FINITE DIMENSIONAL BEHAVIOUR IN DISSIPATIVE PARTIAL DIFFERENTIAL EQUATIONS

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Dissipative partial differential equations have applications throughout the sciences: models of turbulence in fluids, chemical reactions, and morphogenesis in biology can all be written in a general form which allows them to be subjected to a unified analysis. Recent results on these equations show that in many cases they are not as complex as they initially appear, and can be converted into a set of ordinary differential equations. However, most of the relevant references present a bewildering array of terms which can obscure the simple underlying ideas. The main purpose of this paper is to introduce this terminology, motivated by several major results, slowly and by example. Detailed proofs are omitted, but it is hoped that this approach will give a good understanding of and intuitive feel for the subject without recourse to technicalities. Nevertheless, sufficient mathematical detail is included to allow application of these results to many examples.
INTRODUCTION.

Mathematical models are ubiquitous throughout the sciences, and in particular those models related to the change of various systems over time form a very major part of many disciplines. Such models fall into three main categories, of increasing complexity: maps, ordinary differential equations (ODEs), and partial differential equations (PDEs). It is the purpose of this paper to discuss recent techniques which enable partial differential equations to be simplified and converted into ordinary differential equations.

The common property of maps and ordinary differential equations is that their solutions are functions of time alone and contain a priori no spatial information. When the spatial distribution becomes important, it is necessary to use partial differential equations. For example, any model of dispersal requires both a temporal and a spatial element, and the diffusion equation

$$\frac{\partial f}{\partial t} = r \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \equiv r \nabla^2 f,$$

where $r$ is the diffusion coefficient, is fundamental in this context. Suppose that for some non-wandering population the total population size $P$ is expected to obey $dP/dt = g(P)$. A natural way to combine this growth law with the dispersal of the population is to write

$$\frac{\partial f}{\partial t} = r \nabla^2 f + g(f).$$

Such reaction-diffusion equations have many uses throughout the sciences. There are many examples in the field of biology, and a glance through the book by Murray\(^1\) suggests facilitated diffusion through membranes and, esoterically perhaps, a model of the antenna receptor system of the male silk moth *Bombyx mori*. Another obvious application is to chemical reactions in which the spatial concentration of the chemical is important. For example, the Belousov-Zhabotinskii reaction\(^1\), which shows remarkable spatial patterns and colour oscillations, can be modelled by the coupled equations

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u - rv)$$

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - buv,$$
where \( u \) represents the concentration of bromide ions \( Br^- \) and \( v \) the concentration of bromous acid \( HBrO_2 \).

The development of wave fronts and a spatial coherence in this context is related to the use of such equations to model morphogenesis in cells.\(^2\) To understand how an initially identical set of cells can evolve into a patterned state, when the rule governing each cell is identical, Turing\(^3\) showed that the pair of equations

\[
\frac{\partial x}{\partial t} = \mu \frac{\partial^2 x}{\partial s^2} + ax + by \\
\frac{\partial y}{\partial t} = \nu \frac{\partial^2 y}{\partial s^2} + cx + dy,
\]

with \( a, c > 0, b < 0 \) and \( \nu > \mu \) would produce a standing wave from any small displacement of the state \( x = y = 0 \).

There are many pattern formation equations more complicated than Turing’s model. One example is the complex Ginzburg-Landau equation

\[
\frac{\partial u}{\partial t} = Ru + (1 + i\nu)u_{xx} - (1 + i\mu)|u|^2u = 0
\]

used to model finite amplitude instabilities, for example the appearance of convection rolls in a fluid heated from below\(^4\) or the fluctuations about a chemical equilibrium.\(^5\) Related to this is the Kuramoto-Sivashinsky equation\(^6,7\)

\[
u_t + u_{xxxx} + u_{xx} + uu_x = 0,
\]

recent subject of numerous articles on the connection of between PDEs and ODEs. Along with the reaction-diffusion equation and the Navier-Stokes equation this will be used repeatedly as an illustrative example.

The Navier-Stokes equation

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \nabla^2 u - \nabla p \quad \nabla \cdot u = 0,
\]

which gives the velocity \( u(x, t) \) of the fluid throughout some volume \( V \), is the fundamental equation of fluid dynamics. Since it is supposed that this equation determines the fluid
motion, a full understanding of it would provide a solution to the problems of turbulence. At present only the two-dimensional case is fully understood, since there are still problems with the three-dimensional equation as basic as whether the solution exists for all time.\textsuperscript{8}

The mention of turbulence calls to attention the problem with such equations. Consider a general one-dimensional reaction-diffusion equation given by

$$u_t - \nu u_{xx} + f(u) = 0 \quad x \in [0, \pi] \quad x(0) = x(\pi) = 0.$$  \hspace{1cm} (i)

If the solution to this equation can be written as a Fourier series,

$$u(x, t) = \sum_{n=1}^{\infty} c_n(t) \sin nx,$$  \hspace{1cm} (ii)

then substituting (ii) into (i) produces an infinite set of ODEs for the coefficients \{c_n\}. In other words, there are a priori an infinite number of interdependent variables. Working with infinite-dimensional spaces produces some complications, and these are discussed in section I.

Recently it has been possible to show that many partial differential equations are not as complicated as they immediately appear, using two related but distinct approaches. The first idea is to look only at the eventual, recurrent, behaviour of the system. For example, if the viscosity is high enough fluid flowing in a pipe adopts a stable flow pattern: the possibly complex dynamics settle down to something much simpler. This is the idea behind the concept of a global attractor, the mathematical object that describes the eventual behaviour of the system. The theorem presented in section II gives conditions under which a partial differential equation can be shown to have a finite-dimensional attractor. This involves the mathematical concept of compactness, which is discussed with the aid of two simple examples.

A more refined approach yields even more impressive results for a class of equations which can be written in a form similar to (i),

$$du/dt + Au + f(u) = 0,$$
where $A$ is a linear operator (like $-\nabla^2$). Such equations are termed “evolution equations”, and for many of these it is possible to prove the existence of an inertial manifold. This shares many properties with the attractor, but it is a smooth surface rather than (perhaps) a jagged fractal, which means that it is possible to obtain a slaving rule giving the high modes in terms of a finite number of lower modes,

$$c_n = c_n(c_1, \ldots, c_N) + O(e^{-\mu t}) \quad n \geq N + 1$$

and reduce (i) to a finite set of ordinary differential equations which completely determine the dynamics. A sufficient condition for the existence of an inertial manifold depends on the spectrum of $A$ (its eigenvalues $\lambda_n$) and the existence of a large enough gap between two consecutive eigenvalues, so that for some $n$

$$\lambda_{n+1} - \lambda_n > C_f(\lambda_n^\gamma + \lambda_{n+1}^\gamma).$$

(iii)

The constant $C_f$ depends only on $f$, and $\gamma$ is a measure of the number of derivatives in $f$. The general framework that yields the values of $\lambda_n$, $C_f$ and $\gamma$ is introduced in section III.

Using this framework, section IV contains the inertial manifold result and discusses its application to three examples. Even when the spectrum of $A$ does not satisfy the gap condition (iii), it is possible to construct approximate inertial manifolds that have many similar properties, and in conclusion several explicit forms for such an approximation are given.

A similar approach to some of the necessary results from functional analysis covered below can be found in the Renardy & Rogers’ excellent book. General, if technical, references for much of the material are the books by Constantin et al., Hale, Henry, Ladyzhenskaya, and Temam.
I. INFINITE DIMENSIONAL SYSTEMS AND HILBERT SPACES.

In finite-dimensional systems of ordinary differential equations, it is standard to consider the “phase space”, a perhaps high dimensional space whose co-ordinates are given by the set of time-dependent variables. For example, a general autonomous system of ordinary differential equations

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n \]

has the natural phase space \( \mathbb{R}^n \). For other problems the phase space will be some subset of \( \mathbb{R}^n \); for the pendulum

\[ \ddot{x} = -\sin^2 x \]

setting \( y = \dot{x} \) yields

\[ \dot{x} = y \quad \dot{y} = -\sin^2 x, \]

and the phase space is \( S^1 \times \mathbb{R} \), where \( S^1 \) (the circle) is for the angle and \( \mathbb{R} \) for the angular velocity.

An important mathematical property of a finite-dimensional space \( E \) is that any closed bounded subset \( X \) of \( E \) will be compact, and this is frequently exploited in the analysis of ordinary differential equations. The most useful characterisation of compactness in this context is that any sequence in \( X \) will possess a convergent subsequence, so for example if \( f \) is continuous and there exists a sequence of points \( x_n \) in \( X \) with \( f(x_n) = 1/n \), then there exists a point \( x_\infty \in X \) with \( f(x_\infty) = 0 \). That such a deduction is not automatic can be seen by considering \( X = (0,1) \) and \( f(x) = x \). One of the difficulties faced in the analysis of partial differential equations is that the phase space is infinite-dimensional and does not have the property that any closed bounded subset is compact.

It was shown in the introduction that the reaction-diffusion equation given by

\[ u_t - \nu u_{xx} + f(u) = 0 \quad x \in [0,\pi] \quad x(0) = x(\pi) = 0. \quad (1.1) \]
is equivalent to an infinite set of ODEs for the coefficients of the Fourier expansion

\[ u(x, t) = \sum_{n=1}^{\infty} c_n(t) \sin nx. \]

Since there are an infinite number of co-ordinates, the phase space cannot be a subset of \( \mathbb{R}^n \). Indeed, the solution to (1.1) at any time \( s \) is a function \( f(x) = u(x, s) \), so the solutions move in a space of functions, or simply “a function space”. Typical function spaces are, for example, the space \( C^0 \) consisting of all continuous functions, \( C^1 \) consisting of all functions that are differentiable and have continuous first derivative, and in general

\[ C^r = \{ f : d^n f/dx^n \text{ exists and is continuous for } 0 \leq n \leq r \}. \]

A natural choice for the phase space of (1.1) would be the space of twice-differentiable functions \( C^2 \), since the equation involves second derivatives of \( u \). However, to facilitate comparison with the theory of ordinary differential equations, it is most useful to extend the meaning of (1.1) from such a space of smooth functions to a Hilbert space, which shares many features in common with \( \mathbb{R}^n \).

