Tuesday, Lecture 2, Ronald van Luijk, rmluijk@gmail.com

Please let us know about mistakes in these notes!

Much of this lecture is directly copied from Chapter 6 of Bjorn Poonen's notes [6].

1. Del Pezzo Surfaces

Let k be any field and X a smooth, projective, geometrically irreducible variety over k. Let $\overline{k}$ be an algebraic closure of k. Let ω_X be the canonical sheaf on X.

Definition 1.1. *The variety X is called rational if $X_{\overline{k}}$ is birationally equivalent to $\mathbb{P}^n_{\overline{k}}$ for some n.*

Definition 1.2. *The variety X is called a Fano variety if $\omega_X^{\otimes(-1)}$ is ample. In other words, if $-K$ is ample, where K is any canonical divisor.*

Definition 1.3. *A del Pezzo surface is a Fano variety of dimension 2. Its degree is the self-intersection number $K \cdot K$, where K is any canonical divisor.*

Note that if $-K$ is very ample, then it determines an embedding of X into $\mathbb{P}^n$ for some n, under which $-K$ corresponds to a hyperplane section H. The degree of this embedding is H^2, which explains why we call $(-K)^2$ the degree of a del Pezzo surface X.

Proposition 1.4. *Suppose k has characteristic 0 and X is a rational surface. Then X is k-birational to either a del Pezzo surface or a conic bundle over a conic.*

In this lecture we will focus on the geometry of del Pezzo surfaces.

Proposition 1.5. *The degree of a del Pezzo surface is positive.*

Proof. Note that a very ample divisor H intersects every effective curve with positive intersection number, so in particular $H^2 > 0$. Therefore, if D is any ample divisor, so nD is very ample for some positive integer n, then $0 < (nD)^2 = n^2D^2$, so $D^2 > 0$. The statement now follows from the fact that $-K$ is ample. □

Theorem 1.6. *Suppose that k is algebraically closed and that X is a del Pezzo surface. Then either X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, or X is the blow-up of $\mathbb{P}^2$ at distinct points $P_1, \ldots, P_r$ in general position, where $0 \leq r \leq 8$. Here general position means that none of the following hold: (1) three of the P_i lie on a line, (2) six of the P_i lie on a conic, (3) eight of the P_i lie on a singular cubic, with one of these eight points at the singularity. The degree of $\mathbb{P}^1 \times \mathbb{P}^1$ is 8. If X is the blow-up of $\mathbb{P}^2$ in r points, then its degree is $9 - r$. Conversely, any blow-up of $\mathbb{P}^2$ in at most 8 points in general position is a del Pezzo surface.*

Proof. The fact that X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up at $r \leq 8$ points is proven for instance in [4], Thm. 24.4. The first two conditions of general position are also proved there. The third condition is [4], Thm. 26.2 and Rem. 26.3. The degrees are easy to compute. For the converse, see [1], Thm. 1. □

The full proof of Theorem 1.6 is not very deep, but too long to present in this hour. Instead we will break up the proof in parts, which will show that assuming parts of the conclusion, we can prove the rest fairly easily. The following proposition therefore reclaims only part of the previous theorem.

Proposition 1.7. *Suppose that k is algebraically closed and that X is a del Pezzo surface. Then either X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, or X is the blow-up of $\mathbb{P}^2$ at distinct points $P_1, \dots, P_r$.*

Proof. See [4], Thm. 24.4. □

For the rest of this lecture, let X be a del Pezzo surface with canonical divisor K.

Proposition 1.8. *Suppose $\tilde{X} = Bl_P(X)$ is the blow-up of a smooth surface X at a point P, with corresponding map $\pi\colon \tilde{X} \to X$. Let K_X and $K_{\tilde{X}}$ be canonical divisors on X and $\tilde{X}$ respectively and let E denote the exceptional curve on $\tilde{X}$ above P. Then E is isomorphic to $\mathbb{P}^1$, and $K_{\tilde{X}}$ is linearly equivalent with $\pi^* K_X + E$. Moreover, the map $\rho\colon \operatorname{Pic} X \oplus \mathbb{Z} \to \operatorname{Pic} \tilde{X}$ sending (D, n) to $\pi^* D + nE$ is an isomorphism. We have $E^2 = -1$, and for all $C, D \in \operatorname{Pic} X$ we have $(\pi^* C) \cdot (\pi^* D) = C \cdot D$ and $(\pi^* C) \cdot E = 0$. In particular, we have $K_{\tilde{X}} \cdot E = -1$.*

Proof. See [3], Propositions V.3.1-3. □

Exercise 1. *Suppose the setting of Proposition 1.8. Assume that you already know $E^2 = -1$. Prove that the map ρ is an isomorphism. (Hint: use that leaving out a closed subset of codimension 2 does not change the Picard group, while leaving out an irreducible closed subset of codimension 1 gives an exact sequence you have seen before.) Now also prove the rest of the theorem.*

