PDE FOR FINANCE LECTURE NOTES (SPRING 2012)

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1. PDE occurring in Finance

The aim of this section is to show how partial differential equations (PDE) occur naturally when considering problems in finance.

1.1. **Starting point.** Let y_t be the state (e.g. the price) of a system (e.g. a stock) at time t. We assume that y_t is modelled by the stochastic differential equation (SDE)

$$\mathrm{d}y_t = F(y_t, t)\,\mathrm{d}t + G(y_t, t)\,\mathrm{d}\mathcal{W}_t. \tag{1.1}$$

Here $\mathcal{W}_t$ is one-dimensional standard Brownian motion. An example is geometric Brownian motion, where $F(x,t) = \mu x$ and $G(x,t) = \sigma x$, with $\sigma, \mu > 0$. We obtain

$$\mathrm{d}y_t = \mu y_t\,\mathrm{d}t + \sigma y_t\,\mathrm{d}\mathcal{W}_t.$$

This is the simplest model for the time evolution of a stock price.
Now let $\Phi : \mathbb{R} \to \mathbb{R}^n$ be a *payoff function*, which at the moment can be any function. For stock prices, which are positive, we want $\Phi : \mathbb{R}^+ \to \mathbb{R}$, but we can think of other assets that can take negative values. For example, with $\Phi(x) = x$ the payoff would be just the price of the stock, while for $\Phi(x) = (x - C)^+ \equiv \max\{0, x - C\}$ the payoff would be that of a European option with strike price C. Let us assume that at some time $t > 0$, the state of the asset is x, so this means $y_t = x$.
Question: What is the *expected payout*

$$u(x,t) = \mathbb{E}_{y_t = x}(\Phi(y_T)) \tag{1.2}$$

at a final time T?
Answer: $u(x,t)$ is the solution to the PDE

$$\partial_t u(x,t) + F(x,t)\partial_x u(x,t) + \frac{1}{2} G(x,t)^2 \partial_x^2 u(x,t) = 0, \tag{1.3}$$

with *final value condition* $u(x,T) = \Phi(x)$.
(1.3) already shows the basic features of a PDE: It poses the problem to find a function where the partial derivatives balance in a certain way, at each point (x,t), and which fulfils some condition on the boundary (here: final time) of the domain. Usually, provided sufficiently many boundary conditions are given, there is at most one function that satisfies a PDE. This property of uniqueness needs to be proved in many cases, however.

Let us derive (1.3). For this, we rewrite (1.2) as

$$0 = u(x,t) - \mathbb{E}_{y_t=x}(\Phi(y_T)).$$

We want to plug the path y_t of the asset price into u. Notice that $u(x,t) = \mathbb{E}_{y_t=x}(u(y_t,t))$, and $\mathbb{E}_{y_t=x}(\Phi(y_T)) = \mathbb{E}_{y_t=x}(u(y_T,t))$. The last equality follows from $u(x,T) = \mathbb{E}_{y_T=x}(\Phi(y_T)) = \Phi(x)$. We obtain

$$0 = \mathbb{E}_{y_t=x}(u(y_T,T) - u(y_t,t)) = \mathbb{E}_{y_t=x}\Big(\int_t^T \mathrm{d}u(y_s,s)\Big). \tag{1.4}$$

If the function $s \mapsto u(y_s,s)$ would be differentiable, this would follow just from the fundamental theorem of calculus, and the meaning of $\mathrm{d}u(y_s,s)$ would be $\frac{d}{ds}u(y_s,s)\,\mathrm{d}s$. Since the non-smooth path y_s is plugged into u, we however need Itô calculus. We find

$$\begin{aligned}\mathrm{d}u(y_s,s) =& \Big(F(y_s,s)\partial_y u(y_s,s) + \frac{1}{2}G(y_s,s)^2\partial_y^2 u(y_s,s) + \partial_t u(y_s,s)\Big)\,\mathrm{d}s\\ &+ G(y_s,s)\partial_y u(y_s,s)\,\mathrm{d}\mathcal{W}_s.\end{aligned}$$

Since $\mathbb{E}\Big(\int_t^T G(y_s,s)\,\mathrm{d}\mathcal{W}_s\Big) = 0$ for every adapted process y_s (in particular our process is adapted!), it follows that in order to fulfil (1.4), it is enough that the first line of the equation above vanishes after taking the integral over time and the expectation. But the first line is just the PDE (1.3), so if u fulfils that, then also (1.4) holds. It is not difficult to see that on the other hand, if we want (1.4) hold for all t, then u needs to fulfil the PDE.