Many of the abstract spaces in analysis come about by generalising properties of \( \mathbb{R}^n \). For example, a metric space is based on ideas of distance, and a normed space on ideas of length. The concept of a topological space arises from notions of open and closed sets within a metric space. A Hilbert space is based on the more mathematical notions of the scalar (dot) product and the completeness of the Euclidean spaces. The analogue of the scalar product is the inner product \( (u, v) \), and the modulus or norm of \( u \), still denoted \( |u| \), is derived (as for \( \mathbb{R}^n \)) from the inner product via \( |u|^2 = (u, u) \). Completeness means that any Cauchy sequence in \( H \) (a sequence \( \{x_n\} \) such that given \( \epsilon > 0 \) there exists an \( n_0(\epsilon) \) with \( |x_n - x_m| \leq \epsilon \) for all \( n, m \geq n_0 \)) converges to a point in \( H \); this is an essential property (perhaps the essential property) of the real numbers, and it is this property that the spaces \( C^r \) of continuous functions lack when one attempts to make them into inner product spaces.
The similarity to Euclidean spaces is not illusory, and in fact all finite-dimensional Hilbert spaces are isomorphic to $\mathbb{R}^n$, and all infinite-dimensional separable Hilbert spaces (those which contain a countable dense subset) are isomorphic to $l^2$, the space of square summable sequences. This isomorphism onto $l^2$ provides a way of defining a co-ordinate system on $H$ so that it looks like “$\mathbb{R}^\infty$”. Thinking of infinite-dimensional Hilbert spaces as “large” Euclidean spaces, with a warning about issues of compactness, can help to clarify much of the literature.

The most obvious inner product of two functions is the integral of their product,

$$(u,v) = \int_{\Omega} u(x)v(x) \, dV$$

which generates the corresponding norm

$$|u| \text{ or } \|u\|_2 = \left( \int_{\Omega} |u|^2 \, dV \right)^{\frac{1}{2}}.$$

However, if one tries to proceed by defining this inner product on $C^0$ (for example), the problem is that it is possible to construct a sequence in $C^0$ which is convergent under the norm (1.3), but whose limit does not lie in $C^0$, for example, the sequence $f_n(x)$ defined by

$$f_n(x) = \begin{cases} -1 & x < -1/n \\ nx & |x| \leq 1/n \\ 1 & x > 1/n, \end{cases}$$

converges to a discontinuous function. Adding to $C^0$ all the functions that can be produced as the limit of such sequences produces the space $L^2$.

$L^2$, the space of functions that are square integrable (in the sense of Lebesgue), is the fundamental Hilbert space for the study of partial differential equations. For a domain $\Omega$, smoothly bounded in $\mathbb{R}^n$, $L^2(\Omega)$ is defined by

$$L^2(\Omega) = \{u | \int_{\Omega} |u(x)|^2 \, dV < \infty\}$$

and is a Hilbert space when equipped with the inner product (1.2) and the corresponding norm (1.3).
Theorem 1.1\textsuperscript{17} states two important facts about $L^2$. The first is that it is indeed the completion of $C^0$ under the norm (1.3), and that it is therefore complete itself. The second is that $L^2(S^1)$ is the same as the space of all functions on the circle defined by a (suitably convergent) Fourier series.

**Theorem 1.1.**

1. $L^2(\Omega)$ is the completion of $C^0(\Omega)$ with respect to the norm (1.3). Hence $L^2$ is complete.
2. $L^2(S^1)$ is the space formed by all Fourier series of the form

   \[ \{ u : u = \sum_{k=-\infty}^{\infty} c_k z^k, \sum |c_k|^2 < \infty \}. \]

Theorem 1.1 therefore justifies, in some sense, the introduction of the space $L^2$ into the study of partial differential equations as both the natural completion of the space of continuous functions, and by its relationship to Fourier series. Importantly, it also enables the extension of the definition of differential operators like $-\partial^2/\partial x^2$, and thus the very concept of a solution of a differential equation, to functions in $L^2$ and not just smooth functions (it would be of little use if there was no way to attach a sensible meaning to $-\nu u_{xx}$ for any $u \in L^2$, having just shown that $L^2$ is the natural setting for the study of equations involving such terms).

As an example, take from (1.1) the linear operator $A$ given by $Au = -u_{xx}$, and note that the boundary conditions must be specified to determine the properties of $A$: here they are $u(0) = u(\pi) = 0$. If they were $u'(0) = u'(\pi) = 0$ the eigenfunctions would be $\cos nx$ instead of $\sin nx$.

First consider the action of $A$ on a $C^2$ function $u$ with $u(0) = u(\pi) = 0$ and expand $u$ as a Fourier series,

\[ u(x, t) = \sum_{n=1}^{\infty} c_n(t) \sin nx. \] (1.4)

To find the second derivative of $u$, differentiate term by term to give

\[ Au = \sum_{n=1}^{\infty} n^2 c_n(t) \sin nx. \] (1.5)
Now, using theorem 1.1 any function \( u \) in \( L^2 \) with \( u(0) = u(\pi) = 0 \) can be written in the form (1.4), and so it is possible to define \( Au \) by (1.5) for any \( u \in L^2 \) even when \( u \) itself is not differentiable.

Such an extension is possible for any positive symmetric operator \( A \) defined on a dense subspace \( E \) of \( H \). This is guaranteed by Friedrichs’ theorem. It is straightforward to show that, as an operator on functions in \( C^2 \) with compact support in \((0, \pi)\), \( A \) is symmetric (integrating by parts twice)

\[
(A\phi, \psi) = - \int \phi''(x)\psi(x) \, dx = - \int \phi(x)\psi''(x) \, dx = (\phi, A\psi)
\]

and positive (integrating by parts once)

\[
(A\phi, \phi) = - \int \phi''(x)\phi(x) \, dx = \int [\phi'(x)]^2 \, dx \geq 0.
\]

Thus (1.1) can be written as an ordinary differential equation on \( L^2([0, \pi]) \),

\[
\frac{du}{dt} + \nu Au + f(u) = 0.
\] (1.6)

In general, solutions of this equation will be functions in \( L^2 \) and are referred to as “weak” solutions of (1.1); taking the inner product of (1.1) with a test function \( v \in C_0^\infty \) gives

\[
(du/dt, v) + (\nu Au, v) + (f(u), v) = 0
\]

for all \( v \in C_0^\infty \),

and using the self-adjointness of \( A \) this can be written as

\[
\frac{d}{dt} \langle u, v \rangle + (\nu u, Av) + (f(u), v) = 0
\]

for all \( v \in C_0^\infty \)

or

\[
\frac{d}{dt} \langle u, v \rangle - (\nu u, v_{xx}) + (f(u), v) = 0
\]

for all \( v \in C_0^\infty \)

which makes sense for all those \( u \) for which \( f(u) \in L^2 \). Such a “weak” solution is equivalent to a solution of the integral equation

\[
u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}f(u(s)) \, ds
\]
known as the “variation of constants formula”. This formulation of the problem is very useful for further analysis and is used repeatedly throughout the literature.

For discussing the “smoothness” of functions in $L^2$ a generalised notion of the derivative can be introduced in a manner related to weak solutions. If $u$ is $C^1$ and $v$ is in $C_0^\infty$, after an integration by parts the identity

$$(\frac{\partial u}{\partial x}, v) = \int \frac{\partial u}{\partial x} v \, dx = -\int u \frac{\partial v}{\partial x} \, dx = -(u, \frac{\partial v}{\partial x})$$

holds. When $u$ is not smooth enough to have a ‘true’ derivative, the distribution derivative of $u$, $Du$, is defined as the distribution that satisfies

$$(Du, v) = -(u, \frac{\partial v}{\partial x}) \quad \forall \ v \in C_0^\infty.$$ 

Just as the spaces $C^r$ of continuously differentiable functions are built up from standard differentiation and $C^0$, so that (for $r \geq 1$)

$$C^r = \{f | df^n/dx^n \in C^0, 1 \leq n \leq r\},$$

it is possible to build analogous spaces based on the distribution derivative and $L^2$. These are the Sobolev spaces $H^m(\Omega)$ and are defined (for $\Omega \subset \mathbb{R}$) by

$$H^m(\Omega) = \{u | D^u u \in L^2, 1 \leq n \leq r\}.$$ 

When $\Omega$ is a set of higher dimension, the natural requirement is that all derivatives of the appropriate order are in $L^2$. For example, if $\Omega \subset \mathbb{R}^2$, then $H^1(\Omega)$ requires $D_x u \in L^2$ and $D_y u \in L^2$. It is convenient for this purpose to adopt multi-index notation, where $\alpha$ is taken to be a vector of integers,

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n,$$

and the expression $D^\alpha$ denotes a mixed derivative

$$D^\alpha u = D_1^{\alpha_1} \ldots D_n^{\alpha_n} u,$$
with $D_i u$ the distribution derivative with respect to $x_i$. Finally, the “size” of $\alpha$, $[\alpha]$, is just the sum of all the entries in the vector,

$$[\alpha] = \alpha_1 + \cdots + \alpha_n.$$ 

A general form for the spaces $H^m(\Omega)$ is therefore

$$H^m(\Omega) = \{u | D^\alpha u \in L^2, [\alpha] \leq m\}.$$ 

These are Hilbert spaces when equipped with the inner product

$$(u, v)_{H^m} = \sum_{[\alpha] \leq m} (D^\alpha u, D^\alpha v)$$

and the corresponding norm

$$\|u\|_{H^m} = \left(\sum_{[\alpha] \leq m} |D^\alpha u|^2\right)^{1/2}.$$ 

The most frequently used of these spaces are $H^1(\Omega)$,

$$H^1(\Omega) = \{u : u \in L^2(\Omega), D_i u \in L^2(\Omega)\},$$

and $H^1_0(\Omega)$,

$$H^1_0(\Omega) = \text{the closure of } C_0^\infty(\Omega) \text{ in } H^1.$$ 

A useful inequality (the Poincaré inequality, valid for $\Omega$ sufficiently regular\textsuperscript{14})

$$|u| \leq c(\Omega) \sum_i |D_i u| \quad \forall u \in H^1_0(\Omega)$$

means that the alternative (and simpler) inner product and norm

$$(u, v) = \sum_i (D_i u, D_i v) \quad \|u\| = \sum_i |D_i u|$$

also make $H^1_0$ a Hilbert space.
As suggested by the above introduction, the Sobolev spaces are the natural generalisation (or completion) of the spaces $C^r$ with respect to their own norms (defined by 1.7), just as $L^2$ is the natural generalisation of $C^0$. This is the content of the first part of theorem 1.2.\[14\]

The second part concerns the translation of results obtained in the Sobolev space setting back into that of continuous functions: if $Du \in L^2$, the function $u$ is in fact continuous. This can be used to show that any function which has distribution derivatives of all orders in $L^2$ must in fact be infinitely differentiable. Since most of the analysis of partial differential equations is conducted in Sobolev spaces, this is extremely important in that it makes such results relevant to the study of “normal” (classical) solutions.

