MA908 Partial Differential Equations in Finance

Exercise Sheet 4: More on the heat equation

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1. Let u solve the heat equation

$$\partial_t u = \partial_x^2 u \quad \text{for } 0 < x < 1 \text{ and } t > 0, \qquad u(0,t) = u(1,t) = 0, u(x,0) = 1.$$

   (a) Interpret u as the value of a suitable double barrier option.

   (b) Express $u(x,t)$ as a Fourier series.

   (c) How many terms of the Fourier series are needed at $t = 1/100$ to get one percent accuracy of the solution?

2. **Deriving the fundamental solution.**

   (a) Show that if u solves the heat equation, then $u(cx, c^2 t)$ also solves it, for any $c > 0$.

   (b) Now take the simplest type of functions that have the scaling property given in a): assume $u(x,t) = U(|x|^2/t)$ for some function $U : \mathbb{R} \to \mathbb{R}^m$, $x \in R^n$ and $t > 0$. Show that $\partial_t u = \Delta u$ if and only if

$$4zU''(z) + (2n + z)U'(z) = 0 \quad \text{for } z > 0.$$

   (c) Show that the general solution for $U(z)$ is given by

$$U(z) = a \int_0^z e^{-s/4} s^{-n/2} ds + b$$

   for any constants a, b.

   (d) Show, in dimension $n = 1$, that

$$u(x,t) = \partial_x U(x^2/t) = (2x/t)U'(x^2/t)$$

   is also a solution of the heat equation and show for suitable choice of a that this leads to the fundamental solution $\Phi(x,t) = (4\pi t)^{-1/2} \exp(-x^2/4t)$.

3. **Growth estimates via energy methods** Let u solve the heat equation on a bounded domain $D \subset \mathbb{R}^n$ with smooth boundary:

$$\partial_t u = \Delta u \text{ for } x \in D, t > 0, \qquad u(x,0) = g(x) \text{ for } x \in D, \qquad u(x,t) = h(x,t) \text{ for } x \in \partial D, t > 0.$$

   (a) Define $E(t) = \int_D (u(x,t))^2 \, \mathrm{d}x$. Show that if $h = 0$ above, then $\partial_t E(t) \leqslant 0$. Hint: You will need the integration by parts formula

$$\int_D u \Delta u \, \mathrm{d}x = -\int_D |\nabla u|^2 \, \mathrm{d}x + \int_{\partial D} u(x) \nabla u \cdot \boldsymbol{n}(x) \, \mathrm{d}S(x),$$

   where $\boldsymbol{n}$ is the outer normal vector to ∂D.

   (b) Use this to give a short proof of uniqueness for the solution to the heat equation.

   (c) Now consider the equation

$$\begin{aligned} &\partial_t u = \Delta u + \lambda u && x \in D, t > 0 \\ &u(x,0) = f(x) && x \in D \\ &u(x,t) = 0 && x \in \partial D, t > 0, \end{aligned}$$

   with $\lambda > 0$. Use this to derive a growth estimate for $E(t)$; you can use the Gronwall Lemma that says that if $f'(x) \leqslant \alpha f(x)$, then $f(x) \leqslant f(0)\, \mathrm{e}^{\alpha x}$ (can you prove that?).

(d) Now specialize the situation of c) to $D = [0, 1]$ and $f(x) = \sin(\pi x)$. Find the exact solution (e.g. by Fourier series), and compare the exact behaviour of $E(t)$ with the one that comes from the bound you found in c). How different are they?