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Viscous Cahn–Hilliard Equation II. Analysis*

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The viscous Cahn–Hilliard equation may be viewed as a singular limit of the phase-field equations for phase transitions. It contains both the Allen–Cahn and Cahn–Hilliard models of phase separation as particular cases; by specific choices of parameters it may be formulated as a one-parameter (say α) homotopy connecting the Cahn–Hilliard ($\alpha=0$) and Allen–Cahn ($\alpha=1$) models. The limit $\alpha=0$ is singular in the sense that the smoothing property of the analytic semigroup changes from being of the type associated with second order operators to the type associated with fourth order operators. The properties of the gradient dynamical system generated by the viscous Cahn–Hilliard equation are studied as α varies in $[0, 1]$. Continuity of the phase portraits near equilibria is established independently of $\alpha \in [0, 1]$ and, using this, a piecewise, uniform in time, perturbation result is proved for trajectories. Finally the continuity of the attractor is established and, in one dimension, the existence and continuity of inertial manifolds shown and the flow on the attractor detailed.

1. Introduction

In this paper we prove various analytical results concerning the viscous Cahn–Hilliard (VCH) equation in dimension $d=1$, 2 or 3 namely:

$$\begin{aligned} (1-\alpha)\, u_t &= \Delta w, \qquad && x \in \Omega, \quad t>0, \\ \alpha u_t &= \Delta u + f(u) + w, \qquad && x \in \Omega, \quad t>0, \end{aligned} \tag{1.1}$$

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together with the boundary conditions

$$u = w = 0 \qquad \text{for} \quad x \in \partial\Omega, \quad t > 0, \tag{1.2}$$

and initial condition

$$u(x, 0) = u_0(x), \qquad x \in \Omega. \tag{1.3}$$

Throughout the paper $\alpha \in [0, 1]$ and Ω is a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$. We make the following assumption about the function $f(\cdot)$:

Assumption (F). The function $f(\cdot)$ has the form

$$f(u) = \sum_{j=0}^{2p-1} b_j u^j, \qquad b_{2p-1} < 0$$

where $p = 2$ if $d = 3$ and $p < \infty$ otherwise.

The equation arises as a model of phase transitions and is derived in [17]. Note that the typical function $f(\cdot)$ arising in applications is

$$f(u) = \gamma^{-1}[u - u^3] \tag{1.4}$$

and that this satisfies Assumption (F).

Our aim in this paper is two-fold. First, by setting $\alpha = 0$ and $\alpha = 1$ in (1.1)–(1.3) we obtain two distinct models of phase separation namely the Cahn–Hilliard model of spinodal decomposition and the Allen–Cahn model of grain boundary migration. It is of interest to understand how these models are related and the homotopy parameter α enables us to do this. Note that the model (1.1)–(1.3) itself arises as a singular limit of the phase-field equations for phase separation: see [3]. Secondly, for $\alpha = 1$ the dynamics of the resulting reaction-diffusion equation are very well understood. It is an interesting question in the theory of differential equations to extend this knowledge to other equations and the model (1.1)–(1.3) enables some steps to be made in this direction. Numerical results showing the insensitivity of the global attractor for (1.1)–(1.3) to changes in α may be seen in [3].

In Section 2 we describe an existence and regularity theory for the equation, based on the theory of analytic semigroups in [10, 18, 15]. In Section 3 we consider the continuity with respect to α of phase portraits near equilibria; our approach is based on a formulation for trajectories of evolution equations as boundary value problems in time and is motivated by [14]. In Section 4 we use the results of Section 3 to prove a shadowing-type result for trajectories of the viscous Cahn–Hilliard equation, again with respect to variations in the parameter α. The form of results is very closely related to, and motivated by, the work of Babin and Vishik [1]; however the results do differ slightly in form and non-trivially in proof and may therefore be of independent interest. In Section 5 we consider the existence of a global attractor and discuss its continuity with respect to α using the results of Hale [9]; furthermore, (in one dimension) we apply a theorem of Mischaikow [16] which enables us to study the dynamics on the attractor with respect to variation in the parameter α. In Section 6 we also work exclusively in one dimension and prove existence and perturbation results for an inertial manifold. Numerical data presented in [2] indicate that results similar to those proved here also hold for (1.1) subject to Neumann boundary conditions.

Throughout this paper C denotes a generic constant independent of α, but possibly depending upon other quantities. The notation C_α is used to denote a constant depending upon $\alpha \in (0, 1]$ which may become unbounded as $\alpha \to 0$.

2. Existence and Regularity

In this section we formulate (1.1)–(1.3) as an ordinary differential equation in a Banach space and apply semi-group theory (cf. [10, 18]) to prove existence and regularity results together with continuity results, in α, for trajectories. Let $(\cdot, \cdot)$ and $|\cdot|$ denote, respectively, the inner product and norm of $L^2(\Omega)$. We define the linear operator $A = -\Delta$ with domain of definition $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. By spectral theory we may also define the spaces $\dot{H}^s = \mathcal{D}(A^{s/2})$ with norms $|v|_s = |A^{s/2}v|$ for real s. It is well known that $\dot{H}^s$ is a subspace of $H^s(\Omega)$ and that $|\cdot|_s$ and $\|\cdot\|_s \equiv \|\cdot\|_{H^s(\Omega)}$ are equivalent on $\dot{H}^s$. In particular $|v|_1 = \|A^{1/2}v\| = |\nabla v|$ is equivalent to $\|v\| :=$ $\|v\|_1$ on $H^1(\Omega)$ and

$$\|v\|_s \leqslant C\,|v|_s \qquad \forall v \in \dot{H}^s, \quad s \geqslant 0. \tag{2.1}$$

We use the notation

$$\mathcal{B}(v, r) := \{u \in H_0^1(\Omega) : \|u - v\| \leqslant r\},$$
$$\mathcal{B}^2(v, r) := \{u \in \mathcal{D}(A) : \|u - v\|_2 \leqslant r\}.$$

We define G: $L^2(\Omega) \to \mathcal{D}(A)$ to be the Green's operator for A. Thus

$$v = Gf \Leftrightarrow Av = f.$$

Finally we introduce the invertible operator $B_\alpha: L^2(\Omega) \to L^2(\Omega)$ defined by

$$B_\alpha := \alpha I + (1-\alpha)\, G \tag{2.2}$$

and the operator $A_\alpha := B_\alpha^{-1} A$ whose domain of definition is, for $\alpha > 0$, $\mathscr{D}(A_\alpha) = \mathscr{D}(A)$ so that $\mathscr{D}(A_\alpha^{s/2}) = \dot{H}^s$ and for $\alpha = 0$, $\mathscr{D}(A_0) = \dot{H}^4$. It is convenient to use the notation $|v|_B := (v, B_\alpha v)^{1/2}$. Clearly, for $\alpha > 0$ there exists C such that

$$\alpha^{1/2}\, |v| \leqslant |v|_B \leqslant C\, |v| \qquad \forall v \in L^2(\Omega) \tag{2.3}$$

and for $\alpha = 0$, $|v|_B = |v|_{-1} := \|v\|_{H^{-1}(\Omega)}$. Furthermore for each $\alpha \in (0, 1]$ and $\beta \geqslant 0$, $B_\alpha^\beta: L^2(\Omega) \to L^2(\Omega)$ is bounded and has a bounded inverse.

It follows that (1.1)–(1.2) may be written as the abstract initial value problem

$$B_\alpha u_t + Au = f(u), \tag{2.4}$$

or equivalently

$$u_t + B_\alpha^{-1} Au = B_\alpha^{-1} f(u), \tag{2.5}$$

with

$$u(0) = u_0. \tag{2.6}$$

Note that, since B_α^{-1} is bounded from $L^2(\Omega)$ into itself for each $\alpha > 0$, Eq. (2.5) is qualitatively of second-order in space for $\alpha > 0$, although it also has a non-local character. In contrast, for $\alpha = 0$ the equation is of fourth-order in space and local in character. Thus $\alpha = 0$ is a singular limit for the equation.

Under Assumption (F) it may be shown (see, for example, [8]) that $f(u)$ satisfies the following estimates: for all $u, v \in \mathscr{B}(0, R)$ there exists $C = C(R) > 0$ such that

$$|f'(u)\, w| \leqslant C\, \|w\|, \tag{2.7}$$

$$|f(u) - f(v)| \leqslant C\, \|u - v\|, \tag{2.8}$$

$$|f''(u)\, wz| \leqslant C\, \|w\| \cdot \|z\|, \tag{2.9}$$

$$|A(f'(u)\, w)| \leqslant C(|w|_3 + |u|_3\, \|w\|), \tag{2.10}$$

$$|f'(u)\, v|_{-1} \leqslant C\, |v|. \tag{2.11}$$

Let the set of equilibria of (2.4) or (2.5) be denoted by $\mathscr{E}$ so that

$$\mathscr{E} = \{v \in \dot{H}^2 : Av = f(v)\} \tag{2.12}$$

and $\mathscr{E}$ is clearly independent of α. Then, under Assumption (F) and smoothness of $\partial\Omega$, there is a constant $C > 0$:

$$|v|_3 \leqslant C \qquad \forall v \in \mathscr{E}. \tag{2.13}$$

Throughout the paper we assume that $\mathscr{E}$ contains only hyperbolic equilibria so that $\mathscr{E}$ contains N distinct points. Equation (2.4) has the α independent Lyapunov functional $V \in C(H_0^1(\Omega), \mathbb{R})$ defined by

$$V(v) := \tfrac{1}{2}|v|_1^2 - (F(v), 1), \tag{2.14}$$

where $F(u) := \int^u f(s)\, ds$. Solutions of (2.4) clearly satisfy

$$|u_t|_B^2 + \frac{d}{dt} V(u) = 0. \tag{2.15}$$

In [9] it is shown that there exist $c_i > 0$, $i = 1, 2$ such that

$$V(v) \geqslant c_1\, |v|_1^2 - c_2 \qquad \forall v \in H_0^1(\Omega). \tag{2.16}$$

It also follows from Assumption (F) that $\exists C = C(R) > 0$ such that

$$V(v) \leqslant C \qquad \forall v \in \mathscr{B}(0, R). \tag{2.17}$$

Hence, under Assumption (F), equation (2.15) yields the a priori estimate

$$|u(t)|_1 \leqslant C(R) \qquad \forall t \geqslant 0 \tag{2.18}$$

for solutions of (2.4) with $u_0 \in \mathscr{B}(0, R)$. This fact is used to establish global existence of solutions to (2.4).

