2.5. **The heat equation on an interval.** Recall that vanilla options correspond to the heat equation on the full space, while barrier options correspond to the heat equation on the half space. We will now study the heat equation on an interval, which corresponds to double barrier options. Surprisingly, this will turn out to be easier than the cases considered before. We will need the following fact:
Theorem: Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $f(0) = f(1) = 0$. Then for all $x \in [0,1]$,

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(k\pi x) \qquad \text{with} \qquad a_k = 2\int_0^1 f(x)\sin(k\pi x)\,\mathrm{d}x. \tag{2.19}$$

We will not prove this Theorem, but make some remarks:
(i): The theorem is even valid for discontinuous function, but then the sum does not converge for all $x \in [0,1]$. Instead, the (squared) area between the function f and the partial sums becomes arbitrarily small (convergence in L^2).
(ii): Equation (2.19) is called *Fourier series* (more precisely: Fourier-sine-series) of f, and the a_n are the *Fourier coefficients.* A more common variant of (2.19) is

$$f(x) = \sum_{k=0}^{\infty} \tilde{a}_k\, \mathrm{e}^{2\pi \mathrm{i} kx} \qquad \text{with} \qquad \tilde{a}_k = \int_0^1 f(x)\, \mathrm{e}^{-2\pi \mathrm{i} kx}\,\mathrm{d}x.$$

This does not need the condition that $f(0) = f(1) = 0$; however, if we do have that condition, (2.19) is simpler and we will stick with that.
(iii): The proof uses orthogonality of the set $\{\sin(k\pi x) : k \geqslant 1\}$ in the space of square integrable functions. More concretely, you can check that

$$2\int_0^1 \sin(k\pi x)^2\,\mathrm{d}x = 1, \qquad \text{and} \qquad \int_0^1 \sin(k\pi x)\sin(m\pi x)\,\mathrm{d}x = 0,$$

for $0 < k < m$. This already shows that (2.19) works for the sine functions themselves and eventually leads to the proof.
Let us now use the theorem to solve

$$\begin{aligned} &\partial_t u = \frac{1}{2}\partial_x^2 u \qquad \text{for } t > 0, 0 < x < 1, \\ &u(x,0) = g(x), \quad u(0,t) = \phi_0(t), \quad u(1,t) = \phi_1(t). \end{aligned} \tag{2.20}$$

Let us start with the easiest case $\phi_0 = \phi_1 = 0$. Then for each $t > 0$, we can express the solution $u(x,t)$ of (2.20) using (2.19). Thus,

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\sin(k\pi x) \tag{2.21}$$

for some coefficients $a_k(t)$. By the initial condition, we must have

$$a_k(0) = 2\int g(x)\sin(k\pi x)\,\mathrm{d}x,$$

for all k. By the differential equation, we must have

$$\sum_{k=1}^{\infty} \partial_t a_k(t) \sin(k\pi x) = \sum_{k=1}^{\infty} (-\frac{1}{2} k^2 \pi^2 \sin(k\pi x)) a(t).$$

Now the orthogonality of the sine functions that we mentioned above really means that we cannot build one of them out of a finite or infinite number of different ones. So, for the equality above to hold, the individual terms for each k have to agree, and we find $\partial_t a_k = -k^2 \pi^2 a_k$. with solution

$$a_k(t) = a_k(0) \, \mathrm{e}^{-\frac{1}{2} k^2 \pi^2 t}.$$

Wrapping up, we find

$$u(x,t) = \sum_{k=1}^{\infty} g_k \, \mathrm{e}^{-\frac{1}{2} k^2 \pi^2 t} \sin(k\pi x), \qquad \text{with } g_k = 2 \int_0^1 g(y) \sin(k\pi y) \, \mathrm{d}y. \tag{2.22}$$

Note that the term $\mathrm{e}^{-k^2 \pi^2 t}$ decays extremely quickly as t and k get large. This means that for large t (i.e. for maturities far in the future), the first few terms of the sum give a very good approximation of the solution.
Let us now write the solution in form of a Green's function: we write

$$G(x,y,t) = 2 \sum_{k=1}^{\infty} \mathrm{e}^{-\frac{1}{2} k^2 \pi^2 t} \sin(k\pi x) \sin(k\pi y),$$

and then have

$$u(x,t) = \int_0^1 G(x,y,t) g(y) \, \mathrm{d}y.$$

G is the Green's function for the heat equation on the interval, and in the same way as for the half space, one can see that the solution of (2.20) with nonzero boundary conditions ϕ_0 and ϕ_1 and initial condition $g(x) = 0$ is given by

$$u(x,t) = \int_0^t \partial_y G(x,y,t)|_{y=0} \phi_0(s) \, \mathrm{d}s - \int_0^t \partial_y G(x,y,t)|_{y=1} \phi_1(s) \, \mathrm{d}s.$$

The general solution for nonzero initial and boundary conditions can then be found, as in the half-space case, by adding the two types of solutions that we have found above.