1.2. **Vector valued diffusions.** We now study several assets $\boldsymbol{y}_s = (y_s^{(1)}, \dots, y_s^{(n)})$, that solve the system of SDE

$$\mathrm{d}y_s^{(i)} = F_i(\boldsymbol{y}_s, s) + \sum_{j=1}^n G_{ij}(\boldsymbol{y}, s)\,\mathrm{d}\mathcal{W}_s^{(j)},$$

where the $\mathcal{W}_s^{(j)}$ are independent Brownian motions. Let $\Phi(\mathbb{R}^n \to \mathbb{R})$ be again a payoff function. Then

$$u(\boldsymbol{x},t) = \mathbb{E}_{\boldsymbol{y}_t=\boldsymbol{x}}(\Phi(\boldsymbol{y}_T))$$

is the solution of the PDE

$$\partial_s u(\boldsymbol{x},s) + \mathscr{L}u(\boldsymbol{x},s) = 0 \qquad \text{for } t < s < T,$$

with final condition $u(\boldsymbol{x},T) = \Phi(\boldsymbol{x})$, and where

$$\mathscr{L} = \sum_{i=1}^n F_i \partial_{x_i} + \frac{1}{2}\sum_{i,j,k=1}^n G_{ik}G_{kj}\partial_{x_i}\partial_{x_j} \tag{1.5}$$

is called the *generator* of the diffusion $\boldsymbol{y}_t$. The derivation is entirely parallel to the one above and uses the multi-dimensional Itô formula.

1.3. **Discounting and the Black-Scholes PDE.** We now allow for some discounting. We study

$$u(x,t) = \mathbb{E}_{y_t=x}\Big(\mathrm{e}^{-\int_t^T b(y_s,s)\,\mathrm{d}s}\,\Phi(y_T)\Big), \tag{1.6}$$

where $b : \mathbb{R}^2 \to \mathbb{R}$ is a discounting function, and $\Phi : \mathbb{R} \to \mathbb{R}$ is again a payoff function.

We claim that if y_t solves the SDE (1.1), then u as given above in (1.6) is the solution of the PDE

$$\partial_t u + F\partial_x u + \frac{1}{2}G^2\partial_x^2 u - bu = 0, \tag{1.7}$$

with final condition $u(x,T) = \Phi(x)$. Note that although we have not written the arguments of the functions u, F, G and b above, they are still functions and not numbers. We will use this shorter notation often.

To justify our claim, let us again first note that (1.6) is equivalent to $u(y,T) = \Phi(y)$ and

$$0 = \mathbb{E}_{y_t=x}\Big(u(y_T,T)\,\mathrm{e}^{-\int_t^T b(y_s,s)\,\mathrm{d}s} - u(y_t,t)\Big),$$

with the same justification as above. Writing $v((y_r)_{r \leqslant s}) = \mathrm{e}^{-\int_t^r b(y_r,r)\,\mathrm{d}r}$, (which depends now on more than one point in time, but is still adapted), we can transform this into

$$\begin{aligned} 0 &= \mathbb{E}_{y_t=x}\Big(u(y_T,T)v((y_r)_{r \leqslant T}) - u(y_t,t)v((y_r)_{r \leqslant T})\Big) \\ &= \mathbb{E}_{y_t=x}\Big(\int_t^T \mathrm{d}\,(u(y_s,s)v((y_r)_{r \leqslant s}))\Big). \end{aligned}$$

Now we have to apply the Itô formula as above. This is left as an exercise.