Let $S_\alpha(\cdot, \cdot)$: $\mathbb{R}^+ \times H_0^1(\Omega) \to H_0^1(\Omega)$ denote the solution operator to (2.4) so that $u(t) = S_\alpha(t, u_0)$. We denote by $DS_\alpha(\cdot, \cdot)$ and $\partial_t S_\alpha(\cdot, \cdot)$ the Fréchet derivatives of $S_\alpha(t, u)$ with respect to u and t respectively.

Using the theory in [10] for $\alpha > 0$, the results of [8] for $\alpha = 0$ and similar results concerning the derivative of the solution operator, we have the following existence and regularity theorem for solutions of (2.4).

THEOREM 2.1. *For any* $u_0 \in \mathscr{B}(0, R)$ *there exists for each* $\alpha \in [0, 1]$ *a unique solution* $u(t)$ *to* (2.4) *such that* $u(t) \in C([0, T]; H^1(\Omega)) \cap C^1((0, T];$

$L^2(\Omega))$ *for every* $T>0$. *Furthermore* $S_\alpha \in C^1(\mathbb{R}^+ \times H_0^1(\Omega),\ H_0^1(\Omega))$ *and there exist constants* $C_i(T, R, \beta)$ *and* $C_i^\alpha(T, R, \beta)$ *for* $i = 1, 2, 3, 4$ *such that*

$$
\begin{aligned}
|S_0(t, u_0)|_\beta &\leqslant C_1 t^{-(\beta-1)/4} && \forall t \in (0, T], \quad \forall \beta \in [1, 4] \\
|S_\alpha(t, u_0)|_\beta &\leqslant C_1^\alpha t^{-(\beta-1)/2} && \forall t \in (0, T], \quad \forall \beta \in [1, 2], \quad \alpha \in (0, 1] \\
|\partial_t S_0(t, u_0)| &\leqslant C_2 t^{-3/4} && \forall t \in (0, T] \\
|\partial_t S_\alpha(t, u_1)| &\leqslant C_2^\alpha t^{-1/2} && \forall t \in (0, T], \quad \alpha \in (0, 1] \\
|\partial_t DS_0(t, u_0)\, w| &\leqslant C_3 t^{-3/4}\, \|w\| && \forall t \in (0, T] \\
|\partial_t DS_\alpha(t, u_0)\, w| &\leqslant C_3^\alpha t^{-1/2}\, \|w\| && \forall t \in (0, T], \quad \alpha \in (0, 1]. \\
\|DS_0(t, u_0)\, w\| &\leqslant C_4\, \|w\| && \forall t \in (0, T] \\
\|DS_\alpha(t, u_0)\, w\| &\leqslant C_4^\alpha\, \|w\| && \forall t \in (0, T], \quad \alpha \in (0, 1].
\end{aligned}
$$

Corollary 2.1. *For each* $\alpha \in [0, 1]$, $S_\alpha(\cdot, \cdot)$ *is a* C^1 *gradient semigroup for which orbits of bounded sets are bounded and which is completely continuous and asymptotically smooth.*

Proof. Let E be a bounded set in $H_0^1(\Omega)$. By (2.15)–(2.17) it follows that $\{S_\alpha(t, \eta) : \eta \in E\}$ is uniformly bounded in $H_0^1(\Omega)$ for all $t \geqslant 0$. Furthermore, from Theorem 2.1 we deduce that $\{S_\alpha(t, \eta) : \eta \in E\}$ is uniformly bounded in $\|\cdot\|_2 \equiv \|\cdot\|_{H^2}$ for all $t \geqslant 1$. Thus the semigroup is completely continuous in the terminology of [9], p. 36. Corollary 3.2.2 of [9] shows that the semigroup is asymptotically smooth. Finally, Theorem 2.1 shows that each orbit $S_\alpha(t, \eta)$ $(\eta \in H_0^1(\Omega))$ is pre-compact and hence, by (2.15), $S_\alpha(\cdot, \cdot)$ defines a gradient system in the sense of Definition 3.8.1, [9]. ∎

The next lemma is of interest when studying the singular limit $\alpha \to 0$. It estimates the smoothing properties of $\exp(-A_\alpha t)\, B_\alpha^{-1}$ in an α independent way.

Lemma 2.2. *Let* $(\gamma - \beta) \in [-2, 2]$. *Then there exists a constant* $K > 0$ *such that*

$$
\|e^{-A_\alpha t} B_\alpha^{-1} u\|_\gamma \leqslant \frac{K\, \|u\|_\beta}{t^{(\gamma-\beta)/4 + 1/2}} \qquad \forall \alpha \in [0, 1], \quad \forall t > 0. \tag{2.19}
$$

Proof. Let A have eigenfunctions $\{\phi_j\}_{j=1}^\infty$ and eigenvalues $\{\lambda_j\}_{j=1}^\infty$ and set

$$
u = \sum_{j=1}^\infty a_j \phi_j. \tag{2.20}
$$

Define

$$\psi_j = \lambda_j^2/(\alpha\lambda_j + 1 - \alpha), \tag{2.21}$$

noting that these are the eigenvalues of A_α.

Let $\lambda(\psi)$ be the positive root of

$$\lambda^2 - \alpha\psi\lambda - (1-\alpha)\,\psi = 0 \tag{2.22}$$

so that

$$\lambda = \tfrac{1}{2}(\alpha\psi + [\alpha^2\psi^2 + 4(1-\alpha)\,\psi]^{1/2}). \tag{2.23}$$

Note that, from (2.21) it follows that, since $\alpha \in [0, 1]$, we have

$$\psi_j \geqslant \min(\lambda_j, \lambda_j^2)$$

and hence that there exists $\psi^* > 0$ such that

$$\psi_j \geqslant \psi^* > 0 \qquad \forall j \in N, \quad \forall \alpha \in [0, 1].$$

Calculation shows that

$$\begin{aligned} \|e^{-A_\alpha t} B_\alpha^{-1} u\|_\gamma^2 &= \sum_{j=1}^{\infty} a_j^2 \psi_j^2 \lambda_j^{\gamma-2} \exp(-2\psi_j t), \\ \|u\|_\beta^2 &= \sum_{j=1}^{\infty} a_j^2 \lambda_j^\beta. \end{aligned} \tag{2.24}$$

Thus it is sufficient to find $K > 0$ such that, for all $s \in [-2, 2]$,

$$g(\psi) := \lambda(\psi)^{s-2}\,\psi^2 \exp(-2\psi t)\, t^{s/2+1} \leqslant K \qquad \forall \psi \geqslant \psi^*,\ \alpha \in [0, 1],\ t > 0. \tag{2.25}$$

From (2.23) it follows that, for $\alpha \in [0, 1]$,

$$\lambda(\psi) \geqslant \alpha\psi, \qquad \lambda(\psi) \geqslant (1-\alpha)^{1/2}\,\psi^{1/2} \quad \forall \psi \in \mathbb{R}^+. \tag{2.26}$$

Now let

$$q = \max_{\sigma \geqslant 0} [\sigma^{s/2+1} e^{-2\sigma}]$$

noting that $q < \infty$ since $s \geqslant -2$. From the first bound in (2.26) and from (2.25) we deduce that, since $s \leqslant 2$,

$$\begin{aligned} g(\psi) &\leqslant (\alpha\psi)^{s-2}\,\psi^2 \exp(-2\psi t)\, t^{s/2+1} \\ &= \alpha^{s-2}\psi^{s/2-1}(\psi t)^{s/2+1} \exp(-2\psi t) \\ &\leqslant \alpha^{s-2}(\psi^*)^{s/2-1}\, q. \end{aligned}$$

Similarly, but using the second bound in (2.26),

$$\begin{aligned} g(\psi) &\leqslant [(1-\alpha)^{1/2}\,\psi^{1/2}]^{s-2}\,\psi^2 \exp(-2\psi t)\, t^{s/2+1} \\ &= (1-\alpha)^{s/2-1}\,(\psi t)^{s/2+1} \exp(-2\psi t) \\ &\leqslant (1-\alpha)^{s/2-1}\, q. \end{aligned}$$

Putting these two bounds together gives

$$g(\psi) \leqslant \min(\alpha^{s-2}(\psi^*)^{s/2-1}, (1-\alpha)^{s/2-1})\, q \qquad \forall \psi \geqslant \psi^*, \quad \forall t > 0.$$

Now, since $s \leqslant 2$, we have

$$\zeta := \max_{\alpha \in [0,1]} \min(\alpha^{s-2}(\psi^*)^{s/2-1}, (1-\alpha)^{s/2-1}) < \infty$$

and we deduce that (2.25) holds with $K = \zeta q$. This establishes the lemma. ∎

Using Lemma 2.2 we may prove the following perturbation result for trajectories of (2.4).