Corollary 1.9. *If the del Pezzo surface X is the blow-up of $\mathbb{P}^2$ at distinct points $P_1, \dots, P_r$, then $r \leq 8$ and the degree of X equals $9 - r$.*

Proof. The canonical divisor $K_{\mathbb{P}^2}$ on $\mathbb{P}^2$ is linearly equivalent to $-3L$ where L is any line, so $K_{\mathbb{P}^3}^2 = 9L^2 = 9$. The degree follows from Proposition 1.8. From Proposition 1.5 we find $9 - r > 0$, so $r \leq 8$. □

Proposition 1.10. *If C is a closed integral curve on X, then $C^2 \geq -1$. Furthermore, equality implies $p_a(C) = 0$ and $K_X \cdot C = -1$.*

Proof. The adjunction formula gives $C^2 + C \cdot K = 2p_a(C) - 2 \geq -2$, while $C \cdot K < 0$ since $-K$ is ample. □

If we have $C^2 = -1$, then by Proposition 1.10, the integral curve C can be blown down, so we call C an exceptional curve.

Proposition 1.11. *Suppose Y is the blow-up of $\mathbb{P}^2$ in r distinct points, exactly s of which lie on the line L. Then the strict transform of L on Y has self-intersection $L^2 = 1 - s$.*

Proof. Let $\pi\colon Y \to \mathbb{P}^2$ denote the blow-up. Let $E_1, \dots, E_s$ denote the exceptional curves above the s points on L. Let L' denote the strict transform of L on Y. Then $\pi^* L = L' + \sum_{i=1}^s E_i$, so $1 = L^2 = (\pi^* L)^2 = (L' + \sum E_i)^2 = L'^2 + 2\sum L' \cdot E_i + \sum E_i^2 = L'^2 + 2s - s$, so $L'^2 = 1 - s$. □

Corollary 1.12. *If the del Pezzo surface X is the blow-up of $\mathbb{P}^2$ in r points, then no three of these points are colinear.*

Proof. Follows from Propositions 1.11 and 1.10. □

Exercise 2. *Show that if the del Pezzo surface X is the blow-up of $\mathbb{P}^2$ in r points, then no six of them lie on a conic.*

Exercise 3. *Show that if the del Pezzo surface X is the blow-up of $\mathbb{P}^2$ in 8 points, then they do not lie on a singular cubic that has its singularity at one of the 8 points.*

The conditions of Corollary 1.12 and Exercises 2 and 3 together are summarized by saying that the r points must be in general position. Conversely, one can show that if $r \leq 6$ points on $\mathbb{P}^2$ are in general position, then the blow-up of $\mathbb{P}^2$ in those points is indeed a del Pezzo surface (see [4], Thm. 24.5). For $r = 6$ we get the famous cubic surfaces in $\mathbb{P}^3$, see [3], section V.4. For $r = 7$ or $r = 8$, life gets significantly more complicated because then the anticanonical sheaf is no longer very ample, just ample (c.f. [4], Rem. 26.3). For a proof that even in this case, the converse mentioned in Theorem 1.6 is still true, see [1], Thm. 1.

If X is the blow-up of $\mathbb{P}^2$ in $r \leq 8$ points, then Proposition 1.11 states that the strict transform of the line through any two of the points is an exceptional curve. The following exercise gives more exceptional curves.

Exercise 4. *Suppose X is the blow-up of $\mathbb{P}^2$ in $r \leq 8$ points $P_1, \ldots, P_r$ in general position. Then the strict transforms of the following curves in $\mathbb{P}^2$ are all exceptional curves:*

(1) *a line through 2 of the P_i,*
(2) *a conic through 5 of the P_i,*
(3) *a cubic passing through 7 of the P_i such that one of them is a double point (on that cubic),*
(4) *a quartic passing through 8 of the P_i such that three of them are double points,*
(5) *a quintic passing through 8 of the P_i such that six of them are double points,*
(6) *a sextic passing through 8 of the P_i such that seven of them are double points and one is a triple point.*

Note that all these curves satisfy $(-K) \cdot C = 1$, so if $-K$ is very ample, i.e., $r \leq 6$, then under the embedding into $\mathbb{P}^n$ that it determines, all these exceptional curves correspond to lines. Together with the fibers above the points blown up, the curves described in this exercise are in fact an exhaustive list of all exceptional curves on X (see [4], Thm. 26.2). Proving this requires a better understanding of the Picard group of X, which we will work on next.