Remark: The backwards heat equation

Let us close with a remark about the backwards heat equation, given by $\partial_t u = -\partial_x^2 u$. Why do we never investigate it? The reason can be seen from equation (2.22): This would change to

$$u(x,t) = \sum_{k=1}^{\infty} g_k \, \mathrm{e}^{+\frac{1}{2} k^2 \pi^2 t} \sin(k\pi x).$$

But the exponential term now grows when k grows, which means that unless g_k decays very quickly as $k \to \infty$, the 'solution' will be infinite for every positive time. This is related to the fact that the heat equation has such good smoothing properties: it converts e.g. any bounded function at time $t = 0$ into an infinitely differentiable function at $t > 0$ (just check on the solution formula); conversely, the backwards heat equation will be un-solvable for any initial condition that is not infinitely often differentiable, and will be in fact un-solvable for most other initial conditions. So we don't treat it.

3. Uniqueness and the Maximum Principle

So far, we have focused on finding solutions to the heat equation by giving explicit formulae. Here, we will investigate whether these solutions are the only ones there are. This question is of practical importance, as we can see when we recall how we approached the problem of option pricing: we found that
1) the option price must solve a PDE;
2) we can find a solution to the PDE.
But who guarantees that the solution we found actually has any connection to the option price? Without uniqueness, it might well be that the option price is given by a solution to the PDE that we did not find. To be sure that by finding a solution to the PDE we have found the correct option price, we therefore need uniqueness. Fortunately, it holds. A first step to proving it is the following result:
Proposition: Consider the heat equation on a bounded open domain $D \subset \mathbb{R}^n$:

$$(*) \begin{cases} \partial_t u = \Delta u \text{ for } x \in D, t > 0; \\ u(x,0) = g(x), \qquad u(y,t) = \phi(y) \text{ for } y \in \partial D. \end{cases}$$

Assume that

(A) the only solution to $(*)$ with $g = 0$ and $\phi = 0$ is the constant solution $u = 0$.

Then, the solution of $(*)$ is unique for all g and all ϕ.

Proof. Assume that u and $\tilde{u}$ solve $(*)$ with the same initial and boundary conditions. Then $u - \tilde{u}$ solves $(*)$ with zero initial and boundary conditions, by linearity. By assumption (A), this means that $u - \tilde{u} = 0$, and thus $u = \tilde{u}$. □

It remains to show that assumption (A) is valid. For this we use the **Maximum Principle**:
Theorem: Let $D \subset \mathbb{R}^n$ be a bounded open set. Let $f : D \times [0,T] \to \mathbb{R}$ satisfy the inequality

$$\partial_t f \leqslant \Delta f(x,t) \qquad \text{for all } x \in D, 0 < t < T.$$

Then the maximum of f over the set $\overline{D} \times [0,T]$ (overline means closure) is taken either on the spacial boundary $\partial D \times [0,T]$, or at the initial boundary $D \times \{0\}$.
Remarks: a) it follows that the maximum can neither be in the interior of the space-time domain, nor at the final time (except when it is also on the boundary of the space domain).
b) From the maximum principle, (A) follows easily: since u solves $\partial_t u = \Delta u \leqslant \Delta u$, and since $u = 0$ on the boundary, we find that $u(x,t) \leqslant 0 =$ the value at the boundary. And, since $\partial_t(-u) = \Delta(-u) \leqslant \Delta(-u)$, and $-u = 0$ on the boundary, we find that $-u(x,t) \leqslant 0$. So we must have $u(x,t) = 0$.

Proof of the Theorem. Fist let us see what happens when f fulfils the strict inequality $\partial_t f < \Delta f$, i.e. $\partial_t f - \Delta f < 0$. If f is maximal, inside $D \times [0,T]$, say at (y,t), then $\partial_t f(y,t) = 0$, since at the maximum all partial derivatives vanish. Also, $\partial^2_{y_i} f(y,t) \leqslant 0$, since at the maximum all second partial derivatives are negative or zero. So, $\partial_t f - \Delta f \geqslant 0$, which contradicts the strict inequality. Thus f cannot have a maximum in the interior. Now for the final time, if we have a maximum we must have $\partial_t f(y,T) \geqslant 0$, since otherwise we could follow the slope into the interior of the domain and find points where f is even larger. Thus the same argument as above shows that f cannot have a maximum here, either.
Having sorted the case where the strict inequality holds, let us now consider f with only $\partial_t f - \Delta f \leqslant 0$. We define $f_\varepsilon(x,t) = f(x,t) - \varepsilon t$. Then

$$\partial_t f_\varepsilon(x,t) = \partial_t f(x,t) - \varepsilon \leqslant \Delta f(x,t) - \varepsilon < \Delta f(x,t) = \Delta f_\varepsilon(x,t).$$