THEOREM 2.3. *Let $\xi, \xi^\alpha \in \mathscr{B}(0, R)$ and let $\alpha, \varepsilon, \alpha+\varepsilon \in [0, 1]$. There are constants $C_1 = C_1(\alpha, R, T)$ and $C_2 = C_2(R, T)$ such that, for all $t \in (0, T]$,*

$$\begin{aligned} &\|S^\alpha(t, \xi^\alpha) - S^{\alpha+\varepsilon}(t, \xi^{\alpha+\varepsilon})\| \leqslant C_1[\,|\xi^\alpha - \xi^{\alpha+\varepsilon}| + \varepsilon], \qquad \forall \alpha \in (0, 1], \\ &\|DS^\alpha(t, \xi)\, w - DS^{\alpha+\varepsilon}(t, \xi)\, w\| \leqslant C_1 \varepsilon\, \|w\|, \qquad \forall \alpha \in (0, 1], \\ &\|S^\varepsilon(t, \xi^\varepsilon) - S^0(t, \xi^0)\| \leqslant \frac{C_2}{t^{1/2}}\,[\,|\xi^\varepsilon - \xi^0| + \varepsilon] \\ &\|DS^\varepsilon(t, \xi)\, w - DS^0(t, \xi)\, w\| \leqslant \frac{C_2 \varepsilon}{t^{1/2}}. \end{aligned} \tag{2.27}$$

Proof. We recall the equations

$$B_\alpha u_t^\alpha + Au^\alpha = f(u^\alpha), \qquad u^\alpha(0) = \xi^\alpha \tag{2.28}$$

$$B_\alpha v_t^\alpha + Av^\alpha = df(u^\alpha)\, v^\alpha, \qquad v^\alpha(0) = w \tag{2.29}$$

satisfied by $S^{\alpha}(t, \xi^{\alpha})$ and $DS^{\alpha}(t, \xi^{\alpha})\, w$ respectively. Note that, if

$$\theta(t) := u^{\alpha+\varepsilon}(t) - u^{\alpha}(t) \tag{2.30}$$

then (2.28) gives

$$B_{\alpha+\varepsilon}\theta_t + A\theta = f(u^{\alpha+\varepsilon}) - f(u^{\alpha}) + \varepsilon(G - I)\, u^{\alpha}_t.$$

Applying the variation of constants formula we obtain

$$\begin{aligned}\|\theta(t)\| &\leqslant \|e^{-A_{\alpha+\varepsilon}t}\theta(0)\| \\ &\quad + \left\| \int_0^t e^{-A_{\alpha+\varepsilon}(t-s)} B^{-1}_{\alpha+\varepsilon} [\, f(u^{\alpha+\varepsilon}) - f(u^{\alpha}) + \varepsilon(G - I)\, u^{\alpha}_t]\, ds \right\|\end{aligned} \tag{2.31}$$

If $\alpha > 0$ then boundedness of $B^{-1}_{\alpha+\varepsilon}$ and equivalence of $D(A^{\beta}_{\alpha})$ and $D(A^{\beta})$ gives us

$$\|\theta(t)\| \leqslant \|\theta(0)\| + \int_0^t \frac{C_{\alpha}}{(t-s)^{1/2}} [\, |f(u^{\alpha+\varepsilon}) - f(u^{\alpha})| + \varepsilon\, |u^{\alpha}_t|]\, ds \tag{2.32}$$

Applying the Lipschitz condition (2.8) on f and Theorem 2.1 we find that

$$\begin{aligned}\|\theta(t)\| &\leqslant \|\theta(0)\| + \int_0^t \frac{C_{\alpha}\, \|\theta(s)\|\, ds}{(t-s)^{1/2}} + \varepsilon \int_0^t \frac{C_{\alpha}\, ds}{(t-s)^{1/2}\, s^{1/2}} \\ &\leqslant \|\theta(0)\| + C_{\alpha}\varepsilon + \int_0^t \frac{C_{\alpha}\, \|\theta(s)\|\, ds}{(t-s)^{1/2}}.\end{aligned} \tag{2.33}$$

By application of the Gronwall lemma in [8], the first result follows.

If $\alpha = 0$ then (2.31) and Lemma 2.2 yields

$$\|\theta(t)\| \leqslant \|\theta(0)\| + \int_0^t \frac{C}{(t-s)^{3/4}} [\, |f(u^{\varepsilon}) - f(u^{0})| + \varepsilon\, |u^{0}_t|]\, ds \tag{2.34}$$

The Lipschitz condition (2.8), Lemma 2.2 and Theorem 2.1 gives

$$\begin{aligned}\|\theta(t)\| &\leqslant \|\theta(0)\| + \int_0^t \frac{C\, \|\theta(s)\|\, ds}{(t-s)^{3/4}} + \varepsilon \int_0^t \frac{C\, ds}{(t-s)^{3/4}\, s^{3/4}} \\ &\leqslant \|\theta(0)\| + \frac{C\varepsilon}{t^{1/2}} + \int_0^t \frac{C\, \|\theta(s)\|\, ds}{(t-s)^{3/4}}.\end{aligned} \tag{2.35}$$

Application of the Gronwall lemma in [8] gives the third result.

We now consider the estimates on the derivative of the solution operator. Defining

$$\phi(t) := v^{\alpha+\varepsilon}(t) - v^{\alpha}(t) \tag{2.36}$$

yields from (2.29)

$$B_{\alpha+\varepsilon}\phi_t + A\phi = df(u^{\alpha+\varepsilon})\, v^{\alpha+\varepsilon} - df(u^{\alpha})\, v^{\alpha} + \varepsilon(G-I)\, v_t^{\alpha}. \tag{2.37}$$

For simplicity we consider the case $\alpha = 0$. We obtain, by Lemma 2.2,

$$\begin{aligned}
\|\phi(t)\| &\leqslant \|\phi(0)\| + \int_0^t \frac{C}{(t-s)^{3/4}} \left[|df(u^{\varepsilon})\, v^{\varepsilon} - df(u^0)\, v^0| + \varepsilon\, |(G-I)\, v_t^0| \right] \\
&\leqslant \|\phi(0)\| + \int_0^t \frac{C}{(t-s)^{3/4}} \left[|[df(u^{\varepsilon}) - df(u^0)]\, v^0| + |df(u^{\varepsilon})(v^{\varepsilon} - v^0)| \right] \\
&\quad + \varepsilon \int_0^t \frac{C\, |(G-I)\, v_t^0|\, ds}{(t-s)^{3/4}}.
\end{aligned}$$

By (2.7), (2.9) and Theorem 2.1 we have

$$\begin{aligned}
\|\phi(t)\| &\leqslant \|\phi(0)\| + \int_0^t \frac{C}{(t-s)^{3/4}} \left[\|\theta(s)\|\, \|v^0(s)\| + \|\phi(s)\| \right] ds \\
&\quad + \varepsilon \int_0^t \frac{C\, \|w\|\, ds}{(t-s)^{3/4}\, s^{3/4}}.
\end{aligned}$$

Using the third bound from this theorem with $\xi^{\varepsilon} = \xi^0$ and noting that $\phi(0) = 0$ we have, from Theorem 2.1,

$$\begin{aligned}
\|\phi(t)\| &\leqslant \int_0^t \frac{\varepsilon C\, \|w\|\, ds}{(t-s)^{3/4}\, s^{1/2}} + \frac{\varepsilon\, \|w\|}{t^{1/2}} + \int_0^t \frac{C\, \|\phi(s)\|\, ds}{(t-s)^{3/4}} \\
&\leqslant \frac{C\varepsilon\, \|w\|}{t^{1/4}} + \frac{C\varepsilon\, \|w\|}{t^{1/2}} + \int_0^t \frac{C\, \|\phi(s)\|\, ds}{(t-s)^{3/4}}
\end{aligned} \tag{2.38}$$

The Gronwall lemma of [8] gives the result. The bound for $\alpha \neq 0$ follows similarly. ∎

LEMMA 2.4. *For all* $u_0 \in \mathscr{B}(0, R)$ *there exists a constant* $C = C(R) > 0$ *such that, for all* $\alpha \in [0, 1]$

$$\frac{1}{2} \frac{d}{dt} |B_{\alpha} u_t|^2 \leqslant C\, |u_t|_B^2, \qquad t > 0. \tag{2.39}$$

Proof. Using the regularity from [10] and [8] it follows that u_{tt} exists for $t>0$ so that u_t satisfies the equation

$$B_\alpha u_{tt} + A u_t = f'(u)\, u_t, \qquad t>0. \tag{2.40}$$

Taking inner products with $B_\alpha u_t$ yields

$$\frac{1}{2}\frac{d}{dt}|B_\alpha u_t|^2 + \alpha\, \|u_t\|^2 + (1-\alpha)\, |u_t|^2$$
$$\leqslant |f'(u)\, u_t|_B\, |u_t|_B \leqslant \frac{\varepsilon^2}{2}\, |f'(u)\, u_t|_B^2 + \frac{1}{2\varepsilon^2}\, |u_t|_B^2. \tag{2.41}$$

Now

$$|f'(u)\, u_t|_B^2 = \alpha\, |f'(u)\, u_t|^2 + (1-\alpha)\, |f'(u)\, u_t|_{-1}^2.$$

By (2.7) and (2.11) we have

$$|f'(u)\, u_t|_B^2 \leqslant C(R)[\alpha\, \|u_t\|^2 + (1-\alpha)\, |u_t|^2]. \tag{2.42}$$

Thus, by choice of ε sufficiently small, the result follows. ▮

3. Neighbourhood of an Equilibrium Point

Here we prove perturbation results, with respect to α, for phase portraits of (2.4) near equilibria. Consider Eq. (2.4) in the neighbourhood of a hyperbolic equilibrium point $\bar{u} \in \mathscr{E}$. By introducing $v = u - \bar{u}$ we obtain

$$B_\alpha v_t + \mathscr{L} v = h(v), \qquad v(0) = v_0 \tag{3.1}$$

where $\mathscr{L} = A - df(\bar{u})$, $h(v) = f(\bar{u} + v) - f(\bar{u}) - df(\bar{u})\, v$.