The third part goes some way towards addressing the problem of compactness that occurs in infinite-dimensional spaces and has been discussed briefly above. Although a closed bounded subset of $L^2$ is not compact, a set of functions whose derivative is also bounded (a closed bounded subset of $H^1$) does form a compact set, provided $\Omega$ is a bounded subset of $\mathbb{R}$ or $\mathbb{R}^2$. What this means is that if bounds can be obtained on both $|u|$ and $|Du|$ there is a hope that the compactness arguments of ODE theory can be applied, and this is done below in section II when discussing global attractors.

The fourth part is one of many useful inequalities which involve an assortment of different norms. This particular inequality relates the maximum value of $u$, $\|u\|_{\infty}$, to its $L^2$ norm $|u|$ and the $L^2$ norm of its derivative $\|u\|$. Doering & Gibbon\[8\] have a good summary of these “calculus inequalities”.

**Theorem 1.2.**

1. $H^m(\Omega)$ is the completion of $C^m(\bar{\Omega})$ with respect to the norm (1.7). (Completeness)
2. $H^1(\Omega) \subset C^0(\Omega)$. (Embedding)
3. If $\Omega$ is a bounded subset of $\mathbb{R}$ or $\mathbb{R}^2$ then the embedding $H^1(\Omega) \subset L^2(\Omega)$ is compact. (Compactness)
4. If $\Omega \subset \mathbb{R}$, for all $u \in H^1(\Omega)$, $\|u\|_{\infty} \leq c(\Omega)\|u\|_1^{\frac{1}{2}}|u|^{\frac{1}{2}}$. (Agmon’s inequality)
So despite their somewhat convoluted and abstract definition, the use of Sobolev spaces can be justified as reasonable (part 1), and results that are obtained in this context will be useful in deducing properties about the solutions of partial differential equations concerned with standard notions of smoothness and differentiability (part 2). Furthermore, there is a hope that the use of such spaces may help to overcome the compactness problem (part 3), and there are many useful relationships between the various norms (part 4) which are indispensable in any analysis.

To begin an attempt to reduce the dimension of these problems to more manageable size, the concept of dissipation enables the proof of the existence of a compact global attractor, which in many cases can then be shown to be finite dimensional.

II. DISSIPATION AND THE GLOBAL ATTRACTOR.

Semigroups

The standard task in the theory of partial differential equations is to solve an initial value problem for a given equation or class of equations. In other words, to predict the future behaviour of the system given its current state. For many equations it is possible to show that this can always be done, and that the solution is unique (see section III).

In the theory of ordinary differential equations such an approach can be reversed, and given the current state it is possible to reconstruct the past evolution of the system: for every initial condition $u_0$ there exists a solution $u(t; u_0)$ for all $t \in \mathbb{R}$. It is often convenient to represent this in terms of the group of time transformations (or solution operators) $T(t)$, so that

$$u(t; u_0) = T(t)u_0$$

and the operators $T(t)$ obey the necessary properties, i.e.

$$T(t)T(s) = T(t + s) = T(s)T(t) \quad \lim_{t \to 0} T(t) = T(0) = I.$$

For partial differential equations the situation is not so simple. Indeed, consider the
elementary equation

\[ \frac{du}{dt} = -Au, \]

where \( A \) is the extension of \( -u_{xx} \) to \( L^2 \). \( A \) has an infinite set of eigenvectors \( \sin kx \), with eigenvalues \( \lambda_k = k^2 \). As time runs forward from the initial state \( u_0 \), whose Fourier series is

\[ u_0 = \sum_{k=1}^{\infty} (1/k) \sin kx, \]

(so that \( |u|^2 = \sum k^{-2} < \infty \)) the solution will be

\[ u(t) = \sum_{k=1}^{\infty} (1/k) e^{-k^2 t} \sin kx. \] \hspace{1cm} (2.1)

The norm of (2.1),

\[ |u(t)|^2 = \sum_{k=1}^{\infty} k^{-2} e^{-2k^2 t}, \]

is finite for any \( t \geq 0 \), so that \( u(t) \in L^2 \) for \( t \) positive, but diverges for any \( t < 0 \). It is therefore necessary to restrict attention to evolution forwards in time, and so the analogue of the group of transformations \( T(t) \) is a semigroup of transformations \( S(t) \), defined only for \( t \geq 0 \) on the space \( H \) (see Pazy\(^{21} \) for a detailed study of semigroups). A “strongly continuous” semigroup obeys \( S(t) \) obeys

\[ \lim_{t \to 0^+} S(t) = S(0) = I \quad S(s)S(t) = S(s + t) = S(t)S(s) \quad s, t > 0, \]

and gives the solution \( S(t)u_0 \) of the equation with initial condition \( u(0) = u_0 \) at time \( t \). It is in terms of this general concept of a semigroup that this section will proceed.

**Dissipation and the global attractor**

The concept of energy dissipation is a natural one in physical problems. For the Navier-Stokes equations, where the solution \( u \) represents a velocity field, the energy is proportional to

\[ \int_{\Omega} |u|^2 \, dV, \]

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which is simply the square of the $L^2$ norm of $u$. Dissipation of energy is thus equivalent to decrease of $|u|$, and so in the context of partial differential equations, dissipation is taken to mean the existence of an absorbing set - a bounded set in the phase space which all trajectories eventually enter and never leave. More formally,

**Definition 2.1.** A dissipative system is a semigroup $S(t)$ which possesses an absorbing set, i.e. there exists a bounded set $B$ such that for any bounded set $X$ there is a time $t_0(X)$ such that

$$S(t)X \subset B \quad \text{for all } t \geq t_0(X).$$

(2.2)

Since all trajectories eventually enter the absorbing set $B$, all the interesting dynamics will occur within $B$. Usually, however, there is an even smaller set on which the asymptotic dynamics take place, and this set is termed the attractor:

**Definition 2.2.** The (global) attractor is a compact invariant set $\mathcal{A}$ to which all trajectories tend:

$$S(t)\mathcal{A} = \mathcal{A} \quad \text{and} \quad \text{dist}(u(t), \mathcal{A}) \to 0 \text{ as } t \to \infty.$$  

For finite-dimensional systems, the set $\omega(B)$ given by

$$\omega(B) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B}$$

(where $\overline{X}$ denotes the closure of $X$ in $H$) is known to give the global attractor.\textsuperscript{22} However, the proof uses compactness properties, and so the infinite-dimensional case requires an additional condition, that the absorbing set $B$ be itself compact. One can then work within the absorbing set and apply all the traditional arguments.\textsuperscript{11,13,14} Additionally, Mallet-Paret\textsuperscript{23} has proved that such attractors have finite dimension, i.e. that they are homeomorphic to a subset of $\mathbb{R}^n$ for some $n$ (see also Ma\~{n}e\textsuperscript{24}).

**Theorem 2.3.** Suppose that the semigroup $S(t)$ defined on a Hilbert space $H$ possesses a compact absorbing set. Then it has a compact global attractor, which is the maximal bounded attractor in $H$ and is connected. Furthermore, the attractor is finite-dimensional.
To show that the absorbing set $B$ is compact it suffices to show that it is bounded in a space compactly embedded in $H$. For example, theorem 1.2 states that $H^1$ is compactly embedded in $L^2$, so a bounded set in $H^1$ is a compact subset of $L^2$. To show that a set $X$ is bounded in $H^1$ it is necessary to show that

$$|u| \leq \rho_0 \quad \text{and} \quad |Du| \leq \rho_1$$

for all $u \in X$. Thus for an absorbing set to be bounded in $H^1$, one must find asymptotic bounds on $|u|$ and $|Du|$,

$$\limsup_{t \to \infty} |u| \leq \rho_0$$
$$\limsup_{t \to \infty} |Du| \leq \rho_1.$$

(Although a bound on $|u|$ follows from one on $|Du|$ via Poincaré’s inequality (1.8), it is usually necessary to use the bound on $|u|$ to derive that for $|Du|$.)

Thus under conditions often simple to verify, partial differential equations can be shown to possess a finite-dimensional compact attractor.

**Two examples of global attractors**

The first example will be the reaction-diffusion equation (1.1) with a particular choice of the nonlinear term $f$. The equation with $f(u) = u^3 - u$,

$$u_t - \nu u_{xx} + u^3 - u = 0 \quad x \in [0, \pi] \quad x(0) = x(\pi) = 0,$$

(2.3) is called the “Chaffee-Infante” equation, and the simple nonlinear term shortens the analysis.