1.4. **The connection with the Black-Scholes PDE (BSPDE).** The PDE (1.7) is formally equal to the BSPDE. If we specialize (1.1) to geometric Brownian motion, then

$$\mathrm{d}y_t = \mu y_t\,\mathrm{d}t + \sigma y_t\,\mathrm{d}\mathcal{W}_t.$$

We choose a constant discounting function b (interest rate), and then (1.7) becomes

$$\partial_t u + \mu x\partial_x u + \frac{1}{2}\sigma^2 x^2\partial_x^2 u - bu = 0. \tag{1.8}$$

In comparison, the classical BSPDE would read

$$\partial_t u + \frac{1}{2}\sigma^2 x^2\partial_x^2 u + b(x\partial_x u - u) = 0. \tag{1.9}$$

The final conditions for a European call option would be $\Phi(x) = (x - C)^+$. The equations (1.8) and (1.9) are of the same structure, but the coefficients differ unless $b = \mu$. This is connected with the fact that unless we are in a risk-neutral world, the naive option pricing formula that just tries to take the expected discounted payout

$$u(x,t) = \mathbb{E}_{y_t=x}(\,\mathrm{e}^{-b(T-t)}\,\Phi(y_T))$$

as the option price gives the wrong result.

1.5. **Derivation of the Black-Scholes PDE.** We work here again with constant interest rate, which we now will call r instead of b. The no-arbitrage principle means that the price $P(t, x)$ of an option must be given by a self-financing trading strategy, for if it were not, a trader using this strategy would have an arbitrage opportunity. The strategy is determined by the amount a_t of stock and the amount b_t of risk-less bound that the trader holds at time t. It has to replicate the payout $\Phi(y_T)$ at maturity, and since at maturity the payout is equal to the option price for any pricing strategy, we have

$$a_T y_t + b_T \, \mathrm{e}^{rT} = P(y_T). \tag{1.10}$$

Since the strategy is self-financing, the only way the portfolio can change in value is that the stock goes either up or down, and by the growing capital in the bonds. Thus

$$\mathrm{d}(a_t y_t + b_t \, \mathrm{e}^{rt}) = a_t \, \mathrm{d}y_t + r b_t \, \mathrm{e}^{rt} \, \mathrm{d}t. \tag{1.11}$$

Our aim is to find the equation for $P(t, x)$ so that

$$P(t, y_t) = a_t y_t + b_t \, \mathrm{e}^{rt}, \tag{1.12}$$

i.e. the option price is exactly given by the value of the trading strategy portfolio for all times. Differentiating this both sides of the last equation, using the Itô formula and the SDE for $\mathrm{d}y_t$ gives for the left hand side:

$$\mathrm{d}P(t, y_t) = \partial_t P \, \mathrm{d}t + \partial_x P (F \, \mathrm{d}t + G \mathrm{d}\mathcal{W}_t) + \frac{1}{2} G^2 \partial_x^2 P \, \mathrm{d}t,$$

and for the right hand side (using (1.11) instead of the Itô formula):

$$\mathrm{d}(a_t y_t + b_t \, \mathrm{e}^{rt}) = a_t \, \mathrm{d}(F \, \mathrm{d}t + G \mathrm{d}\mathcal{W}_t) + r b_t \, \mathrm{e}^{rt} \, \mathrm{d}t.$$

Equating the $d\mathcal{W}_t$ terms means that we must have

$$a_t(y_t) = \partial_x P(t, y_t),$$

while equating the $\mathrm{d}t$ terms means that

$$\partial_t P + \frac{1}{2} G^2 \partial_x^2 P - r b_t \, \mathrm{e}^{rt} = 0$$

By (1.12), $b_t = (P(y_t, t) - a_t y_t) \, \mathrm{e}^{-rt} = (P(y_t, t) - \partial_x P(y_t, t) y_t) \, \mathrm{e}^{-rt}$, and plugging this into the last equation finally means that P must satisfy

$$\partial_t P + \frac{1}{2} G^2 \partial_x^2 P + r(x \partial_x P - P) = 0,$$

with final condition $P(x, T) = \Phi(x)$. This is the BSPDE (for general F, not only for geometric Brownian motion!). Let us compare this with the expected discounted payoff function given by (1.7), which we write slightly differently as

$$\partial_t u + \frac{1}{2} G^2 \partial_x^2 P + r\Big(\frac{1}{r} F \partial_x u - u\Big) = 0.$$

We see that the naive approach will only work when $F(x) = rx$, and not for any other F. However, this situation can be forced to happen by using the so-called Girsanov transformation. This topic is beyond the scope of the current course.