Below we prove that $\mathscr{L}_\alpha := B_\alpha^{-1}\mathscr{L}$ is sectorial. Hence we may define projections $P_\alpha\colon H_0^1(\Omega) \to Y_\alpha$ and $Q_\alpha\colon H_0^1(\Omega) \to Z_\alpha$ associated with those parts of the spectrum of $\mathscr{L}_\alpha$ with negative and positive real parts respectively. Since $\bar{u}$ is hyperbolic we have $H_0^1(\Omega) = Y_\alpha \oplus Z_\alpha$. We will drop explicit α dependence in P, Q, Y and Z except where necessary.

Lemma 3.1. *The operator $\mathscr{L}_\alpha$ is sectorial for every $\alpha \in [0, 1]$. For $\alpha \in (0, 1]$ we have $D(\mathscr{L}_\alpha^\beta) \equiv \dot{H}^{2\beta}$ and, for $\alpha = 0$, $D(\mathscr{L}_0^\beta) \equiv \dot{H}^{4\beta}$, for all $\beta \in [0, 1]$.*

Proof. For $\alpha \in (0, 1]$, we have

$$\mathscr{L}_\alpha - A_\alpha = B_\alpha^{-1}\mathscr{L} - B_\alpha^{-1} A = -B_\alpha^{-1}\, df(\bar{u}). \tag{3.2}$$

But, by (2.7),

$$|B_\alpha^{-1}\, df(\bar{u})(A_\alpha)^{-1/2}\, w| = |B_\alpha^{-1}\, df(\bar{u})\, A^{-1/2} B_\alpha^{1/2} w|$$
$$\leqslant C_\alpha\, |df(\bar{u})\, A^{-1/2} B_\alpha^{1/2} w| \leqslant C_\alpha\, \|A^{-1/2} B_\alpha^{1/2} w\| \tag{3.3}$$
$$\leqslant C_\alpha\, |B_\alpha^{1/2} w| \leqslant C_\alpha\, |w|. \tag{3.4}$$

Hence by [10], Corollary 1.4.5 and Theorem 1.4.8, we have that $\mathscr{L}_\alpha$ is sectorial and also that

$$D(\mathscr{L}_\alpha^\beta) = D(A_\alpha^\beta) = \dot{H}^{2\beta} \qquad \forall \beta \in [0, 1].$$

For $\alpha = 0$ we have

$$\mathscr{L}_0 - A^2 = -A\, df(\bar{u}). \tag{3.5}$$

By (2.10), recalling that $\mathscr{E}$ is bounded in $H^3(\Omega)$ by (2.13),

$$|(\mathscr{L}_0 - A^2)\, A^{-3/2} w| = |A\, df(\bar{u})\, A^{-3/2} w|$$
$$\leqslant C\, |A^{-3/2} w|_3 + C\, |\bar{u}|_3\, |A^{-3/2} w|_1$$
$$\leqslant C\, |w| + CK\, |A^{-1} w| \leqslant C\, |w|. \tag{3.6}$$

By [10], Corollary 1.4.5 and Theorem 1.4.8, we deduce that $\mathscr{L}_0$ is sectorial and $D(\mathscr{L}_0^\beta) \equiv D(A^{2\beta}) \equiv \dot{H}^{4\beta}$ as required. ▮

Using Lemma 3.1 it follows from [10], Theorems 1.5.3 and 1.5.4, that $\exists \delta$, $K > 0$ such that for all $\alpha \in [0, 1]$

$$\begin{aligned} \|e^{+\mathscr{L}_\alpha t} y\| &\leqslant K e^{-\delta t}\, \|y\| \qquad \forall y \in Y_\alpha, \\ \|e^{-\mathscr{L}_\alpha t} z\| &\leqslant K e^{-\delta t}\, \|z\| \qquad \forall z \in Z_\alpha. \end{aligned} \tag{3.7}$$

We define

$$L_\alpha(t) := e^{-\mathscr{L}_\alpha t}, \qquad G_\alpha(v, t) = \int_0^t L_\alpha(t-s)\, B_\alpha^{-1} h(\bar{S}_\alpha(s, v))\, ds \tag{3.8}$$

where

$$\bar{S}_\alpha(\tau, v) := S_\alpha(\tau, \bar{u} + v) - \bar{u}, \qquad \forall \tau \geqslant 0. \tag{3.9}$$

We also set $L_\alpha := L_\alpha(T)$, $G_\alpha(\cdot) := G_\alpha(\cdot, T)$. Thus, if $V_n = v(nT)$ then (3.1) yields

$$V_{n+1} = L_\alpha V_n + G_\alpha(V_n). \tag{3.10}$$

By (3.7) it follows that, for any $a < 1$ there exists $T^* = T^*(\bar{u}) > 0$ such that, for all $T \geqslant T^*$, $\alpha \in [0, 1]$

$$\begin{aligned} \|L^{-1}v\| &\leqslant a\,\|v\| \qquad \forall v \in Y_\alpha \\ \|Lv\| &\leqslant a\,\|v\| \qquad \forall v \in Z_\alpha \end{aligned} \tag{3.11}$$

It is shown in [14] that there exists a function $K: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $K(\rho) \to 0_+$ as $\rho \to 0_+$ such that

$$|h(v) - h(w)| \leqslant K(\rho)\,\|v - w\| \qquad \forall v, w \in P(0, \rho). \tag{3.12}$$

Thus, for any $\alpha > 0$ there exists $K_1 = K_1(\alpha) > 0$ such that

$$|B_\alpha^{-1}[h(v) - h(w)]| \leqslant K_1 K(\rho)\,|A^{1/2}[v - w]|. \tag{3.13}$$

Note that $K_1(\alpha)$ is unbounded as $\alpha \to 0$. Also, for $\alpha = 0$, we have

$$|A_0^{-1/2} B_0^{-1}[h(v) - h(w)]| \leqslant K(\rho)\,|A_0^{1/4}[v - w]|. \tag{3.14}$$

This shows that Assumption 4.5 of [19] is satisfied with $\xi = 0$, $\beta = \frac{1}{2}$ for $\alpha > 0$ and $\xi = -\frac{1}{2}$, $\beta = \frac{1}{4}$ for $\alpha = 0$ and $g(\cdot) := B_\alpha^{-1}h(\cdot)$.

We now consider the boundary value problem

$$B_\alpha v_t + \mathscr{L}v = h(v), \qquad P_\alpha v(\tau) = \xi, \qquad Q_\alpha v(0) = \eta. \tag{3.15}$$

The following two results have similar proofs. We give only the proof of the second in detail.

THEOREM 3.2. *Let* $\alpha_0 \in (0, 1]$. *Then there exist constants* $C = C(\alpha_0) > 0$, $\rho^* = \rho^*(\alpha_0) > 0$ *such that, for all* $\rho \in (0, \rho^*)$, $\alpha, \beta \in (\alpha_0, 1]$, $\tau > T^*$ *and* $\xi \in Y_\alpha$, $\eta \in Z_\alpha$ *with* $\|\xi\|$, $\|\eta\| \leqslant \rho/2$ *there is a solution* $v^\alpha(t)$ *of* (3.15), *unique in* $\overline{B(0, C\rho)}$, *and a point* $w \in H_0^1(\Omega)$ *satisfying*

$$\sup_{T^* \leqslant t \leqslant \tau} \|v^\alpha(t) - \bar{S}_\beta(t, w)\| \leqslant C\,|\alpha - \beta|. \tag{3.16}$$

THEOREM 3.3. *Let* $\alpha = 0$. *There exist constants* C, ρ^*, $\varepsilon^* > 0$ *such that, for all* $\rho \in (0, \rho^*)$, $\varepsilon \in (0, \varepsilon^*)$, $\tau > T^*$ *and* $\xi \in Y_0$, $\eta \in Z_0$ *with* $\|\xi\|$, $\|\eta\| \leqslant \rho/2$

there is a solution $v^0(t)$ *of* (3.15) *unique in* $\overline{B(0, C\rho)}$, *and a point* $w \in H_0^1(\Omega)$ *satisfying*

$$\sup_{T^* \leqslant t \leqslant \tau} \|v^0(t) - \bar{S}_\varepsilon(t, w)\| \leqslant C\varepsilon. \tag{3.17}$$

Proof of Theorems 3.2 and 3.3. We consider Theorem 3.3 first. We apply Corollary 4.14 of [19] to get existence and uniqueness. The perturbation result follows from Theorem 4.18 of [19]. To apply the theory of [19], Chap. 4 we need to establish two things: (i) the existence of a C^1 semigroup $S_\alpha(\cdot, \cdot)$ with C^1 dependence on α uniformly on bounded sets of $H_0^1(\Omega)$ and on bounded time intervals disjoint from the origin; (ii) we need to show that $g(\cdot) := B_\alpha^{-1} h(\cdot)$ satisfies Assumption 4.5 of [19]. Point (i) follows from Theorems 2.1 and 2.3 and point (ii) is established prior to this theorem.