To follow the time evolution of $|u|$, take the inner product with $u$,

$$\frac{1}{2} \frac{d}{dt} |u|^2 = -2\nu |u_x|^2 + \int_0^\pi u^2 - u^4 \, dx$$

where $(u, u_{xx}) = -(u_x, u_x)$ after an integration by parts. Now complete the square,

$$u^4 - u^2 + \frac{1}{4} \geq 0,$$

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to yield

\[ \frac{1}{2} \frac{d}{dt} |u|^2 \leq -2\nu |u_x|^2 + \frac{\pi}{4}. \]  

(2.4)

Considering the eigenfunction expansion (1.5) gives $|u_x| \geq |u|$, and so

\[ \frac{1}{2} \frac{d}{dt} |u|^2 \leq -2\nu |u|^2 + \frac{\pi}{4}, \]

giving

\[ \limsup_{t \to \infty} |u|^2 \leq \frac{\pi}{8\nu} = \rho_0^2, \]

and hence

\[ |u(t)| \leq \rho_0' \sim \nu^{-1/2} \quad \text{for some } \rho_0' > \rho_0 \quad \text{when } t > t_0(|u_0|). \]  

(2.5)

A bound on the derivative follows from setting $v = u_x$ and considering

\[ v_t + \nu v_{xx} + 3u^2v - v = 0 \]

in a similar fashion to (2.3). Then

\[ \frac{1}{2} \frac{d}{dt} |v|^2 \leq (1 - \nu)|v|^2 - (3u^2v, v) \]

\[ \leq (1 - \nu)|v|^2, \]

since $(3u^2v, v) \geq 0$. An asymptotic bound on $|v|$ follows using the uniform Gronwall lemma:\^{14}

**Lemma 2.4 (Uniform Gronwall).** Let $g, h, \text{ and } y$ be three positive locally integrable functions on $(t_0, \infty)$ such that $dy/dt$ is locally integrable on $(t_0, \infty)$ and which satisfy

\[ dy/dt \leq gy + h \quad \text{for } t \geq t_0 \]

and the uniform bounds

\[ \int_t^{t+r} g(s) \, ds \leq a_1, \quad \int_t^{t+r} h(s) \, ds \leq a_2, \quad \int_t^{t+r} y(s) \, ds \leq a_3 \quad \text{for } t \geq t_0 \]
where \( r, a_1, a_2, \) and \( a_3 \) are positive constants. Then

\[
y(t + r) \leq \left( \frac{a_3}{r} + a_2 \right) \exp(a_1), \quad \text{for all } t \geq t_0.
\]

The lemma is clearly applicable with \( t_0 = t_0(|u_0|) \) as in (2.5); from (2.4) the integral of \(|v|^2\) is bounded for \( t \geq t_0; \)

\[
\int_t^{t+r} \frac{d}{dt}|u|^2 \, ds + \int_t^{t+r} 2\nu|v|^2 \, ds \leq \frac{\pi}{4}
\]

so that

\[
\int_t^{t+r} |v|^2 \leq \frac{\pi r}{8\nu} + \frac{\rho_0^2}{2\nu} = a_3,
\]

and it is immediate that \( a_1 = 2(1 - \nu)r \) and \( a_2 = 0 \). Therefore \(|u_x|\) is eventually bounded, with \(|u_x| \sim \nu^{-1}\).

Thus \( u \) is eventually contained in a bounded set in \( H_0^1 \), which is a compact subset of \( L^2 \). Theorem 2.3 then applies to ensure the existence of a finite-dimensional compact global attractor.

Another simple example is the complex Ginzburg-Landau equation with periodic boundary conditions,

\[
\begin{align*}
  u_t &= Ru + (1 + i\nu)u_{xx} - (1 + i\mu)|u|^2u = 0 \quad u(x + 1, t) = u(x, t), \quad (2.6)
\end{align*}
\]

used to model finite amplitude instabilities, for example the appearance of convection rolls in the Rayleigh-Bénard problem. Note that in this complex setting, the norm of \( u \) is derived from the inner product

\[
(u, v) = \int_0^1 u\bar{v} \, dx.
\]

The analysis follows that presented by Doering et al., given here for the simpler case when \(|\mu| \leq \sqrt{3}\).

Define the “mass” \( M \) and “energy” \( E \) by

\[
M = \int_0^1 |u|^2 \, dx = \|u\|_2^2, \quad E = \int_0^1 |u_x|^2 \, dx = \|u_x\|_2^2,
\]
take the inner product of (2.8) with $\bar{u}$

$$\frac{1}{2} \frac{dM}{dt} = RM + \int_0^1 (1 + iv)u_{xx}\bar{u}\,dx - \int_0^1 (1 + i\mu)|u|^4\,dx$$

and take the real part to give

$$\frac{1}{2} \frac{dM}{dt} = RM - \int_0^1 |u_x|^2\,dx - \int_0^1 |u|^4\,dx,$$

where the second term on the right-hand side has been integrated by parts. Now use the Cauchy-Schwarz inequality

$$\int_\Omega fg\,dV \leq \left(\int_\Omega f^2\,dV\right)^{1/2} \left(\int_\Omega g^2\,dV\right)^{1/2}$$

(2.7)

with $f = 1$ and $g = |u|^2$ to obtain

$$M^2 \leq \int_0^1 |u|^4\,dx$$

so that

$$\frac{1}{2} \frac{dM}{dt} \leq RM - E - M^2$$

$$\leq RM - M^2.$$  

Thus while $M > R(1 + \epsilon)$, $M(t)$ is decreasing exponentially, and so

$$\limsup_{t\to\infty} |u|^2 \leq R,$$

which means that

$$\{ u : |u| \leq R(1 + \epsilon) = \rho \} \text{ is absorbing.}$$

(2.8)

Taking the derivative of (2.6) and setting $v = u_x$ yields

$$v_t = Rv + (1 + iv)v_{xx} + 2(1 + i\mu)v|u|^2 + (1 + i\mu)u^2\bar{v}_x.$$

As before, multiply by $\bar{v}$, integrate by parts and take the real part,

$$\frac{1}{2} \frac{dE}{dt} = RE - \int_0^1 |u_{xx}|^2\,dx - 2\int_0^1 |u|^2|u_x|^2\,dx - Re\{ (1 + i\mu) \int_0^1 u^2\bar{u}_x\,dx \}.$$  

(2.9)
The final two terms can be taken together and omitted, since if $|\mu| \leq \sqrt{3}$, certainly $|1 + i\mu| \leq 2$ and these two terms are bounded by

$$(-2 + |1 + i\mu|) \int_0^1 |u|^2 |u_x|^2 \leq 0.$$ 

A bound on the quadratic term is obtained by observing that

$$\int_0^1 |u_x|^2 \, dx = -\int_0^1 \bar{u} u_{xx} \, dx \leq \left( \int_0^1 |u|^2 \, dx \right)^{1/2} \left( \int_0^1 |u_{xx}|^2 \, dx \right)^{1/2},$$

integrating by parts and using the Cauchy-Schwarz inequality (2.7). Therefore

$$-\int_0^1 |u_{xx}|^2 \, dx \leq -E^2 / M.$$

Using this in (2.9) gives

$$\frac{1}{2} \frac{dE}{dt} \leq RE - E^2 / M$$

and so eventually (by (2.8))

$$\frac{1}{2} \frac{dE}{dt} \leq RE - E^2 / \rho$$

giving

$$\limsup_{t \to \infty} |u_x|^2 \leq \rho^2.$$ 

Therefore $|u|$ and $|u_x|$ are eventually bounded; there exists an absorbing set in $H^1$, and so the complex Ginzburg-Landau equation has a finite-dimensional compact attractor. Many other equations are amenable to such an analysis.\textsuperscript{11,14}

For one class of equations possessing a compact absorbing set, the dissipative evolution equations, it is possible to go further and embed the attractor into a finite-dimensional manifold (or surface) of solutions. As mentioned in the introduction, this requires further analytical tools, and in particular a measure of the number of derivatives that occur in the nonlinear terms.
III. DISSIPATIVE EVOLUTION EQUATIONS.

An equation in the form of (1.6),

\[ \frac{du}{dt} + \nu Au + f(u) = 0 \]

is called an evolution equation. The factor \( \nu \) (which will occasionally be omitted in what follows) allows the dissipative term \( A \) to be adjusted, and can thus be interpreted as a “viscosity”. In line with considering \( A \) a purely viscous damping term, all the eigenvalues of \( A \) are required to be positive, so that if \( f = 0 \) any initial condition decays to zero (cf. 2.1). Many important partial differential equations can be written in this form, and three examples will now be given, to be used later as illustrations. In each case, the equation can also be shown to be dissipative.

Exercise 3.1. The simplest and best understood example is the one-dimensional reaction-diffusion equation

\[ u_t - \nu u_{xx} + f(u) = 0 \quad x \in [0, \pi] \quad x(0) = x(\pi) = 0. \]

See the introduction for a discussion of the role of such equations in various branches of chemistry and biology. Higher dimensional forms of this equation (with \( u \in \Omega \subset \mathbb{R}^n \)) will be briefly considered later.

This equation has already been discussed in some detail above. The space \( H \) can be taken to be \( L^2([0, \pi]) \), and the linear operator \( A \) is given by the extension of

\[ Au = -u_{xx} \quad x \in [0, \pi] \quad x(0) = x(\pi) = 0 \]

from \( C^2 \) to \( L^2 \). The eigenfunctions of \( A \) are \( \sin nx \) with corresponding eigenvalues \( n^2 \),

\[ Aw_n = \lambda_n w_n \quad w_n = \sin nx \quad \lambda_n = n^2. \]

Note that the eigenvalues \( \lambda_n \to \infty \) as \( n \to \infty \), a feature common to all the problems here. (This is expressed mathematically by saying that the linear term is unbounded, i.e. there
is no constant $C$ such that $|Au| \leq C|u|$ for all $u$.) The nonlinear term $f(u)$ is exactly the $f(u)$ in (3.2). By methods similar to those used in section II for the specific case of the Chaffee-Infante equation, many reaction-diffusion equations can be shown\textsuperscript{14} to be dissipative both in the space $L^2$ and in $H^1$,

$$\limsup_{t \to \infty} |u| \sim \nu^{-1/2} \quad \limsup_{t \to \infty} |u_x| \sim \nu^{-1}.$$ 

**Exercise 3.2.** The one-dimensional Kuramoto-Sivashinsky equation with periodic boundary conditions

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0 \quad u(x + L, t) = u(x, t), \quad (3.3)$$

has been used to model cellular instabilities in flame fronts, thin liquid films on planes, and as a general scalar model for the onset of chaos in films and interfaces. It is has attracted much attention recently as a paradigm of low-dimensional behaviour in infinite-dimensional systems\textsuperscript{26–34}.