Proposition 1.13. *Suppose X is the blow-up of $\mathbb{P}^2$ in $r \leq 8$ points $P_1, \ldots, P_r$ in general position. Let E_i denote the exceptional curve above P_i and let L denote the pull back of a line in $\mathbb{P}^2$. Then $\operatorname{Pic} X$ is generated by L and the E_i, with $-K_X \sim 3L - \sum E_i$. The intersection numbers are given by $L^2 = 1$, $L \cdot E_i = 0$, and $E_i \cdot E_j = -\delta_{ij}$.*

The intersection pairing turns the Picard group into a unimodular lattice of signature $(1, r)$. In particular this means that if we know a divisor up to numerical equivalence, then we know its divisor class. This will prove extremely useful in studying not only the geometry of the surface, but also the arithmetic. For instance, we get a representation of the absolute Galois group $\operatorname{Gal}(\overline{k}/k)$ into the automorphism group of this lattice. Better yet, since the class $[K]$ is fixed under galois, we get a representation into the automorphism group of the orthogonal complement of $[K]$, which is a root lattice and subject of the next lecture.

Proposition 1.14. *Suppose C is an integral curve on X satisfying $C^2 = -1$. Then there is no other effective divisor D that is linearly equivalent to C.*

Proof. Suppose D is an effective divisor that linearly equivalent to C. If D does not have C in its support, then D and C intersect in finitely many points, so $D \cdot C \geq 0$, which contradicts $D \cdot C = C^2 = -1$. We conclude that D does have C in its support, so $D - C$ is an effective divisor that is linearly equivalent to 0, so $D - C = 0$ and $D = C$. □

Proposition 1.14 is often phrased by saying that curves with negative selfintersection do not move.

We have already seen that if C is an integral curve on X with $C^2 = -1$, then we also have $(-K) \cdot C = 1$, so by the adjunction formula we have $p_a(C) = 0$ and so C is an exceptional curve. We now also define the notion of an exceptional divisor class, namely as a class D satisfying $D^2 = -1$ and $(-K) \cdot D = 1$. Every exceptional curve represents an exceptional divisor class and Proposition 1.14 states that every exceptional divisor class contains at most one exceptional curve. In fact, a stronger statement is true.

Proposition 1.15. *Every exceptional divisor class contains exactly one exceptional curve.*

Proof. Let D be any divisor in an exceptional divisor class. Note that for rational surfaces we have $\chi(\mathcal{O}_X) = 1$. Therefore, Riemann-Roch on surfaces gives

$$h^0(\mathcal{L}(D)) - h^1(\mathcal{L}(D)) + h^0(\mathcal{L}(K - D)) = \frac{1}{2} D \cdot (D - K) + 1 + p_a(X) = 1.$$

From $(-K) \cdot (K - D) = -K^2 - 1 < 0$ we see that no effective divisor is linearly equivalent to $K - D$, so $h^0(\mathcal{L}(K - D)) = 0$ and we get $h^0(\mathcal{L}(D)) \geq 1$, which implies that there is indeed an effective divisor D' that is linearly equivalent to D. Since $(-K)$ intersects every irreducible curve in the support of D positively, and $(-K) \cdot D = 1$, we see that D is irreducible, and therefore an exceptional curve. By Proposition 1.14 it is the only exceptional curve in the class. □

Proposition 1.16. *Suppose the del Pezzo surface X is the blow-up of $\mathbb{P}^2$ in $r \leq 8$ points $P_1, \ldots, P_r$ in general position. Then the fibers above the P_i and the curves in Exercise 4 are exactly all the exceptional curves on X.*

Proof. By Proposition 1.15 it suffices to count the number of exceptional divisor classes in the Picard lattice. As every divisor class is uniquely represented by $D = aL - \sum_{i=1}^r b_i E_i$ for some a and b_i, we are counting the solutions to the equations $1 = D \cdot (-K) = 3a - \sum b_i$ and $-1 = D^2 = a^2 - \sum b_i^2$. Setting $b_0 = 1$ for convenience, the first equation becomes $3a = \sum_{i=0}^r b_i$, under which the second becomes equivalent to $\sum_i (a - 3b_i)^2 = 18$. This clearly gives finitely many solutions, all of which can be enumerated easily. □

Exercise 5. *Determine the number of exceptional curves on a del Pezzo surface of degree d for each d (getting two possibilities for degree 8).*

References

[1] Michel Demazure. Surfaces de Del Pezzo - II. *Séminaire sur les Singularités des Surfaces*, 777:viii+339, 1980. eds. Demazure, Michel and Pinkham, Henry Charles and Teissier, Bernard, Held at the Centre de Mathématiques de l'École Polytechnique, Palaiseau, 1976–1977.

[2] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[3] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[4] Yu. I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.

[5] J. Milne. *Class Field Theory*. http://www.jmilne.org/math/CourseNotes/math776.html.

[6] B. Poonen. *Rational points on varieties*. http://math.berkeley.edu/~poonen/math274.html.