The proof of Theorem 3.2 is similar. The dependence on α_0 follows from the non-uniformity of $K_1(\alpha)$ in (3.13) as $\alpha \to 0$. ∎

In addition to the problem (3.15) we also study the problem: find $v(t)$ with $\|Pv\|$, $\|Qv\| \leqslant \rho$, $\forall t \geqslant 0$ satisfying:

$$B_\alpha v_t + \mathscr{L} v = h(v), \quad Q_\alpha v(0) = \eta \in Z, \quad \|\eta\| \leqslant \rho/2, \qquad \forall t \geqslant 0. \tag{3.18}$$

This problem corresponds to constructing the stable set for (3.1). Also by applying Theorem 4.19 in [19] we may prove the following:

THEOREM 3.4. *Let* $\alpha_0 \in (0, 1)$. *Then there exist constants* $C = C(\alpha_0) > 0$, $\rho^* = \rho^*(\alpha_0) > 0$ *such that, for any* $\rho \in (0, \rho^*)$, $\alpha, \beta \in (\alpha_0, 1]$, $\tau > T^*$ *and* $\eta \in Z_\alpha$ *with* $\|\eta\| \leqslant \rho/2$ *there is a solution* $v^\alpha(t)$ *of* (3.18), *unique in* $\overline{B(0, C\rho)}$, *and a point* $w \in H_0^1(\Omega)$ *satisfying*

$$\sup_{t > T^*} \|v^\alpha(t) - \overline{S_\beta}(t, w)\| \leqslant C\,|\alpha - \beta|. \tag{3.19}$$

THEOREM 3.5. *Let* $\alpha = 0$. *There exist constants* C, ρ^*, $\varepsilon^* > 0$ *such that, for all* $\rho \in (0, \rho^*)$, $\varepsilon \in (0, \varepsilon^*)$, $\tau > T^*$ *and* $\eta \in Z_0$ *with* $\|\eta\| \leqslant \rho/2$ *there is a solution* $v^0(t)$ *of* (3.18), *unique in* $\overline{B(0, C\rho)}$, *and a point* $w \in H_0^1(\Omega)$ *satisfying*

$$\sup_{t > T^*} \|v^0(t) - \overline{S_\varepsilon}(t, w)\| \leqslant C\varepsilon. \tag{3.20}$$

4. Uniform in Time, Piecewise Approximation of Trajectories

Here we prove uniform in time, piecewise approximation of trajectories with respect to perturbations in α. In [1] similar results are proved: applications of results in [1] gives perturbations which are uniformly of size $\mathcal{O}(\alpha^q)$, for some $q<1$, and for which the approximating trajectories are finite dimensional. In contrast, our approach gives perturbations which are uniformly of size $\mathcal{O}(\alpha)$, but for which the piecewise approximating trajectories are not necessarily finite dimensional.

Throughout this and the next section we assume that $\mathcal{E}$ contains only hyperbolic equilibria. These are then isolated, finite in number and labelled $\{\bar{u}_i\}_{i=1}^{M}$. Recall that the set $\mathcal{E}$ is bounded in H^3 by (2.13).

Definition 4.1. For any $\rho>0$ the open set $Q(\rho)\subset\mathcal{D}(A)$ is defined by

$$Q(\rho)=\{\eta\in\mathcal{D}(A):|A\eta-f(\eta)|_0<\rho\}. \tag{4.1}$$

Lemma 4.1. *There exist $K_1>0$ and $\rho_0>0$ such that for $\rho<\rho_0$*

$$Q(\rho)=\bigcup_{i=1}^{M} Q_i \tag{4.2}$$

where $Q_i\cap Q_j=\emptyset$ for $i\neq j$, $Q_i\subset\mathcal{B}^2(\bar{u}_i, K_1\rho)$ and

$$K_1\rho<\delta_I=\min_{\substack{\bar{u},\,\bar{v}\in\mathcal{E}\\ \bar{u}\neq\bar{v}}}\{|\bar{u}-\bar{v}|, |\bar{u}-\bar{v}|_2\}$$

Proof. The separation of the equilibria in $L^2(\Omega)$ and $H^2(\Omega)$ follows from the fact that they are hyperbolic and lie in a bounded set in $H^3(\Omega)$ by (2.13).

Now note that there exists ρ_1 such that for $\rho<\rho_1$

$$Q(\rho)\subset\bigcup_{i=1}^{M}\mathcal{B}^2(\bar{u}_i;\delta_I/3).$$

This follows since, assuming the contrary, it holds that there exists a sequence $\{\rho_j\}$ converging to zero and a sequence $\{u_j\}$ such that $u_j\in Q(\rho_j)$ but $u_j\notin\bigcup_{i=1}^{M}\mathcal{B}^2(\bar{u}_i;\delta_I/3)$. However it holds that

$$|u_j|_1^2=(Au_j-f(u_j), u_j)+(f(u_j), u_j) \tag{4.3}$$

and by the Assumption (F) on $f(\cdot)$ it follows that the $\{u_j\}$ lie in a bounded set in $H_0^1(\Omega)$. Thus $\{u_j\}$ has a weakly converging subsequence in H_0^1 whose limit u_0 satisfies

$$Au_0 - f(u_0) = 0 \qquad \text{in } H^{-1}(\Omega) \tag{4.4}$$

and $u_0 \notin \mathscr{B}^2(\bar{u}_i; \delta_{I/3})$ for any $\bar{u}_i \in \mathscr{E}$. This is a contradiction.

We seek a solution $v(w)$, for a fixed $\bar{u} \in \mathscr{E}$, in the ball $\mathscr{B}^2(\bar{u}; \delta_{I/3})$ of

$$H(v, w) = 0 \tag{4.5}$$

where $H = \mathscr{D}(A) \times L^2(\Omega) \to L^2(\Omega)$ is defined by

$$H(v, w) := Av - f(v) - w. \tag{4.6}$$

Let $B(0, \rho)$ denote the closed ball of radius ρ in $L^2(\Omega)$. Using the implicit function theorem we construct a solution $v(w)$ for all $w \in B(0, \rho_0)$ which: (i) satisfies $v(0) = \bar{u}$ and (ii) is continuously differentiable with respect to w.

Let $Z = \mathscr{D}(A) \times L^2(\Omega)$ be equipped with the standard product norm: for $z = \{y, \omega\} \in Z$

$$\|z\| = |y|_2 + |\omega|_0. \tag{4.7}$$

The Frechet derivative $dH(\cdot, \cdot)\colon Z \to L^2(\Omega)$ is given by

$$dH(y, \omega) := \begin{pmatrix} DH(y, \omega) \\ H_\omega(y, \omega) \end{pmatrix} \tag{4.8}$$

where

$$DH(y, \omega) = A - df(y)\, I \qquad \text{and} \qquad H_\omega(y, \omega) = -I, \tag{4.9}$$

and $I\colon L^2(\Omega) \to L^2(\Omega)$ is the identity. Set

$$\|dH(y, \omega)\| = \sup_{\|z\| = 1} |dH(y, \omega)\, z|. \tag{4.10}$$

It follows that for $v_i \in \mathscr{B}^2(\bar{u}; \delta_{I/3})$, $i = 1, 2$ by (2.9)

$$\begin{aligned} \|dH(v_1, \omega_1) - dH(v_2, \omega_2)\| &= \sup_{\|z\| = 1} |(dH(v_1, \omega_1) - dH(v_2, \omega_2))\, z| \\ &\leqslant \sup_{|y|_2 \leqslant 1} |(df(v_1) - df(v_2))\, y| \\ &\leqslant C \sup_{|y|_2 \leqslant 1} \|v_1 - v_2\| \cdot \|y\| \\ &\leqslant C \sup_{|y|_2 \leqslant 1} |v_1 - v_2|_2. \end{aligned} \tag{4.11}$$

Hence it follows that

$$\|dH(v_1, \rho_1) - dH(v_2, \rho_2)\| \leqslant C\,|v_1 - v_2|_2, \tag{4.12}$$

and the continuous differentiability of H follows. The invertibility of $DH(\bar{u}, 0)$ is a consequence of the hyperbolicity of $\bar{u}$. Hence the implicit function theorem yields a C^1 function $\mathscr{F}\colon B(0, \rho_0) \to \mathscr{D}(A)$ such that $v = \mathscr{F}(w)$ solves (4.5) for w near 0 and satisfies $\bar{u} = \mathscr{F}(0)$. Since $\mathscr{F}$ is C^1 we deduce that

$$|v - \bar{u}|_2 \leqslant K_1\,|w|_0$$

for all $w \in B(0, \rho_0)$ and some $K_1 > 0$. If $|w|_0 < \rho \leqslant \rho_0$ then

$$|v - \bar{u}|_2 \leqslant K_1\,\rho.$$

But $|w|_0 < \rho$ if and only if $v \in Q(\rho)$; the required result follows. ∎

We now prove a "finite time of arrival" result for trajectories from a bounded set in H_0^1 into an H^2 neighbourhood of $\mathscr{E}$.