This equation can be approached in two ways, depending on how the linear term $A$ is chosen. The simplest method is to take $A$ on $L^2$ as the extension of

$$Au = u_{xxxx} \quad u(x + L, t) = u(x, t), \quad (3.4)$$

obtain the eigenfunctions and eigenvalues

$$Aw_n^i = \lambda_n w_n^i \quad w_n^i = \sin 2n\pi x/L \quad w_n^c = \cos 2n\pi x/L \quad \lambda_n = (2n\pi/L)^4, \quad (3.5)$$

and set $f(u) = u_{xx} + uu_x$.

A more refined approach is to include the term $u_{xx}$ in $A$ as far as is possible, but if $A$ is defined by $Au = u_{xxxx} + u_{xx}$ there is a problem, since the eigenvalues corresponding to $w_n$ and $v_n$ become

$$\lambda_n = (2n\pi/L)^4 - (2n\pi/L)^2$$

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which are negative for \( n < L/2\pi \) and so \( A \) is no longer positive. By subtracting off the first few Fourier modes it is possible to rectify this.\(^{35}\)

Absorbing sets for both \( u \) in both \( L^2 \) and \( H^1 \) exist, i.e. there are asymptotic bounds on both \(|u|\) and \(|Du|\). The existence of such absorbing sets for odd initial conditions was known in the mid-80s,\(^{32}\) but their existence (with reasonable bounds) for all initial conditions is a recent result,\(^{26}\)

\[
\limsup_{t \to \infty} |u| \sim L^{8/5} \quad \limsup_{t \to \infty} |u_x| \sim L^{56/25}.
\]

**Exercise 3.3.** \( \bullet \) The Navier-Stokes equation on a bounded domain \( \Omega \) in \( \mathbb{R}^2 \) is

\[
uu_t - \nu \nabla^2 u + u \cdot \nabla u + \nabla p = g(t) \quad \nabla \cdot u = 0 \quad x \in \Omega \quad u = 0 \text{ on } \partial \Omega. \quad (3.6)
\]

Equation (3.6) is not immediately in the form (3.1): the equation \( \nabla \cdot u = 0 \) is unusual, and the term \( \nabla p \) seems somewhat aberrant. However, functions \( \nabla \phi \) are orthogonal in \( L^2(\Omega, \mathbb{R}^n) \) to functions \( u \) with \( \nabla \cdot u = 0 \) and \( u = 0 \) on \( \partial \Omega \): observe that

\[
\int_\Omega \nabla \cdot (\phi u) \, dV = \int_\Omega u \cdot \nabla \phi \, dV + \int_\Omega \phi (\nabla \cdot u) \, dV
\]

and therefore if \( \nabla \cdot u = 0 \), by the divergence theorem

\[
\int_{\partial \Omega} \phi u \cdot n \, dS = \int_\Omega u \cdot \nabla \phi \, dV
\]

and now using the boundary condition \( u = 0 \) on \( \partial \Omega \),

\[
\int_\Omega u \cdot \nabla \phi \, dV = 0.
\]

As one would therefore hope, \( L^2(\Omega, \mathbb{R}^n) \) can be split into two orthogonal spaces

\[
H_u = \text{Cl}\{u \in C^1(\Omega, \mathbb{R}^n) | \nabla \cdot u = 0, \ u = 0 \text{ on } \partial \Omega\}
\]

\[
H_p = \text{Cl}\{v \in C^1(\Omega, \mathbb{R}) | v = \nabla \phi \text{ for some } \phi\},
\]

where “\( \text{Cl} \)” denotes the closure in \( L^2(\Omega) \).
Now define $P$ to be the projector onto the space $H_u$. Then (3.6) becomes
\[ u_t - \nu P \nabla^2 u + Pu \cdot \nabla u = Pg(t) \]
and define $A$ on $H_u$ using Friedrichs’ theorem, as the extension of
\[ Au = -P \nabla^2 u \quad x \in \Omega \quad \nabla \cdot u = 0 \quad u = 0 \text{ on } \partial\Omega. \]
For constant forcing $g(t) = g_0 \in H_u$, set $f(u) = Pu \cdot \nabla u - Pg_0$ and the equation is now in the required form,
\[ \frac{du}{dt} + \nu Au + f(u) = 0, \]
with $H = H_u$. For more details see the books by Constantin & Foias\textsuperscript{36} and Temam.\textsuperscript{37}

As mentioned in the introduction, the Navier-Stokes equation in $\mathbb{R}^3$ is notoriously hard to treat, and it is not even known whether unique solutions exist for all time.\textsuperscript{8} However, the equation in two dimensions is much more tractable and possesses a compact absorbing set and hence a finite-dimensional global attractor,\textsuperscript{11,13,36,37,38}

**Generalised Fourier series and the domain of $A$**

In all these cases, $A$ is a self-adjoint positive operator with a compact inverse, i.e.
\[ (A\phi, \psi) = (\phi, A\psi) \quad (\phi, A\phi) \geq \lambda_1 |\phi|^2, \]
and $A^{-1}X$ is compact if $X$ is bounded. This enables the spectral theory of linear operators to be used and guarantees\textsuperscript{14} the existence of a complete set of orthonormal eigenfunctions $w_n$ with eigenvalues $\lambda_n$,
\[ \langle w_m, w_n \rangle = \delta_{mn} \quad A w_n = \lambda_n w_n, \]
where $\lambda_n$ are ordered to be increasing with $n$ and satisfy $\lambda_n \to \infty$ as $n \to \infty$. Any vector $u \in H$ can then be written as the generalised Fourier series
\[ u = \sum_{j=0}^{\infty} (u, w_j) w_j = \sum_{j=0}^{\infty} c_j w_j. \quad (3.7) \]
The norm of $u$ is

$$|u| = \left( \sum_{j=0}^{\infty} c_j^2 \right)^{\frac{1}{2}} \quad (3.8)$$

and $Au$ is given by

$$Au = \sum_{j=0}^{\infty} \lambda_j c_j w_j. \quad (3.9)$$

For

$$\frac{du}{dt} + Au + f(u) = 0$$

to make sense in $H$, $Au$ must be in $H$. That this is not immediate can be seen by considering the action of $-\partial^2/\partial x^2$ on $L^2$. Indeed, if

$$u_0 = \sum_{k=1}^{\infty} (1/k) \sin kx,$$

then

$$-\partial^2 u_0 / \partial x^2 = \sum_{k=1}^{\infty} k \sin kx,$$

whose $L^2$ norm is infinite. It is therefore necessary to define a space of all those functions $u$ for which $Au$ is still in $H$. This is termed the domain of $A$ and is written $D(A)$:

$$D(A) \equiv \{ u : |Au| < \infty \},$$

where $|u|$ is the norm in $H$ (which is usually $L^2$). For example, the operator $A$ which is the extension of $-u_{xx}$ from $C_0^{\infty}(0, \pi)$ to $L^2(0, \pi)$ has domain $H_2 \cap H_0^1$, since $D^2 u$ must be in $L^2$ (so $u \in H^2$) and $u(0) = u(\pi) = 0$, so $u \in H_0^1$.

Use of the Fourier expansions (3.7-9) gives the equivalent definition

$$D(A) \equiv \{ u | u = \sum c_j w_j, \sum \lambda_j^2 |c_j|^2 < \infty \}. \quad (3.10)$$

**Fractional power spaces.**

For the Kuramoto-Sivashinsky equation (3.3), the nonlinear term $u_{xx} + uu_x$ is less smooth than the function $u$ itself. Nonlinear terms containing derivatives are common in
partial differential equations, and it is useful to be develop a formalism for coping with such questions of smoothness. This is done by defining fractional powers of the operator $A$.

When (in the notation of (3.4,5)) $u = \sum c_n w_n$, $A u$, the extension of $\partial^4 u/\partial x^4$ to $L^2$, is given by

$$Au = \sum \lambda_j c_j w_j^n = \sum (2n\pi/L)^4 c_n \sin(2n\pi x/L).$$

Define now $A^{1/2}$ according to

$$A^{1/2} u = \sum \lambda_j^{1/2} c_j w_j^n,$$

and then

$$A^{1/2} u = \sum (2n\pi/L)^2 c_n \sin(2n\pi x/L),$$

which is equal to $-\partial^2 u/\partial x^2$ when $u \in C^2$. Therefore $A^{1/2}$ is another “differential operator”, and $D(A^{1/2})$ includes the space of functions twice differentiable.

In general, the operators $A^\alpha$ are defined (cf. 3.9) by

$$A^\alpha u = \sum \lambda_j^\alpha c_j w_j$$

where $u = \sum c_j w_j$

and their domains (cf. 3.10) by

$$D(A^\alpha) = \{u|u = \sum c_j w_j, \sum \lambda_j^{2\alpha} |c_j|^2 < \infty\}. \tag{3.11}$$

The spaces $D(A^\alpha)$ are called (for obvious reasons) fractional power spaces. They are Hilbert spaces for the inner product and norm

$$(u,v)_\alpha = (A^\alpha u, A^\alpha v) \quad \|u\|_\alpha = |A^\alpha u|$$

and $A^\alpha$ maps $H$ isomorphically onto $D(A^\alpha)$. Further, since $A^{-1}$ is compact, $D(A^{s+\epsilon})$ is compactly embedded in $D(A^s)$, for any $\epsilon > 0$.

So for the Kuramoto-Sivashinsky equation with $A$ as in (3.4), one writes

$$f : D(A^\alpha) \to D(A^{\alpha-1/2}),$$
i.e. $f(u)$ contains second derivatives of $u$ and is hence less regular than $u$ itself. The choice of the value of $\alpha$ is not obvious \textit{a priori}; it is usually determined by the requirement that the equation is known to be dissipative in that space. For example, the Kuramoto-Sivashinsky equation is dissipative in $H$ and $D(A^{1/4})$, so it is natural to choose $\alpha = 0$ or $\alpha = 1/4$. That both these choices lead to negative power spaces is not a difficulty: $D(A^{-1/2})$ can simply be defined by (3.11). Which value of $\alpha$ is used will depend on the problem and the analysis that is possible in each case. Often it is best to use the higher value, since results with $u \in D(A^{1/4})$ will imply greater regularity of the solutions.

