MA908 Partial Differential Equations in Finance

Exercise Sheet 5: Deterministic optimal control

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1. Consider the control problem of optimal consumption. The equation for the wealth $y(t)$ at time t is
$$\partial_s y(s) = ry(s) - \alpha(y(s), s), \qquad y(t) = x > 0,$$
where $\alpha(y(s), s)$ is the control, in this case the rate of consumption. We have the natural constraints $\alpha \geqslant 0$ and $y \geqslant 0$. We consider a power law utility rate, which gives rise to the utility function
$$u(x,t) = \max_{\alpha} \int_t^{\tau} \mathrm{e}^{-\rho s}\, \alpha(y(s), s)^{\gamma}\, \mathrm{d}s, \qquad (*)$$
with $0 < \gamma < 1$, and $\tau = \inf\{s \geqslant t_0 : y_s = 0 \text{ or } s \geqslant T\}$.
   (a) Show that u has the scaling property $u(\lambda x, t) = \lambda^{\gamma} u(x,t)$. (Hint: for $y(t) = x$, let $\alpha_x(t)$ be the control that achieves the maximum on the right hand side of $(*)$. For $y(t) = \lambda x$, consider the (possible less than optimal) control $\lambda \alpha_x(t)$. Deduce that $u(\lambda x, t) \geqslant \lambda^{\gamma} u(x,t)$. Now consider $1/\lambda$ instead of λ, and repeat the above steps.)

   (b) Derive the HJB-equation. Follow the steps given in the lecture, but include the discounting term $\mathrm{e}^{-\rho s}$ that was absent there. Your resulting equation should read
$$\partial_t u + \max_{\alpha \geqslant 0} \left((rx - \alpha)\partial_x u + \mathrm{e}^{-\rho t}\, \alpha^{\gamma} \right) = 0,$$
with final condition $\alpha(x, T) = 0$.

   (c) By a), we know that u must be of the form $u(x,t) = g(t)x^{\gamma}$ (why?). Use this to get an equation for α and g. Your equation for g should read
$$\partial_t g + r\gamma g + (1-\gamma)(\mathrm{e}^{\gamma t}\, g)^{1/(1-\gamma)} g = 0, \qquad (**)$$
with final condition $g(T) = 0$. Show that if g solves $(**)$, then $G = \mathrm{e}^{\rho t}\, g$ solves
$$\partial_t G + (r\gamma - \rho)G + (1-\gamma)G^{\gamma/(\gamma-1)} = 0 \qquad (***),$$
and $H = G^{1/(1-\gamma)}$ solves $\partial_t H - \mu H + 1 = 0, \qquad H(T) = 0$, with $\mu = (\rho - r\gamma)/(1-\gamma)$.

   (d) Solve the equation for H, find the optimal $g(t)$, and the optimal $\alpha(t)$.

2. Consider (a generalized version of) the control problem of the previous section, but with discounting to time t instead of time 0. More precisely, consider the ODE
$$\partial_s y(s) = f(\boldsymbol{y}(s), \boldsymbol{\alpha}(s)), \qquad \boldsymbol{y}(t) = \boldsymbol{x},$$
($\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \boldsymbol{\alpha} \in \mathbb{R}^m$), and the optimal utility
$$v(x,t) = \max_{\boldsymbol{\alpha}} \left(\int_t^T \mathrm{e}^{-\rho(s-t)}\, h(\boldsymbol{y}(s), \boldsymbol{\alpha}(s))\, \mathrm{d}s + \mathrm{e}^{-\rho(T-t)}\, g(\boldsymbol{y}(T)) \right).$$

   (a) Show that v satisfies
$$\partial_t v - \rho v + H(\boldsymbol{x}, \nabla v) = 0, \qquad \text{with Hamiltonian} \qquad H(\boldsymbol{x}, \boldsymbol{p}) = \max_{\boldsymbol{a} \in \mathbb{R}^m} (f(x,a) \cdot \boldsymbol{p} + h(\boldsymbol{x}, \boldsymbol{a})),$$
and final condition $v(\boldsymbol{x}, T) = g(x)$. Note that this equation is autonomous, i.e. it does not explicitly depend on t. You can repeat the arguments in a), but there is an easier way to see it.

   (b) Now consider the infinite time horizon variant of the above problem, i.e.
$$\tilde{v}(\boldsymbol{x}, t) = \max_{\boldsymbol{\alpha}} \int_t^{\infty} \mathrm{e}^{-\rho(s-t)}\, h(\boldsymbol{y}(s), \boldsymbol{\alpha}(s))\, \mathrm{d}s.$$
Show that $\tilde{v}$ does in fact not depend on t. You will need to assume that for any allowed control $\alpha : s \mapsto \alpha(s)$, with $s \geqslant t$, the map $s \mapsto \alpha(s + \tau)$, with $s \geqslant t - \tau$ is also an allowed control. Conclude (using the first part of the present problem) that $\tilde{v}$ solves the equation
$$-\rho\tilde{v} + H(\boldsymbol{x}, \nabla\tilde{v}) = 0.$$