LEMMA 4.2. *Let E be bounded in $H_0^1(\Omega)$. For any $\rho > 0$ there exists $T^0 = T^0(\rho, E) < \infty$ such that T^0 is a time of arrival for $S_\alpha(\cdot, \cdot)$ from E into $Q(\rho)$. That is, for each $u_0 \in E$ there exists $t \in [0, T^0]$ such that $S_\alpha(t, u_0) \in Q(\rho)$.*

Proof. We observe that if $u(t) \notin Q(\rho)$ for $t \in [t_1, t_2]$ then

$$|B_\alpha u_t(t)|_0^2 = |Au(t) - f(u(t))|_0^2 \geqslant \rho^2. \tag{4.13}$$

Using the fact that there exists $C > 0$:

$$|v|_B^2 \geqslant C\,|B_\alpha v|^2 \qquad \forall v \in L^2(\Omega), \quad \forall \alpha \in [0, 1]$$

it follows that

$$|u_t|_B^2 \geqslant C\rho^2 \qquad \text{for} \quad t \in [t_1, t_2]. \tag{4.14}$$

From (2.15) we obtain

$$V(u(t_2)) - V(u(t_1)) \leqslant -(t_2 - t_1)\, C\rho^2. \tag{4.15}$$

Hence, by (2.16), (2.17)

$$\begin{aligned}(t_2 - t_1) &\leqslant \frac{1}{C\rho^2}\,(V(u(t_1)) - V(u(t_2))) \\ &\leqslant C(|u_0|_1)/\rho^2 \leqslant C(E)/\rho^2\end{aligned} \tag{4.16}$$

Thus any $T^0 > C(E)/\rho^2$ defines a time of arrival. ∎

The following inequality is useful:

LEMMA 4.3. *For any $u_1, u_2 \in \mathscr{D}(A)$ we have a constant $C_f > 0$ such that*

$$V(u_1) - V(u_2) \leqslant (Au_1 - f(u_1), u_1 - u_2) + C_f\, |u_1 - u_2|_0^2. \tag{4.17}$$

Proof. First note that, under Assumption (F), there is a constant C_f such that

$$f'(u) \leqslant 2C_f \qquad \forall u \in \mathbb{R}.$$

Thus, for any $u_1, u_2 \in \mathbb{R}$, there is a $\xi \in \mathbb{R}$ such that

$$\begin{aligned} F(u_2) - F(u_1) &= f(u_1)(u_2 - u_1) + \tfrac{1}{2} f'(\xi)(u_2 - u_1)^2 \\ &\leqslant f(u_1)(u_2 - u_1) + C_f(u_2 - u_1)^2. \end{aligned}$$

Hence, if $u_1(x)$, $u_2(x) \in \mathscr{D}(A)$, since the dimension $d \leqslant 3$,we have that

$$(F(u_2) - F(u_1), 1) \leqslant (f(u_1), u_2 - u_1) + C_f\, |u_2 - u_1|_0^2.$$

Now

$$\begin{aligned} V(u_1) - V(u_2) &= \tfrac{1}{2}(\nabla u_1, \nabla u_1) - \tfrac{1}{2}(\nabla u_2, \nabla u_2) - (F(u_1) - F(u_2), 1) \\ &\leqslant (\nabla u_1, \nabla u_1 - \nabla u_2) + (f(u_1), u_2 - u_1) + C_f\, |u_2 - u_1|_0^2 \\ &= (Au_1 - f(u_1), u_1 - u_2) + C_f\, |u_2 - u_1|_0^2. \end{aligned}$$

This completes the proof. ∎

It is convenient to introduce $E = E(V^*)$ defined by

$$E := \{\eta \in H_0^1 : V(\eta) \leqslant V^*\} \tag{4.18}$$

for any

$$V^* > V_{\min} := \inf_{\eta \in H_0^1(\Omega)} V(\eta).$$

Clearly E is nonempty and bounded in $H_0^1(\Omega)$. Furthermore for each $V^* > V_{min}$ there exist exactly N^* equilibria $\bar{u}_i \in \mathscr{E}$ $i \in [1, N^*]$ contained in E and $N^* \in [1, M]$.

LEMMA 4.4. *Let $u(t) \in E$ solve (2.4) in (t_1, t_2) and $\rho \in (0, \rho_0)$. If $u(t) \notin Q(\rho)$ $\forall t \in [t_1, t_2]$ and if there exists $\bar{u}_k \in \mathscr{E}$ such that $u_i := u(t_i) \in \partial Q_k$, $i = 1, 2$, then there exists $K_2 \geqslant 1$ such that*

$$u(t) \in \mathscr{B}^2(\bar{u}_k; K_2 \rho) \qquad \forall t \in [t_1, t_2]. \tag{4.19}$$

Proof. From (2.15) we have

$$\int_{t_1}^{t} |u_t(s)|_B^2 \, ds = V(u_1) - V(u(t)). \tag{4.20}$$

Thus, by Lemma 4.3, we have that for $t \in [t_1, t_2]$,

$$\begin{aligned} \int_{t_1}^{t} |u_t(s)|_B^2 \, ds &\leqslant \int_{t_1}^{t_2} |u_t(s)|_B^2 \, ds \\ &\leqslant |Au_1 - f(u_1)|_0 \, |u_1 - u_2|_0 + C_f \, |u_1 - u_2|_0^2. \end{aligned} \tag{4.21}$$

Note that

$$|Au_1 - f(u_1)| \leqslant K_1 \rho \tag{4.22}$$

by Lemma 4.1. Since for $\rho < \rho_0$, by Lemma 4.1, $\partial Q_k(\rho) \in \mathscr{B}^2(\bar{u}_k; K\rho)$ it follows that

$$|u_1 - u_2|_0^2 \leqslant C\rho^2. \tag{4.23}$$

Hence for $t \in [t_1, t_2]$, we have from (4.21), (4.22) and (4.23), the estimate

$$\int_{t_1}^{t} |u_t(s)|_B^2 \, ds \leqslant C\rho^2. \tag{4.24}$$

Applying Lemma 2.4, we find that for $t \in [t_1, t_2]$,

$$\begin{aligned} |Au(t) - f(u(t))|_0^2 = |B_\alpha u_t(t)|_0^2 &\leqslant |B_\alpha u_t(t_1)|_0^2 + C\rho^2 \\ &\leqslant |Au_1 - f(u_1)|_0^2 + C\rho^2 \end{aligned} \tag{4.25}$$

This proves the lemma, by (4.22). ∎

Lemma 4.5. *Let* $u(t) \in E$ *solve* (2.4) *in* (t_1, t_2), $u(t) \notin Q(\rho)$ *for* $t \in [t_1, t_2]$, $u_i = u(t_i) \in \partial Q_i(\rho)$ *for some* $\bar{u}_i \in \mathscr{E}$, $i = 1, 2$ *with* $\bar{u}_1 \neq \bar{u}_2$ *and* $\rho < \rho_0$. *Then there exists* $K_3 > 0$ *such that*

$$V(\bar{u}_2) - V(\bar{u}_1) \leqslant -K_3 \rho \delta_I. \tag{4.26}$$

Proof. Since $u(t) \notin Q(\rho)$ for $t \in [t_1, t_2]$ we have as in the proof of Lemma 4.2, that

$$|u_t|_B \geqslant C \, |B_\alpha u_t|_0 \geqslant C\rho$$

and hence from (2.15)

$$
\begin{aligned}
V(u_2)-V(u_1) &\leqslant -\int_{t_1}^{t_2} |u_t(s)|_B^2 \, ds \\
&\leqslant -C\rho \int_{t_2}^{t_1} |u_t(s)|_B \, ds \\
&\leqslant -C\rho \left| \int_{t_1}^{t_2} u_t(s)\, ds \right|_B \\
&= -C\rho \, |u(t_2)-u(t_1)|_B.
\end{aligned}
$$

But

$$|u(t_2)-u(t_1)|_B \geqslant |\bar{u}_2-\bar{u}_1|_B - |u(t_2) - \bar{u}_2|_B - |u(t_1)-\bar{u}_1|_B \geqslant \delta_I - C\rho.$$

Hence, by choice of ρ sufficiently small, there exists $K>0$:

$$V(u_2)-V(u_1) \leqslant -K\rho\delta_I. \tag{4.27}$$

Thus the lemma is proved by noting that

$$
\begin{aligned}
V(\bar{u}_2)-V(\bar{u}_1) = (V(\bar{u}_2)-V(u_2)) + (V(u_1)-V(\bar{u}_1)) \\
+ (V(u_2)-V(u_1))
\end{aligned}
\tag{4.28}
$$

and applying Lemma 4.3. ∎

LEMMA 4.6. *Let T^0 be the time of arrival for the set E into $Q(\rho)$. There exists ρ_0 such that if $\rho<\rho_0$ then for each $u_0 \in E$ there exist $N_0 = N_0(u_0, \rho) \leqslant N^*$ equilibria enumerated as $\{\bar{u}_i\}_{i=1}^{N_0}$ and N_0 intervals $\{I_i\}_{i=1}^{N_0}$ satisfying:*

(0) *The solution $u(t)$, $t \in [0, \infty)$, enters precisely N_0 distinct components $\{Q_i(\rho)\}_{i=1}^{N_0}$ of $Q(\rho)$;*

(i) *$I_i = [t_i^-, t_i^+] \subseteq \mathbb{R}^+$ where*

$$t_i^- = \inf\{t \colon u(t) \in Q_i(\rho)\}, \qquad t_i^+ = \sup\{t \colon u(t) \in Q_i(\rho)\};$$

(ii) *$I_i \cap I_j = \varnothing \quad i \neq j$;*

(iii) *$|t_i^- - t_{i-1}^+| \leqslant T^0 \quad i = 2, ..., N_0$;*

(iv) *$t_{N_0}^+ = \infty$.*

Furthermore $u(t) \in \mathscr{B}^2(\bar{u}_i, K_2\rho)$, $u(t) \notin \mathscr{B}^2(\bar{u}_j, K_2\rho)$, $j \neq i$ $\forall t \in (t_i^-, t_i^+)$ and there exists $C>0$:

$$V(\bar{u}_{i+1}) - V(\bar{u}_i) \leqslant -C\rho\delta_I, \qquad i = 1, 2, ..., N_0 - 1.$$

Proof. By [9], for any $u_0 \in E$, $\alpha \in [0, 1]$, we have that there exists $\bar{u} \in \mathscr{E}$ such that

$$\lim_{t \to \infty} S_\alpha(t, u_0) = \bar{u},$$

since (2.4) forms a gradient system by Corollary 2.1. Clearly for any u_0 and ρ there exist $N_0 = N_0(u_0, \rho)$ equilibria and N_0 intervals such that (i) and (iv) hold.