Once the value of $\alpha$ has been selected, the “general” problem is

$$
\frac{du}{dt} + Au + f(u) = 0 \quad f : D(A^\alpha) \to D(A^\beta).
$$

(3.12)

It is natural to consider initial conditions $u_0$ which lie in $D(A^\alpha)$, since this is the space on which $f$ acts. Furthermore, distances are usually measured in this space (e.g. $|A^\alpha(u - v)|$), since results that are valid for distances in $D(A^\alpha)$ with $\alpha > 0$ will certainly be valid in $H$. It is for this reason (see below) that semigroups for an equation such as (3.12) are usually defined on $D(A^\alpha)$ rather than on $H$.

**Dissipative evolution equations.**

It was remarked at the beginning of section II that partial differential equations have to be studied by means of semigroups due to the problems arising from any attempt to extend the solution backwards in time. The existence of solutions forward in time follows for the examples above based on two properties of $f$.

The first is that the derivatives occurring in the nonlinear term are not as high as those in the linear term $A$. For example, if $A$ is the Laplacian $-\nabla^2$, $f$ contains only first derivatives (or less). In the formalism of (3.12) this can be expressed as $\alpha - \beta < 1$. The second is Lipschitz continuity of $f$, necessary even in the theory of ordinary differential equations for existence and uniqueness results.\textsuperscript{15} What this requires is that for any $u, v \in U$,
a bounded set in $D(A^\alpha)$, there exists a constant $k_U$ such that

$$|A^\beta(f(u) - f(v))| \leq k_U|A^\alpha(u - v)|. \quad (3.13)$$

Such continuity clearly follows if $f \in C^1$, but is not automatic if $f$ is only $C^0$.

Dissipative equations of this type possess unique solutions defined for all time,$^{12, 39}$ given by a strongly continuous semigroup $S(t)$ defined on $D(A^\alpha)$, i.e. for an initial condition $u_0 \in D(A^\alpha)$, there exists a solution $u(t, u_0) = S(t)u_0$ defined for all $t > 0$. For $t > 0$ this solution is in fact $^{40}$ contained in $D(A^{1+\beta})$.

Although the nonlinear term is locally Lipschitz, there is no reason why the Lipschitz constant $k_U$ in (3.13) should not become unbounded as the size of $U$ increases. This complicates the analysis of the equation, since there is no straightforward estimate of $|A^\beta(f(u) - f(v))|$ when $u$ and $v$ are general elements of $D(A^\alpha)$. However, it is known that the set $B$ is absorbing, and therefore that all the asymptotic behaviour occurs within it. In other words, the dynamics outside $B$ are transient, and there is no loss if the nonlinear term is somehow truncated to be zero here. This will then lead to a uniform Lipschitz constant. (A similar truncation is almost invariably employed in finite-dimensional invariant manifold theory, see for example Kelley.$^{41}$)

Suppose therefore that $B$ is contained in a sphere in $D(A^\alpha)$ of radius $\rho$, which will be denoted by $\Omega_\rho$,

$$\Omega_\rho \equiv \{u : |A^\alpha u| \leq \rho\}.$$ 

Then choose a $C^\infty$ cut-off function $\theta : \mathbb{R}_+ \to [0, 1]$, satisfying

$$\theta(r) = 1, \ 0 \leq r < 1 \quad |\theta'(r)| \leq 2 \quad \theta(r) = 0, \ r \geq 2,$$

and set $R(u) = \theta(|A^\alpha u|/\rho)f(u)$. The new nonlinear term $R$ agrees with $f$ within $B$, but is zero for $|A^\alpha u| > 2\rho$. The “prepared” equation

$$du/dt + Au + R(u) = 0 \quad (3.14)$$
thus has the same asymptotic dynamics as (3.12), but is linear for large $|A^\alpha u|$. It is straightforward to show\(^\text{14}\) that $R$ is globally bounded,

$$|A^\beta R(u)| \leq C_0 \quad \forall \, u \in D(A^\alpha),$$

and globally Lipschitz as required,

$$|A^\beta(R(u) - R(v))| \leq C_1 |A^\alpha(u - v)| \quad \forall \, u, v \in D(A^\alpha),$$

with

$$C_0 \leq \sup_{\Omega, \rho} |A^\beta f(u)| \quad \text{and} \quad C_1 \leq 2C_0/\rho + k_{\Omega, \rho}.$$

Other “preparations” of the equation are possible.\(^\text{35,42}\) For example, if there is an absorbing set in $L^2$ (radius $\rho_0$) and in $H^1_0$ (radius $\rho_1$), set $R(u) = \theta(|u|/\rho_0)\theta(\|u\|/\rho_1)f(u)$.

This equation has been arrived at by truncating an evolution equation known to be dissipative. It is also possible to show that any such equation is dissipative, with a compact absorbing set.\(^\text{43}\) This means that restriction to such equations such as (3.14) is equivalent to the assumption of the existence of a finite-dimensional global attractor. It is natural to ask how the existence of this finite-dimensional attractor can be used to simplify the equations describing the asymptotic evolution of $u(t)$. This is tackled using the technique of inertial manifolds.
IV. INERTIAL MANIFOLDS.

One problem with the attractors obtained above is clearly exhibited by the famous Lorenz attractor\textsuperscript{44} in $\mathbb{R}^3$ - it has a complicated, twisted fractal structure, and a description of the dynamics “on the attractor” would be difficult to write down. A much tidier description is given by the original equations on $\mathbb{R}^3$.

It would therefore be of great interest if the global attractor of a dissipative partial differential equation such as (3.14) could be embedded in a finite-dimensional smooth manifold of solutions $\mathcal{M}$. If this manifold were positively invariant, i.e.

$$S(t)\mathcal{M} \subset \mathcal{M} \quad t > 0,$$

the equation could then be restricted to this manifold and the asymptotic behaviour obtained: imagine a three-dimensional phase space reduced to the two dimensions of an attracting surface.

Furthermore, examples can be constructed in which the rate of attraction to $\mathcal{A}$ is arbitrarily slow. An exponentially attracting manifold would therefore add further structure to the asymptotic behaviour, and make the description of the dynamics on the manifold more relevant to the transient flow.

To satisfy these criteria, the concept of an inertial manifold was introduced in the mid-80s by Foias \textit{et al.}\textsuperscript{45}

**Definition 4.1.** An inertial manifold $\mathcal{M}$ is a finite-dimensional Lipschitz manifold, which is positively invariant (4.1) and attracts all orbits exponentially, that is

$$\text{dist}(S(t)u_0, \mathcal{M}) \leq C(X)e^{-\mu t} \quad \text{for all } u_0 \in X,$$

where $X$ is a bounded set in $D(A^\alpha)$. The distance in (4.2) is measured in $D(A^\alpha)$.

Although the definition is new, the idea of reducing the dimension of the flow in such a way has been discussed before.\textsuperscript{46–48} However, in identifying inertial manifolds as
a promising area of research Foias et al. have inspired a more thorough investigation into the possibilities and implications of such an attracting manifold.

All inertial manifolds that have been obtained to date have been given as the graph of a function over a finite-dimensional subspace of $H$ (like a surface in $\mathbb{R}^3$ with $z = f(x, y)$). Defining the projection operator $P_n$ as the projection onto the first $n$ eigenfunctions,

$$P_n u = \sum_{i=1}^{n} (u, w_i)w_i,$$

and $Q_n$ as its orthogonal complement,

$$Q_n = I - P_n \quad Q_n u = \sum_{i=n+1}^{\infty} (u, w_i)w_i,$$

this means that there exists a function $\phi : P_n H \to Q_n H$ so that on the inertial manifold,

$$q \equiv Q_n u = \phi(p),$$

where $p = P_n u$. An expression for $\mathcal{M}$ itself is given by

$$\mathcal{M} = \mathcal{G}[\phi] = \{ p + \phi(p) : p \in P_n H \}.$$

Thus in the terminology of much mathematical physics, $^4^9$ the variables $q$ are “slaved” to the variables $p$ through the simple relationship $q = \phi(p)$. This means that the high wavelength/short length scale components are determined by the low wavelength/long length scale components.

Since $q = \phi(p)$ on the inertial manifold, restricting

$$du/dt + A u + R(u) = 0 \quad (4.3)$$

to $\mathcal{M}$ yields the finite-dimensional ordinary differential equation

$$dp/dt + A p + PR(p + \phi(p)) = 0 \quad (4.4),$$

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termed the inertial form. Thus one can say that the dynamics on the inertial manifold is “finite-dimensional”, since any trajectory on $M$ is given by $u(t) = p(t) + \phi(p(t))$, with $p(t)$ a solution of the inertial form (4.4). The hope is that the dynamics on $M$ will determine all the asymptotic dynamics of the PDE. This is indeed the case when any trajectory $u(t)$ of (4.3) can be shown to approach a trajectory $\bar{u}(t)$ on $M$,

$$|A^\alpha(u(t) - \bar{u}(t))| \to 0 \quad \text{as } t \to \infty,$$

and then one says that $M$ is “asymptotically complete”. This property is not automatic (there are counterexamples) but can be shown to follow from the conditions of theorem 4.2.

The condition required by almost all of the current theorems to ensure the existence of an inertial manifold is a sufficiently large gap in the spectrum of the linear operator $A$. There are many different methods of proof, but they all yield the same (or similar) conditions, an interesting illustration of a variety of different mathematical approaches being employed towards the same end. References 10,12,14,27,33,35,42,50,53-60 contain a mixture of existence results for general and particular equations. The following theorem is standard (see any of the above references for a theorem in a similar format); for ease of reference its statement recapitulates the form of the prepared equation derived at the end of the last section, but note that there is a restriction to $\alpha - \beta \leq 1/2$. This is a technical condition which allows for some simplification of the results, and still includes all the examples of section III. (Chow et al. consider the more general case $\alpha - \beta < 1$ and obtain additional results on the smoothness of $\phi$ and its continuity with respect to perturbations of $R$.)

