(X)/k}$, the sheaf ω_X is isomorphic to $\mathcal{L}(D)$ for $D = (\omega)$.*

Exercise 20. *Show that any divisor that is linearly equivalent to a very ample divisor, is in fact itself very ample.*

Exercise 21. *Find all sequences $(d_1, \ldots, d_r)$ with $d_i \geq 2$ such that a canonical divisor on a smooth complete intersection X in $\mathbb{P}^{r+2}$ of hypersurfaces of degree $d_1, \ldots, d_r$ is not very ample. (Compare this to the next lecture.)*

Hodge diamonds

Exercise 22. *Compute the Hodge numbers of $\mathbb{P}^2$.*

Exercise 23. *Compute the Hodge numbers of $C_1 \times C_2$, where C_1 and C_2 are smooth curves of genus g_1 and g_2 respectively.*

Exercise 24. *Compute the Hodge numbers of smooth curves of degree d in $\mathbb{P}^2$.*

Exercise 25 ($\star$). *Compute the Hodge numbers of smooth surfaces in $\mathbb{P}^4$ that are intersections of two hypersurfaces of degree d and e.*