Now choose ρ_0 sufficiently small so that Lemma 4.1 holds and so that

$$\mathscr{B}^2(\bar{u}_i, K_2 \rho) \cap Q_j = \varnothing \qquad \forall i \neq j; \tag{4.29}$$

this can be done by Lemma 4.1. Now note that $u(t) \in \mathscr{B}^2(\bar{u}_i, K_2 \rho)$ $\forall t \in I_i$ by Lemma 4.4 as required and that Lemma 4.1 ensures that (ii) holds. Point (iii) holds by Lemma 4.2 and Lemma 4.5 gives the required estimate on the decrease in $V(\bar{u}_i)$. ▮

It is clear that individual trajectories corresponding to the same initial condition but slightly different values of α may separate exponentially. Hence it is not possible to prove uniform continuity of trajectories with respect to α. We seek such uniform in time perturbation results by weakening the notion of "solution" to allow piecewise continuous solutions in time with a finite number of discontinuities. With this in mind we make two definitions:

Definition 4.2. The function $\bar{u}(t; \alpha)$ is said to be a *piecewise continuous solution* of (2.4) if there exist an integer N, non-negative numbers $\{T_i\}_{i=0}^{N}$ and elements $\{U_i\}_{i=0}^{N-1}$ of $H_0^1(\Omega)$ such that $0 = T_0 < T_1 < T_2 < \cdots < T_N = \infty$ and for $i = 1, ..., N$

$$\bar{u}(t; \alpha) = S_\alpha(t - T_{i-1}, U_{i-1}), \qquad T_{i-1} \leqslant t < T_i.$$

Recall from Lemma 4.1 the constant ρ_0.

Definition 4.3. A piecewise continuous solution of (2.4) is said to be a *combined stabilised trajectory* (c.s.t) if there exists $\rho < \rho_0$ and $\{\bar{u}_j\}_{j=0}^{N-1} \in \mathscr{E}$ such that $U_j \in \mathscr{B}^2(\bar{u}_j; \rho)$ $j = 0, ..., N-1$ and $V(\bar{u}_j) < V(\bar{u}_{j-1})$, $j = 1, ..., N-1$.

The following result proves uniform continuity in time, and across a bounded set of initial data, of piecewise continuous solutions with respect to variation in α.

THEOREM 4.7. *Let* $\alpha_0 \in (0, 1]$, $E \subset H_0^1(\Omega)$. *Then there exists a constant* $C = C(\alpha_0, E)$, $\varepsilon^* = \varepsilon^*(\alpha_0, E)$ *such that, for every* $u_0 \in E$, α, $\alpha + \varepsilon \in [\alpha_0, 1]$ *there exists a c.s.t.* $\tilde{u}(t; \alpha + \varepsilon)$ *such that*

$$\sup_{t \geq 0} \|S_\alpha(t, u_0) - \tilde{u}(t; \alpha + \varepsilon)\| \leq C\varepsilon.$$

Proof. For simplicity consider $u_0 \notin Q$; the case $u_0 \in Q$ can be handled similarly. Define $t_0^- = t_0^+ = 0$ and $t_i^\pm$ as in Lemma 4.6. Given all $\{I_i\}_{i=1}^{N_0}$ we remove all I_j with $|I_j| \leq T_j^*$ where T_j^* is equal to $T^*(\bar{u}_j)$ from Section 3. Relabel $\{I_i\}_{i=1}^{M_0}$, $M_0 \leq N_0$.

Define

$$I_i^* = [t_i^- + T_i^*, t_i^+], \qquad J_i = [t_i^+, t_{i+1}^- + T_{i+1}^*].$$

Note that $|J_i| \leq T_0 + \sum_{j=1}^{N^*} (T_0 + T_j^*)$ where N^* is the total number of equilibria.

To define the c.s.t. we set $N = M_0 + 1$ and

$$T_i = t_i^- + T_i^*, \qquad i = 1, ..., N-1,$$

and $U_i = S_\beta(T_i^*, \bar{u}_i + w)$ where $\bar{u}_i$ is the unique equilibrium point in Q_i, $\beta = \alpha + \varepsilon$ and w are as in Theorem 3.2 for $i \neq M_0$ and as in Theorem 3.4 if $i = M_0$. We take $U_0 = u_0$.

On I_i^* we apply Theorems 3.2 and 3.4 to obtain the required error bound whilst on J_i we apply Theorem 2.3 since $|J_i|$ depends only on E. ∎

A similar proof yields the following:

THEOREM 4.8. *Let* $E \subset H_0^1(\Omega)$. *Then there exist constants* $C = C(E)$ *and* $\varepsilon^* = \varepsilon^*(E)$ *such that, for every* $u_0 \in E$, $\varepsilon \in [0, \varepsilon^*)$ *there exists a c.s.t.* $\tilde{u}(t; \varepsilon)$ *such that*

$$\sup_{t \geq 1} \|S_0(t, u_0) - \tilde{u}(t; \varepsilon)\| \leq C\varepsilon.$$

5. CONTINUITY OF THE ATTRACTOR

As a consequence of the existence of a Lyapunov function and the smoothing properties of Theorem 2.1 and Corollary 2.1 it is straightforward to prove the following by application of the theory in [9], Theorem 3.8.5; recall that we assume throughout this section that all equilibria of (2.4) are hyperbolic.

THEOREM 5.1. *For each* $\alpha \in [0, 1]$ *the semigroup* $S_\alpha(t, \cdot)$ *has a global attractor* $\mathscr{A}_\alpha$. *Furthermore*

$$\mathscr{A}_\alpha = \bigcup_{v \in \mathscr{E}} W(v), \tag{5.1}$$

where $W(\cdot)$ *denotes the unstable set.*

It is of interest to study the relationships between the sets $\mathscr{A}_\alpha$ as α varies in $[0, 1]$. To this end we let $d(A, B)$ denote the Haussdorf distance between two sets A and B in H^1. Thus $d(A, B) = 0$ if and only if the closures of A and B are identical sets.

THEOREM 5.2. *Let* $\alpha_0 \in (0, 1)$. *There is a constant* $C = C(\alpha_0) > 0$ *such that, for all* α, $\alpha + \varepsilon \in [\alpha_0, 1]$

$$d(\mathscr{A}_\alpha, \mathscr{A}_{\alpha+\varepsilon}) \leqslant C\,|\varepsilon|.$$

Furthermore, there is a constant $K > 0$ *such that for all* $\varepsilon \in [0, 1]$

$$d(\mathscr{A}_0, \mathscr{A}_\varepsilon) \leqslant K\varepsilon.$$

Proof. We apply Theorem 4.10.8 in [9]. The required gradient structure and smoothing properties for (H1) follow from Corollary 2.1; (H2)–(H5) are straightforward; the C^0 closeness of solutions from Theorem 2.3 implies (H7) whilst the C^1 closeness from the same theorem, together with the theory of [21], gives the closeness of unstable manifolds required in (H6). ∎

We remark that in one space dimension continuity of the attractor with respect to $\alpha \in [0, 1]$ in the presence of a non-hyperbolic equilibrium point has recently been shown in [7].

In the remainder of this section we consider only the case of dimension $d = 1$ and $\Omega = (0, 1)$. In this case we can say something detailed about the flow on the attractor and (in Section 6) also study existence and smoothness of inertial manifolds.

Theorem 5.2 is concerned only with the continuity of the attractor $\mathscr{A}_\alpha$ considered as a set of points in $H_0^1(\Omega)$. We now discuss the dynamics on the attractor and show, roughly speaking, that there is a subset of $\mathscr{A}_\alpha$ on which the dynamics are independent of $\alpha \in [0, 1]$ in the case where $f(\cdot)$ is given by (1.4). Let

$$D^p = \{z \in \mathbb{R}^p : \|z\| \leqslant 1\}$$

and

$$\partial D^p = \{z \in \mathbb{R}^p : \|z\| = 1\}.$$

Then any $z \in D^p$ may be written as $z = r\xi$ where $\xi \in \partial D^p$ and $r \in [0, 1]$.