**Theorem 4.2.** Consider the equation

$$\frac{du}{dt} + Au + R(u) = 0$$

where $A$ is a positive self-adjoint operator with compact inverse and eigenvalues $\lambda_n$, and the nonlinear function $R$, with $R(u) = 0$ for $|A^\alpha u| > 2\rho$, maps $D(A^\alpha)$ into $D(A^\beta)$ with
$\alpha - \beta \leq \frac{1}{2}$ and satisfies
\[ |A^\beta R(u)| \leq C_0 \quad \text{and} \quad |A^\beta (R(u) - R(v))| \leq C_1 |A^\alpha (u - v)|. \]

Then if the condition
\[ \lambda_{n+1} - \lambda_n > 2C_1 (\lambda_n^{\alpha-\beta} + \lambda_{n+1}^{\alpha-\beta}) \] (4.5)
holds there exists an inertial manifold given as the graph of a Lipschitz function $\phi : P_n H \to Q_n H \cap D(A^{\alpha})$, with $\phi(p) = 0$ for $|A^{\alpha}p| > 2\rho$ and Lipschitz constant at most 1. Furthermore, for all initial conditions $u_0 \in X$, bounded in $D(A^{\alpha})$, there exists a trajectory $\bar{u}(t)$ lying on $\mathcal{M}$ such that
\[ |A^{\alpha}(S(t)u_0 - \bar{u}(t))| \leq C(X)e^{-\mu t}, \]
where
\[ \mu \geq \lambda_{n+1} - 2C_1 \lambda_n^{\alpha-\beta}. \] (4.6)

Three examples

Theorem 4.2 gives sufficient conditions for the existence of an inertial manifold, and a natural question is whether it is actually applicable to any well-known partial differential equations. There is one simple condition that must be satisfied, and in many cases this is easy to check. The three examples 3.1-3.3 should suffice to show the strengths and weaknesses of the theorem.

What the gap condition requires is not only large gaps in the spectrum, but if $\alpha \neq \beta$, these gaps must occur soon enough. As an example, the eigenvalues $\lambda_n \propto n^2$ of the Laplacian in one dimension satisfy (4.5) with $\alpha = \beta$,
\[ \lambda_{n+1} - \lambda_n > 4C_1, \]
eventually, as $\lambda_{n+1} - \lambda_n \propto 2n+1$ (so inertial manifolds will exist for reaction-diffusion equations), but if $\alpha - \beta = \frac{1}{2}$, (4.5) becomes
\[ 2n + 1 > 2C_1(2n + 1), \]
either satisfied for all $n$ if $C_1 < \frac{1}{2}$ or for none if $C_1 > \frac{1}{2}$. This, as will be discussed below, is the essential problem in applying theorem 4.2 to the Navier-Stokes equations.

**Exercise 4.1.** The reaction-diffusion equation

$$\frac{du}{dt} - \nabla^2 u + f(u) = 0$$

with $u = 0$ on the boundary of $[0, L]^n$, and $f(u)$ Lipschitz,

$$|f(u) - f(v)| \leq C_1|u - v|.$$

For $\alpha = \beta = 0$ condition (4.5) becomes

$$\Lambda - \lambda > 4C_1.$$ 

Since the eigenvalues of the Laplacian on such a domain are proportional to

$$m_1^2 + m_2^2 + \ldots + m_n^2 \quad m_i \in \mathbb{N}$$

for $n = 1$ and $n = 2$ this equation always possesses an inertial manifold: this is obvious for $n = 1$ and follows from a result of Richards$^{61}$ for $n = 2$. However, for $n = 3$ there do not exist arbitrarily large gaps in the spectrum of $A$, and theorem 4.2 is no longer applicable. If $f$ is smooth enough an existence result for $[0, 2\pi]^3$ has been obtained by Mallet-Paret & Sell,$^{42}$ but the same authors (& Shao$^{62}$) have found a counterexample for $n = 4$ when the boundary conditions are $\nabla u \cdot n = 0$ (Neumann).

In the one-dimensional case much is known of the behaviour of these inertial manifolds due to the gradient structure of the equations. In particular any changes of dimension in the manifolds arise from changes in stability of the fixed points of the equation.$^{63,64}$

**Exercise 4.2.** The Kuramoto-Sivashinsky equation in one dimension.

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0 \quad u(x + L, t) = u(x, t).$$
The main task for the Kuramoto-Sivashinsky equation is to produce the Lipschitz constant $C_1$ required to test the applicability of the theorem. This requires the use of some of the calculus inequalities mentioned in section I, and much algebra.\textsuperscript{14} However, once the Lipschitz constant has been obtained,

$$|A^{-1/4}(R(u) - R(v))| \leq C_1 |A^{1/4}(u - v)|,$$

where $R(u)$ is the prepared nonlinear term

$$R(u) = f(u)\theta(|Du|/\rho_1),$$

the application of theorem 4.2, with $\alpha = 1/4$ and $\beta = -1/4$, is straightforward. Indeed, recalling that the eigenvalues of the linear term are $(2\pi n/L)^4$ (see (3.5)), the gap condition (4.5) becomes

$$(2\pi)^2 ((n + 1)^4 - n^4)/L^4 > 2C_1(2n^2 + 2n + 1)^2/L^2,$$

guaranteeing the existence of an inertial manifold for all values of $L$. Asymptotically this condition is $n > L^2C_1/2\pi$, which can be used to yield an estimate of the dimension $N$ of the inertial manifold, $N \sim L^3$.\textsuperscript{24} Better estimates than this, e.g. $N \sim L^{2.46}$, are possible.\textsuperscript{33}

**Exercise 4.3.** The 2D Navier-Stokes equation.

$$u_t - \mu \nabla^2 u + u \cdot \nabla u + \nabla p = f \quad \nabla \cdot u = 0 \quad x \in \Omega \quad u = 0 \text{ on } \partial \Omega.$$

Despite the success of theorem 4.2 in dealing with the above cases and many other equations, unfortunately the 2D Navier-Stokes equation, the inspiration for much of this theory, is not amenable to such an analysis. This is because the gaps of the Laplacian on a 2D domain do not satisfy (4.5) with $\alpha - \beta = \frac{1}{2}$, indeed on a general domain the eigenvalues satisfy\textsuperscript{65} $\lambda_n \sim Dn$, and so the gap condition cannot hold. Even on a rectangular domain,\textsuperscript{42,61} the gaps $h = \lambda_{n+1} - \lambda_n$ grow like $\log \lambda_n$, and so the gap condition

$$h > C_2 e^{h/2}$$

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with $C_2(\nu) \to \infty$ as $\nu \to 0$ cannot be satisfied for small $\nu$.

However, in this case (a rectangular domain) Kwak\textsuperscript{66} has managed to obtain an inertial form (a finite set of ordinary differential equations) for the dynamics on the attractor of this equation \textit{without} obtaining an inertial manifold. His method is to transform the Navier-Stokes equation into a generalised reaction diffusion equation, such that the attractor of the new system contains the attractor of the original equation. The theory of Sell & You\textsuperscript{67} for evolution equations with $A$ not self-adjoint then leads to the existence of an inertial manifold for the new system. The inertial form derived from this manifold describes the dynamics on the attractor of the 2D Navier-Stokes equation. (The same procedure has also been applied to a reaction-diffusion equation with an additional convective term.\textsuperscript{68})

This is a pleasing conjunction of inspiration and theory, and in some sense this is the present jewel of the subject, since it shows that in the asymptotic regime the two-dimensional Navier-Stokes equations are essentially finite-dimensional for all values of viscosity $\nu > 0$.

Such a result suggests that within the Navier-Stokes equations there may be “inertial behaviour” waiting to be extracted. The restriction to inertial manifolds which are graphs could be the reason that it has proved so hard to find directly. The possibility of constructing inertial manifolds of more general topology is an exciting and unexplored line of enquiry.

**The gap condition**

A straightforward application of theorem 4.2 to the Navier-Stokes equations fails because of the spectral gap condition. It is thus a natural question to ask to what extent this condition is necessary or sharp.

For the unusual case of the Chaffee-Infante reaction-diffusion equation

$$u_t - \nu u_{xx} + u^3 - u = 0, \quad u(0, t) = u(\pi, t) = 0$$
it is known\textsuperscript{63} that the dimension of the smallest inertial manifold is exactly $n$ when

$$(n + 1)^{-2} < \nu < n^{-2},$$

i.e. $n \sim \nu^{-1/2}$. However, an application of theorem 4.2 to this equation (using the results of section II) gives the dimension of the manifold as $n \sim \nu^{-5/2}$, an extremely inaccurate prediction in the light of the exact result. There is thus good reason to believe that many inertial manifolds will exist outside the range of the theorem (and there are indeed examples where this is so, such as\textsuperscript{42} reaction-diffusion equations on $\mathbb{R}^3$ (as discussed above as example 4.3) and for the dynamics of rotating elastic beams,\textsuperscript{69} where the function $\phi$ giving the inertial manifold can be determined explicitly). However, there are justifications in the general case for considering the gap condition to be desirable, if not essential. The main reason for this is the importance of flow normal hyperbolicity, which follows from the gap condition and guarantees asymptotic completeness\textsuperscript{51} (a similar condition is hidden in Constantin et al.\textsuperscript{35}). This ensures that the rate of convergence towards the manifold is faster than any convergence within the manifold itself.

**Definition 4.3.** The manifold is flow normally hyperbolic if the rate $\kappa$ of convergence of any two trajectories in $\mathcal{M}$,

$$|u_1(t) - u_2(t)| \geq C|u_1(0) - u_2(0)|e^{-\kappa t} \quad u_1, u_2 \in \mathcal{M}$$

is less than $\mu$, the rate of convergence towards $\mathcal{M}$.