1.6. **Boundary value problems.** Let now $D \subset \mathbb{R}^n$ be a subset of the space of possible asset values. We study the SDE

$$\mathrm{d}y_s^{(i)} = F_i(\boldsymbol{y}_s, s)\,\mathrm{d}s + \sum_{j=1}^{n} G_{ij}(\boldsymbol{y}_s, s)\,\mathrm{d}\mathcal{W}_s^{(j)}$$

whenever $\boldsymbol{y}_s \in D$, with starting value $\boldsymbol{y}_t = \boldsymbol{x} \in D$. Let $\tau(\boldsymbol{x})$ be the first time that $\boldsymbol{y}_s$ exits D, or $\tau(\boldsymbol{x}) = T$ if $\boldsymbol{y}_s$ does not leave D before T. (note that $\tau(\boldsymbol{x})$ is random!). Let

$$u(x,t) = \mathbb{E}_{\boldsymbol{y}_t=\boldsymbol{x}}(\Phi(\boldsymbol{y}_{\tau(\boldsymbol{x})}, \tau(\boldsymbol{x}))). \tag{1.13}$$

An example where this is useful is a knock-out option. In this case, D can be an interval $[a,b] \subset \mathbb{R}$, or a half-interval $[a,\infty)$. The option pays nothing (knock-out) if the stock price leaves D before maturity T, and pays $\Phi(\boldsymbol{y}_T, T)$ otherwise. This can be brought into the form above by having $\Phi(\boldsymbol{y}, s) = 0$ whenever $s < T$. The function u then solves the *boundary value problem*

$$\begin{aligned} \partial_t u(\boldsymbol{x}, s) + \mathscr{L} u(\boldsymbol{x}, s) &= 0 \qquad \text{for } \boldsymbol{x} \in D, \\ u(\boldsymbol{x}, t) &= \Phi(\boldsymbol{x}, t) \qquad \text{for } \boldsymbol{x} \in \partial D. \end{aligned} \tag{1.14}$$

Here, $\mathscr{L}$ is given by (1.5).
The derivation of (1.14) is similar to what we had before. (1.13) is equivalent to

$$0 = \mathbb{E}_{\boldsymbol{y}_t=\boldsymbol{x}}\Big(u(\boldsymbol{y}_{\tau(\boldsymbol{x})}, \tau(\boldsymbol{x})) - u(\boldsymbol{y}_t, t)\Big) = \mathbb{E}_{\boldsymbol{y}_t=\boldsymbol{x}}\Big(\int_t^{\tau(\boldsymbol{x})} \mathrm{d}u(\boldsymbol{y}_s, s)\Big).$$

with the condition $u(\boldsymbol{x}, s) = \Phi(\boldsymbol{x}, s)$ whenever $\boldsymbol{x} \in \partial D$. For $s < \tau(\boldsymbol{x})$, we compute $\mathrm{d}u(\boldsymbol{y}_s, s)$ as before using the Itô formula and obtain the PDE as we did several times already. The difference is only the boundary condition, which transfers into the PDE. There is one important caveat, which we have hidden a bit: namely, for the stochastic integrals

$$\mathbb{E}\Big(\int_t^{\tau(\boldsymbol{x})} G_{ij}(\boldsymbol{y}_s, s)\,\mathrm{d}\mathcal{W}_s^{(j)}\Big)$$

to be equal to zero, we need that $\mathbb{E}_{\boldsymbol{y}_t=\boldsymbol{x}}(\tau(\boldsymbol{x})) < \infty$. This is automatic here since we assumed $\tau(\boldsymbol{x}) \leqslant T$, but if we would study an infinite time horizon, this would cause problems, and we would have to be very careful. For more information search the internet with the keyword 'gamblers ruin'.