Consider the flow on D^p generated by the equations

$$\xi_t = Q\xi - \langle Q\xi, \xi \rangle \xi, \qquad \xi(0) \in \partial D^p$$

and

$$r_t = r(1 - r), \qquad r(0) \in [0, 1]$$

where Q is the diagonal matrix $diag\{1, 1/2, 1/3, ..., 1/p\}$. We let $e_j^{\pm} = (0, ..., \pm 1, 0, ...)$ be unit vectors in the jth direction. We denote the flow on D^p by $\Theta(t)$: $D^p \to D^p$. In one dimension with f given by (1.4) and in the case $\alpha = 1$ the work of Henry [10, 11] shows that the flow on the attractor for (2.4) is equivalent to the flow on D^p generated by $\Theta(t)$. We now apply a general result, due to Mischaikow [16], to relate the flows $\Theta(t)$ on D^p and $S_\alpha(\cdot, \cdot)$ on $\mathscr{A}_\alpha$ for $\alpha \in [0, 1)$.

Under the conditions of the following theorem the equilibria are all hyperbolic if $\gamma \in (1/(p+1)^2 \pi^2, 1/p^2\pi^2)$. They then number $2p + 1$ and are labelled $\{\bar{u}_j^{\pm}\}_{j=1}^{p}$ and 0. See [5] for details.

THEOREM 5.3. *Consider Eq.* (2.4) *in dimension* $n = 1$, *with* $\Omega = (0, 1)$, $f(u)$ *given by* (1.4) *and* $\gamma \in (1/(p+1)^2 \pi^2, 1/p^2\pi^2)$. *For every* $\alpha \in [0, 1]$ *there exists an order preserving time reparameterisation of* $S_\alpha(t, \cdot)$, *denoted* $\tilde{S}_\alpha(t, \cdot)$ *and a continuous surjective map* ψ^α: $\mathscr{A}_\alpha \to D^p$, *such that*

$$\tilde{S}_\alpha(t, (\psi^\alpha)^{-1} z_0) = (\psi^\alpha)^{-1} \Theta(t) z_0$$

for every $z_0 \in \mathbb{R}^p$. *Furthermore*

$$(\psi^\alpha)^{-1} e_j^{\pm} = \bar{u}_j^{\pm}, \qquad j = 1, ..., p,$$

for each $\alpha \in [0, 1]$ *and* $(\psi^\alpha)^{-1} 0 = 0$.

Proof. We apply Theorems 1.2 and 2.1 of [16]. Hypothesis (H1) follows from our Theorem 5.2. Hypothesis (H2) follows from [5], together with Theorem 3.1 in [3] which show that the dimension of the unstable manifold of an equilibrium $\bar{u}$ is independent of α. Hypothesis (H3)(i) follows from our Corollary 2.1 and (H3)(ii) follows in a straightforward fashsion from (2.14), (2.15). ∎

6. CONTINUITY OF INERTIAL MANIFOLDS

We now proceed to study the existence and perturbation theory for inertial manifolds. We consider the case $d = 1$ and $\Omega = (0, 1)$ only. Since the singular limit $\alpha \to 0$ is the primary non-standard part of the analysis we shall give full details for the case α near 0 only. The difficulty here is that the operator A_α degenerates from being of second order type to being of fourth order type as $\alpha \to 0$. To overcome this problem use of the Lemma 2.2 is fundamental.

We assume that f satisfies Assumption (F). Using the existence of an attractor, and hence an absorbing set in $H_0^1(\Omega)$, together with the fact that the problem is posed in one dimension with f a polynomial, it follows by use of cut-off functions that the long time dynamics of (2.4) are completely equivalent to the dynamics of the equation

$$B_\alpha u_t + Au = r(u), \qquad u(0) = u_0 \tag{6.1}$$

where

$$\|r(u) - r(v)\| \leqslant L \, \|u - v\| \qquad \forall u, v \in H_0^1(\Omega). \tag{6.2}$$

Thus it is sufficient to study the existence of an exponentially attracting, positively invariant, finite dimensional manifold $\mathscr{M}$ for (6.1) in order to understand inertial manifolds for (2.4). The inertial manifold for (6.1) or (2.4) is defined to be the intersection of $\mathscr{M}$ with a positively invariant set inside which $f(\cdot)$ and $r(\cdot)$ are equivalent. We introduce the projections P and Q defined by

$$u = \sum_{j=1}^{\infty} u_j \phi_j \to Pu = \sum_{j=1}^{q} u_j \phi_j, \qquad Qu = \sum_{j=q+1}^{\infty} u_j \phi_j, \tag{6.3}$$

where the $\{\phi_i\}$ are defined in (2.20).

We let $Y = PH_0^1(\Omega)$, $Z = QH_0^1(\Omega)$ and seek the inertial manifold $\mathscr{M}$ as the graph of a function $\Phi \in C(Y, Z)$. Recall that on the inertial manifold equation (6.1) reduces to the ordinary differential equations in $\mathbb{R}^q$ given by

$$B_\alpha p_t + Ap = Pr(p + \Phi(p)), \qquad p(0) = p_0 \in Y. \tag{6.4}$$

THEOREM 6.1. *There is an integer q such that, for each $\alpha \in [0, 1]$, Eq.* (6.1) *has an inertial manifold $\mathscr{M}_\alpha$. Furthermore there is a constant $K > 0$ such that, for every α, $\alpha + \varepsilon \in [0, 1]$,*

$$d(\mathscr{M}_\alpha, \mathscr{M}_{\alpha+\varepsilon}) \leqslant K \, |\varepsilon|.$$

Proof. To construct the inertial manifold and analyze perturbations to it we employ the Hadamard graph transform approach in [12]. Let

$$L_\alpha := e^{-A_\alpha T}, \qquad N_\alpha(v) = \int_0^T L_\alpha(T - s) \, B_\alpha^{-1} r(S_\alpha(s, v)) \, ds. \tag{6.5}$$

From these we define the map $G(\cdot)=L_\alpha\cdot+N_\alpha(\cdot)$. We show that this mapping has an attractive invariant manifold which perturbs smoothly in α by use of the Main Theorem in [12]. That these manifolds are also invariant and exponentially attracting for the underlying continuous flow follows as in the proof of Theorem 4.2 in [12]. The notation from that paper, with $h=\alpha$ being the perturbation parameter, is used throughout this proof.

Thus it remains to verify Assumptions G, G^h and Conditions C' from [12]. Given $\alpha_0>0$ define

$$b=\sup_{\alpha\in[0,\alpha_0]} e^{-\lambda^{(\alpha)}T},\qquad a=\inf_{\alpha\in[0,\alpha_0]} e^{-\Lambda^{(\alpha)}T},\qquad c=\inf_{\alpha\in[0,\alpha_0]} e^{-\mu^{(\alpha)}T} \tag{6.6}$$

where $\lambda^{(\alpha)}$, $\Lambda^{(\alpha)}$ and $\mu^{(\alpha)}$ are the qth, $(q+1)$st and first eigenvalues of A_α respectively. Straightforward calculation using (6.2), the continuity of the semigroup $S_\alpha(t,\cdot)$ and Lemma 2.2 with $\gamma=\beta=1$ gives the existence of a constant C, independent of α, such that

$$\begin{aligned}\|N_\alpha(u)-N_\alpha(v)\| &\leqslant CT^{1/2}\,\|u-v\|, &&\forall u,v\in H_0^1(\Omega),\quad \forall\alpha\in[0,1],\\ \|N_\alpha(u)\| &\leqslant CT^{1/2}, &&\forall u\in H_0^1(\Omega),\quad \forall\alpha\in[0,1].\end{aligned} \tag{6.7}$$

Proceeding as in the proof of Lemma 4.1 in [12] it follows that, for $\alpha=0$, Conditions C' and Assumptions G hold for sufficiently large q ($=q_0$ say) since $A_0=A^2$ has eigenvalues $\lambda_j=j^4\pi^4$ and here $\beta=\frac{1}{2}$ by virtue of (6.7). For $\alpha\neq 0$ the eigenvalues of A_α are

$$\lambda_j^{(\alpha)}=\frac{j^4\pi^4}{\alpha j^2\pi^2+(1-\alpha)}.$$

It follows that, for any $\sigma>0$, there exists $\alpha_0>0$ such that

$$|\lambda^{(0)}-\lambda^{(\alpha)}|+|\Lambda^{(0)}-\Lambda^{(\alpha)}|\leqslant\sigma\qquad\forall\alpha\in[0,\alpha_0].$$

By continuity we deduce that there exists $\alpha_0>0$ such that Conditions C' and Assumptions G hold for $\alpha\in[0,\alpha_0]$. The existence of an inertial manifold, representable as the graph of a function $\Phi^\alpha\in C(Y,Z)$, follows from Main Theorem in [12]. It also follows that there exists $K(p)>0$ such that, for all α, $\alpha+\varepsilon\in[0,\alpha_0)$,

$$\|\Phi^\alpha(p)-\Phi^{\alpha+\varepsilon}(p)\|\leqslant K(p)\,\varepsilon\qquad\forall p\in Y.$$

This is since Assumptions G^h hold trivially since Assumptions G hold, since $P=P^h$ and since the error estimates of Theorem 2.3 hold. The set convergence of the inertial manifolds for nearby α follows.

The case $\alpha \in [\alpha_0/2, 1]$ may be handled similarly to the case $\alpha \in [0, \alpha_0]$ except that now $\beta = 0$ when following Lemma 4.1 in [12] and the spectrum of A_α grows quadratically. Putting the two overlapping intervals together gives the desired result. ∎

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