Not only does this property guarantee asymptotic completeness, there are $C^1$ counterexamples where this property does not hold and the manifold is not asymptotically complete\textsuperscript{51} This property can also be shown to ensure persistence of the manifold.\textsuperscript{43,70}

The relation of flow normal hyperbolicity to the gap condition can be easily shown. Indeed, the rate of attraction $\mu$ is given by (4.6),

$$\mu \geq \lambda_{n+1} - 2C_1\lambda_{n+1}^{\alpha-\beta}.$$
and a basic estimate of the rate of separation $\kappa$ of trajectories on $\mathcal{M}$ can be simply derived from
\[
dp{p}{t} + Ap + Pf(p + \phi(p)) = 0,
\]
as $\kappa \leq \lambda_n + 2C_1\lambda_n^{\alpha-\beta}$. Flow normal hyperbolicity is then ensured by
\[
\lambda_{n+1} - \lambda_n > 2C_1(\lambda_n^{\alpha-\beta} + \lambda_n^{\alpha-\beta}),
\]
which is precisely the gap condition (4.5). Although this is a coarse derivation, it does indicate that the gap condition and flow normal hyperbolicity are not unrelated.

Despite these comments, the study of general inertial manifolds, whose existence is not derived from a theorem such as 4.2 is still interesting, and it is possible\textsuperscript{43} to derive results valid for all such manifolds. Such general results are fundamental for an investigation of the bifurcation structure of the manifolds for a family of equations such as
\[
du{u}{t} + \nu Au + R(u) = 0
\]
as $\nu$ is decreased, an important but relatively unexplored avenue of enquiry.

**Approximate inertial manifolds and nonlinear Galerkin methods.**

Although theorem 4.2 ensures that there exists some function $\phi$ whose graph is an inertial manifold, its proof is non-constructive and gives no clue as to what this function may be. Furthermore there are important equations, such as the 2D Navier-Stokes equation, for which the existence of an inertial manifold cannot be proved using the current methods. It is clear that for inertial manifolds to be of numerical use it is necessary to develop ways to find approximate inertial manifolds which can be given explicitly, and it turns out that such manifolds can be shown to exist for far more equations than exact inertial manifolds.\textsuperscript{71} Just as an inertial manifold gives rise to the asymptotically exact slaving rule $q = \phi(p)$, an approximate inertial manifold gives an approximate rule $q \approx \psi(p)$.\textsuperscript{72} Such a rule clearly has application to various numerical methods. Although it can be applied
to finite difference methods and finite element analysis, such a rule has found its most natural application in a spectral setting called the nonlinear Galerkin method. (What follows below is only a cursory summary of a rapidly expanding area of research; see for a much fuller list of references.)

The standard Galerkin method for analysing

\[ \frac{du}{dt} + Au + f(u) = 0 \] (4.7)

is to consider the finite-dimensional quantities \( u_m, u_m \in P_mH \), solutions of the truncated (finite-dimensional) equation

\[ \frac{du_m}{dt} + Au_m + P_m f(u_m) = 0. \] (4.8)

The solutions \( u_m \) of (4.8) then converge to the solution \( u \) of (4.7) uniformly on bounded intervals of time and on compact sets in \( H \) as \( m \to \infty \). The use of such Galerkin approximations in both numerical and analytical work is common. The nonlinear Galerkin method extends (4.8) by including some of the neglected terms using an approximate inertial manifold, so that (4.8) becomes

\[ \frac{du_m}{dt} + Au_m + P_m f(u_m + P_2m \psi(u_m)) = 0, \]

where \( \psi \) is the function defining an approximate inertial manifold. The factor \( P_2m \) has to be included since otherwise \( \psi \) produces an infinite-dimensional term. The standard Galerkin approximation now corresponds to the approximate inertial manifold \( \psi = 0 \).

Attention below will be mainly restricted to approximations for the 2D Navier-Stokes equations, reflecting the preoccupations of the literature (see references 77 and 78 for a discussion of approximate inertial manifolds for reaction-diffusion equations).

Two approximations derived by analytic arguments can also be recovered via a more physically interesting approach. From the \( Q \) part of the evolution equation

\[ \frac{dq}{dt} + \nu Aq + Qf(u) = 0, \]
any stationary state (one with $du/dt = 0$) satisfies

$$\nu Aq + Qf(p + q) = 0,$$

and thus the manifold $M^s$ given as the graph of $\phi^s$, where

$$\phi^s(p) = -(\nu A)^{-1}Qf(p + \phi^s(p)) \quad (4.9)$$

contains all the stationary solutions of the equation. Since these are contained in the global attractor and hence in the true inertial manifold (should one exist) the two manifolds are “threaded together”. If the dimension of this approximate manifold $M^s$ is taken to be $m$, i.e. $Q = Q_m$, all trajectories enter a neighbourhood of the manifold of width $c_s\lambda_{m+1}^{-2}$. In other words, by taking the dimension large enough, it can be ensured that the attractor lies within an arbitrarily small distance of the approximate inertial manifold.

However, the function $\phi^s$ is not given explicitly by (4.9). This problem can be circumvented, since it can be shown that if $m$ is large enough the iterates of

$$\phi \mapsto -(\nu A)^{-1}Qf(p + \phi(p))$$

converge to a fixed point which is a solution of (4.9), and so one can use these iterates to approximate this solution. Starting with $\phi = 0$ (a standard Galerkin method) yields a first approximation

$$\phi_0(p) = -(\nu A)^{-1}Qf(p), \quad (4.10)$$

which has the advantage of being analytically simple, and the attractor lies within $c_0\lambda_{n+1}^{-3/2}$ of corresponding manifold $M_0$. Note that the slaving rule (4.10) is that produced by the standard technique of adiabatic elimination.

The second iterate is

$$\phi_1(p) = -(\nu A)^{-1}Qf(p + \phi_0(p))$$

defining a new manifold $M_1$. It turns out that the attractor lies as close to $M_1$ as to $M^s$ itself ($c_1\lambda_{n+1}^{-2}$) and so little should be lost in taking $\phi_1$ in place of $\phi^s$. 

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These ideas have been developed in several ways. One series of papers$^{82-85}$ has been concerned with the idea that the approximate slaving function represents in some way a good model for turbulence, and has compared this approach to a standard statistical model. In a further analytic generalisation, Temam$^{86}$ has proposed a hierarchy of approximate manifolds of ever increasing accuracy. However, attention has increasing turned to the question of whether the nonlinear method is in fact an improvement on the simple truncation, and its future as a practical method for computations.

Initial analytical results, analogous to those for the standard Galerkin method, showed convergence of the finite-dimensional nonlinear Galerkin schemes to the full problem but gave no indication of the rate of convergence.$^{75,87}$ This result has been improved$^{76,88}$ in the case of Dirichlet boundary conditions to show that the nonlinear scheme converges faster than the linear one. If $u_m(t)$ represents the solution of the Galerkin equations and $u(t)$ the solution of the full equation, the estimates can be expressed as bounds for the difference of the $P_m$ components of the solutions over finite time intervals, and are then of the form (for some constant $K(T)$)

$$|u_m(t) - P_m u(t)| \leq K \lambda_m^{-\alpha}, \quad t \in [0, T]$$

with $\alpha = 1$ in the linear case and $\alpha = 3/2$ in the nonlinear case, taking the simplest approximate manifold (4.10). Thus there is a clear hope that the nonlinear Galerkin method will offer better accuracy than standard methods.
A first analytical approach to the utility of the nonlinear Galerkin method was undertaken by Heywood & Rannacher,\textsuperscript{88} who convincingly argue that the approximate slaving rule does not reflect any fundamental physics of the problem, since the improved accuracy results from the incompatibility of the Fourier basis functions with the solution at the boundary. Although this would seem to imply that no improvement can be realised in the spatially periodic case, a more detailed investigation by Jones \textit{et al.}\textsuperscript{89} has shown that when, for whatever reason (e.g. the inclusion of a less regular forcing term in the equation), the solution is not well approximated by the chosen Fourier basis the nonlinear Galerkin method can be expected to converge faster than the standard method.

Several other authors have reported encouraging numerical results.\textsuperscript{81,90–93} However, these techniques have yet to be adopted by those engaged in serious numerical computations, and there is a need to demonstrate unequivocally the gains to be had from an efficient implementation of the nonlinear Galerkin method in terms of computing time. It may be the case that the use of these approximate slaving rules in other contexts (e.g., finite element and finite difference schemes) will prove most productive.

\textsuperscript{89}D.A. Jones, L.G. Margolin, and E.S. Titi, “On the effectiveness of the approximate inertial manifold - a computational study,” Los Alamos Preprint LA-UR-93-3475 (1993).
CONCLUSION.

For a wide class of partial differential equations it is possible to prove the existence of a finite-dimensional global attractor by obtaining simple bounds on the size of the solution (theorem 2.3): in most cases this amounts to proving that $|u|$ and $|Du|$ are eventually bounded, as was done for the two examples of section II. For dissipative evolution equations a simple condition on the spectrum of the linear operator $A$ ensures the existence of an inertial manifold, whose simple form allows the reduction of the PDE to a finite set of ordinary differential equations (theorem 4.2). In computational situations, use of an approximate inertial manifold in the inertial form may lead to faster and more accurate computations than the standard Galerkin method.

The theory of attractors is now very well developed, and the many minor modifications and analytical tricks required to apply it to a variety of particular examples are ever increasing.$^{11,14}$ However, inertial manifold theory has been less exhaustively studied, and there appear to be two outstanding problems. The first is the possibility of inertial manifolds that are not graphs but more general Lipschitz manifolds, which may open up the possibility of obtaining a finite-dimensional description of the behaviour in a variety of equations where this is not presently possible. Secondly, the analysis of properties of general inertial manifolds as a means to studying their bifurcation structure has been almost entirely ignored.

The objective of understanding fully developed turbulence in the 3D Navier-Stokes equations is a long way off as yet.$^{94}$ Until the questions of existence and uniqueness have been solved (although Raugel & Sell $^{95,96}$ have recently made some headway on thin domains in $\mathbb{R}^3$), one cannot even define an appropriate semigroup for the equations. However, Kwak’s result on inertial forms for the 2D equations does give some hope that this goal will eventually be achieved.
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