The last equality is because the term εt that we added does not depend on x. We see that f_ε fulfils the strict inequality and therefore takes its maximum at the boundary. On the other hand, $f_\varepsilon(x,t) \leqslant f(x,t) \leqslant f_\varepsilon(x,t) + \varepsilon T$, and thus

$$\max_{\text{boundary}} f(x,t) \geqslant \max_{\text{boundary}} f_\varepsilon(x,t) = \max_{\text{whole set}} f_\varepsilon(x,t) \geqslant \max_{\text{whole set}} f(x,t) - \varepsilon T.$$

As this is valid for any ε, we can take the limit $\varepsilon \to 0$ in the above inequality and show the claim. □

The maximum principle also holds for unbounded domains, but it does need extra assumptions. Indeed, it can be shown that there are solutions to the whole space heat equation with initial condition $u(x,0) = 0$ that are nonzero for positive times. These are called 'Tikhonov's great blast of heat from infinity', after the Russian mathematician Andrey Nikolayevich Tikhonov who found them. However, some mild growth conditions on the solution at spatial infinity can restore uniqueness.
Theorem: If f solves $\partial_t f = \Delta f$ for $t > 0$, $f(x,0) = 0$ for all $x \in \mathbb{R}^n$, and if $f(x,t) \leqslant M\,\mathrm{e}^{c|x|^2}$ for some $M, c > 0$ and all $x \in \mathbb{R}^n, t > 0$, then we must have $f(x,t) = 0$ for all x, t.

Proof. We first prove it for $t < t_0$ for some t_0 to be fixed shortly. By induction we then know it to hold for $t < 2t_0$, $t < t_0$ et cetera, and thus eventually for all t. So let us assume that f fulfils all the conditions of the theorem. We define

$$g(x,t) = f(x,t) - \frac{\delta}{(t-t_1)^{n/2}}\,\mathrm{e}^{\frac{|x|^2}{4(t_1-t)}}\,,$$

where t_1 and δ are for the moment arbitrary. It can be checked directly that g solves the heat equation. Now write $D_R = \{x \in \mathbb{R}^n : |x| < R\}$ for the ball of radius R. We first establish that when we take R large enough, and t_1 small enough, then on the boundary of the space-time domain $D_R \times [0,t_1]$ we have $g \leqslant 0$. Indeed,

$g(x,0) < f(x,0) = 0$, and for $x \in \partial D$ we have

$$g(x,t) = f(x,t) - \frac{\delta}{(t-t_1)^{n/2}} \mathrm{e}^{\frac{R^2}{4(t_1-t)}} \leqslant M \mathrm{e}^{c|x|^2} - \frac{\delta}{(t-t_1)^{n/2}} \mathrm{e}^{\frac{R^2}{4(t_1-t)}} \leqslant$$
$$\leqslant M \mathrm{e}^{cR^2} - \frac{\delta}{(t-t_1)^{n/2}} \mathrm{e}^{\frac{R^2}{4(t_1-t)}} .$$

For t_1 such that $\frac{1}{4t_1} > C$, the second function grows faster than the first, and thus for R large enough, indeed $g(x,t) < 0$ on ∂D. What's more, this will continue to hold for any $\tilde{R} > R$. So for all $\tilde{R} > R$, and all $t_0 < t_1$, we can apply the finite domain maximum principle to find $g(x,t) \leqslant 0$ on $D_{\tilde{R}} \times [0,t_0]$. Therefore indeed $g(x,t) < 0$ on $\mathbb{R}^d \times [0,t_1]$ (each point of $\mathbb{R}^d$ is in a sufficiently large ball). Translating back to f, we find

$$f(x,t) \leqslant \frac{\delta}{(t_1-t)^{n/2}} \mathrm{e}^{\frac{|x|^2}{4(t_1-t)}} .$$

We have not yet used the parameter δ. Now we see that the above inequality holds foe any $\delta > 0$, and so by taking the limit $\delta \to 0$, we must have $f(x,t) \leqslant 0$ for all $t \leqslant t_0$. The same argument can now be applied to $-f$, yielding $f(x,t) \geqslant 0$ on the same set. In conclusion, we find $f(x,t) = 0$. □